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NOTE ABOUT ŁUKASIEWICZ'S THEOREM CONCERNING THE SYSTEM OF AXIOMS OF THE IMPLICATIONAL PROPOSITIONAL CALCULUS

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In [1][1] Łukasiewicz has proved the following Theorem:

If to Pierce's law CCCpqpp and the law of the syllogism CCpqCCqrCpr we join any thesis of the form $Cp C\alpha\beta$, that is a thesis whose antecedent is a variable and whose consequent is an implication, then we obtain the law of simplification CpCqp and hence a complete system of axioms of the implicational propositional calculus.

Remark I: Concerning this Theorem and its proof it should be noticed:

(a) that Łukasiewicz has established this Theorem using exclusively the rules of substitution and detachment for implication,
(b) that any concrete formula of the form $CpC\alpha\beta$ is a purely implicational thesis of the bi-valued propositional calculus,
(c) that since no substitution instance of the formula of the form $CpC\alpha\beta$ is used in the proof presented by Łukasiewicz, the Theorem holds for any thesis of the form $CpC\alpha\beta$,
(d) that, as Łukasiewicz indicated, if a formula of the form $CpC\alpha\beta$ is not a thesis of the implicational propositional calculus, then the proof of the Theorem still stands, but instead of the system mentioned above we are obtaining a contradiction,
(e) that in [1], p. 310, Łukasiewicz has shown that the theses of the form $CpC\alpha\beta$ cannot be replaced by the law of identity *Cpp*, i.e., that the set of theses {*Cpp*; *CCCpqpp*; *CCpqCqrCpr*} cannot be accepted as an axiom-system of the implicational propositional calculus

and

1. An acquaintance with [1] is presupposed. In [1] there are two serious printing errors, viz. on page 306, line 15, instead of *CCqrCCqrCpr* there should be: *CCpqCCqrCpr*, and on page 307, line 25, instead of 11 $CpCCCsq\alpha sCCsqs$ there should be: 11 $CpCCCsq\alpha CC\beta sCCsqs$.

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(f) that the discussed Theorem is the most general proposition concerning the modifications of the Tarski-Bernays axiom-system in the way, as it is indicated in [1], since it automatically covers all of the particular results which were obtained previously by M. Wajsberg, *cf.* [3], [4], and [1], p. 306.

In this note I shall discuss some elementary consequences of this Theorem which, it seems to me, Łukasiewicz did not observe. Namely, that the Theorem allows us to establish without essential difficulties several axiomatizations of the implicational propositional calculus each of which is akin to the Tarski-Bernays axiom-system, but shorter. For this purpose let us consider the following theses:

A1 *CCpqCCqrCpr*
B1 *CtCCpqCCqrCpr*
C1 *CCpqCtCCqrCpr*
D1 *CCpqCCqrCtCpr*
E1 *CCCpqpp*
F1 *CtCCCpqpp*
G1 *CCCpqpCtp*

It will be shown that each of the following pairs {*A1*; *F1*}, {*B1*; *E1*}, {*A1*; *G1*}, {*C1*; *E1*}, and {*D1*; *E1*} can be accepted as an axiom-system of the implicational propositional calculus. Proof:

(i) Since *F1* holds

E1 *CCpqpp* [*F1*, *t*/*CtCCCpqpp*; *E1*]

and *B1* implies

A1 *CCpqCCqrCpr* [*B1*, *t*/*CtCCpqCCqrCpr*; *B1*]

it follows at once from the Theorem that each of the pairs {*A1*; *F1*} and {*B1*; *E1*} posesses the desired property.

(ii) Let us assume *A1* and *G1*. Then:

G2 *CCCCqrCprsCCpqs* [*A1*, *p*/*Cpq*, *q*/*CCqrCpr*, *r*/*s*; *A1*]
G3 *CCpCprCtCpr* [*G2*, *q*/*Cpr*, *s*/*CtCpr*; *G1*, *p*/*Cpr*, *q*/*r*]
G4 *CtCCCpqpp* [*G3*, *p*/*CCpqp*, *r*/*p*; *G1*, *t*/*CCpqp*]
E1 *CCCpqpp* [*G4*, *t*/*CCCpqpCtp*; *G1*]

Since {*A1*; *G1*} → {*E1*; *G4*}, according to the Theorem {*A1*; *G1*} constitutes an axiom-system of the investigated theory.

(iii) Let us assume *C1* and *E1*. Then:

C2 *CvCCCtCCqrCprsCCpqs* [*C1*, *p*/*Cpq*, *q*/*CtCCqrCpr*, *r*/*s*, *t*/*v*; *C1*]
C3 *CCCtCCqrCprsCCpqs* [*C2*, *v*/*CCpqCtCCqrCpr*; *C1*]
A1 *CCpqCCqrCpr* [*C3*, *t*/*CCCqrCprq*, *s*/*CCqrCpr*; *E1*, *p*/*CCqrCpr*]

Since {*C1*; *E1*} → {*A1*; *C2*}, in accordance with the Theorem the desired proof is carried out.

(iv) Let us assume *D1* and *E1*. Then:

D2	*CCCCqrCtCprsCvCCpqs*	[*D1*, *p/Cpq*, *q/CCqrCtCpr*, *r/s*, *t/v*; *D1*]
D3	*CvCCpCtCprCtCpr*	[*D2*, *q/CtCpr*, *s/CtCpr*; *E1*, *p/CtCpr*, *q/r*]
D4	*CCpCtCprCtCpr*	[*D3*, *v/CCCpqpp*; *E1*]
D5	*CCqrCCpqCpr*	[*D4*, *p/Cpq*, *t/Cqr*, *r/Cpr*; *D1*, *t/Cpq*]
D6	*CCtCCpqpCtp*	[*D5*, *q/CCpqp*, *r/p*, *p/t*; *E1*]
D7	*CCvCtCCpqpCvCtp*	[*D5*, *q/CtCCpqp*, *r/Ctp*, *p/v*; *D6*]
A1	*CCpqCCqrCpr*	[*D7*, *v/Cpq*, *t/Cqr*, *p/Cpr*; *D1*, *t/CCprq*]

Whence $\{D1;\ E1\} \rightarrow \{A1;\ D3\}$ and, therefore, according to the Theorem the proof is complete.

Thus, it follows from (i)-(iv) that each of the pairs of theses mentioned above can be accepted as an axiomatization of the implicational propositional calculus. Q.E.D.

Open Problem: I was unable to prove or to disprove whether *E1* and

K1 *CCpqCCqrCpCtr*

can be accepted as an adequate axiom-system of the implicational propositional calculus.

Remark II: It is worthwhile to notice that the Theorem of Łukasiewicz can be applied not only to the implicational propositional calculus, but also to some other propositional calculi. It follows immediately from the proof of the Theorem, *cf.*, [1], pp. 307-308, and Remark I,

(α) that in order to avoid a contradiction a formula of the form $CpC\alpha\beta$ must be a thesis of the consistent system under consideration,

and

(β) that no substitution instance of such a thesis is used in the proof of the Theorem.

For instance, let us consider the following two cases:

(A) Let us assume the theses *A1, E1,* and

R1 *CqCCNpNqp*

each of which belongs to the classical propositional calculus. Since *R1* has the form $CpC\alpha\beta$, it follows from the Theorem that these three theses imply:

R2 *CpCqp*

Whence, we have at our disposal the implicational propositional calculus and, therefore, we have also:

R3 *CCpCqrCCpqCpr*
R4 *CCpCqrCqCpr*

Then:

R5 *CCNpNqCpq* [*R4*, *p/q*, *q/CNpNq*, *r/p*; *R1*] which means, *cf.* [2], p. 136, that $\{A1; E1; R1\} \rightleftarrows \{R2; R3; R5\}$, i.e., that due the Theorem we are able to establish without any difficulty or long deductions that the theses *A1, E1,* and *R1* constitute an axiom-system of the bi-values propositional calculus.

(B) Let us assume the theses *A1, E1*, and

S1 *CpCNpq*

Since matrix

$\mathfrak{M}1$

	C	0	1	N
*	0	1	1	0
	1	0	1	0

in which 1 is the designated value, verifies *A1, E1,* and *S1* but falsifies *R1* for $p/0$ and $q/1$: $C1CCN0N10 = C1CC000 = C1C10 = C10 = 0$, we know that a system based on the axiom *A1, E1,* and *S1* is a proper subsystem of the classical propositional calculus. Then, due to the Theorem we know automatically that the theses *A1, E1,* and *S1* imply *R2,* i.e., that the system $\{A1; E1; S1\}$ is the implicational propositional calculus, to which *S1* is added as a new axiom.

It is possible to present many other analogous systems, even in the fields of many-valued propositional calculi, but for such examples more explanations and deductions would be required.

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