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HYPERENUMERATION REDUCIBILITY

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There are several ways of reducing a set B to a set A and the differences are mostly related to the basic procedures allowed in the reduction (recursive, arithmetical, etc.) but also to the manner in which the input set A enters in the reduction. In some cases both inputs from A and the complement $\overline{A}$ are allowed in the reduction (as in Turing reducibility) and in other cases the reduction operates only on positive information about A (as in enumeration reducibility). It is important to see that in some cases the positive reduction is actually a generalization of the positive-negative reduction with the same basic procedures. For instance the Turing reduction $B \leq_T A$ can be in fact defined as $C_B \leq_e C_A$ where C_A is (the graph of) the characteristic function of A. This does not mean that the structures induced by the reductions (the so-called degrees) are similar. In fact it is well known there are important differences between the ordering of Turing degrees and partial degrees.

In this paper we present a form, a positive reduction, which we call hyperenumeration reducibility. It is related to hyperarithmetical reduction exactly as enumeration reducibility is related to Turing reducibility. The basic procedures in hyperenumeration reducibility are analytical involving function quantifiers. Since any form of positive reduction is essentially weak this strengthening of the basic procedures seems to be a desirable feature. We attempt, in this paper, a classification of sets of natural numbers in terms of hyperenumeration reducibility. The basic ideas have been first applied to enumeration reducibility and in Section **1** we give the fundamental definitions. The ideas and results in this section are introduced mainly as a motivation for the material in the remaining sections. Hyperenumeration reducibility is defined in Section **2**, and in Section **3** we introduce the fundamental notion of pseudo hyperarithmetical set. The structure of degrees containing such sets is discussed. Finally, we prove there are sets of degrees in which there is no pseudo hyperarithmetical set. The construction here follows a forcing technique which was first introduced by Thomason in [6]. We shall use the notation of Rogers [5].

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The operation $(x \cdot y)$ produces the concatenation of the sequence number x by adding y as a new element. Hence $(\bar{f}(v) \cdot f(v)) = \bar{f}(v + 1)$. We shall assume $\bar{f}(0) = 1$. The recursive function $l(u)$ is such that $l(\bar{f}(v)) = v$.

1 *Enumeration reducibility* We recall that B is enumeration reducible to A in case there is a recursively enumerable set (**RE** set) W such that

$$x \in B \text{ iff } (\exists y)(\langle x, y\rangle \in W \ \& \ D_y \subseteq A)$$

where D_y is the canonical enumeration of finite sets. This relation is written $B \leqslant_e A$ and W is called the basis of the reduction. This relation is a partial order and the degrees induced by this order we shall call e-degrees. The least upperbound of sets A and B is given by the set

$$A \vee B = \{2 \cdot x \colon x \in A\} \cup \{2 \cdot x + 1 \colon x \in B\}$$

Clearly we have

$$A \vee \overline{A} \equiv_e C_A \equiv_e A'$$

where C_A is the characteristic function of A and A' is the jump of A.

Proposition 1 $B \leqslant_e A'$ iff $B \leqslant_1 A'$.

The implication from right to left is clear. From left to right note that since A' is recursively enumerable in A from $B \leqslant_e A'$ we get that B is recursively enumerable in A, hence $B \leqslant_1 A'$.

We shall identify a function F (total or partial) with its graph. This means that $F = \{\langle x, F(x)\rangle \colon x \text{ in the domain of } F\}$. Note that if F is a total function then $\overline{F} \leqslant_e F$.

We shall say that a set A is pseudo recursive (**PR**) in case $\overline{A} \leqslant_e A$. Note that in case A is **RE** set then A is recursive if and only if A is **PR**.

Proposition 2 *The following conditions are equivalent:*

(i) A *is* **PR**
(ii) $A \equiv_e C_A$
(iii) $A \equiv_e A'$

If A is **PR** then clearly $C_A \leqslant_e A$ so (ii) follows. We have already mentioned the equivalence of (ii) and (iii). Now (ii) implies (i) since $\overline{A} \leqslant_e C_A$.

A listing of a set A is a total function F such that A is the range of F. If F is a listing of A then $A \leqslant_e F$. If A is infinite then A has a unique strictly increasing listing F_A. Clearly A is **PR** if and only if $F_A \leqslant_e A$.

Proposition 3 *If* $A \leqslant_T B \leqslant_e A$ *and* B *is* **PR** *then* A *is* **PR**.

Since $A \leqslant_T B$ is equivalent to $C_A \leqslant_e C_B$ we have

$$\overline{A} \leqslant_e C_A \leqslant_e C_B \leqslant_e B \leqslant_e A$$

Corollary *If* B *is* **PR** *and* $A \equiv_m B$ *then* A *is* **PR**.

Note that in the corollary we can replace $\equiv_m$ by any relation that implies both $\equiv_T$ and $\equiv_e$. For instance, the relation $\equiv_p$ in [4]. It is easy to show that retraceable sets and almost recursive sets are **PR** sets. Note also that for any set A, $A \leqslant_1 A'$ and $\overline{A} \leqslant_1 A'$ hence $A' \leqslant_e \overline{A'}$, so $\overline{A'}$ is always **PR**. But then A' is never **PR** since otherwise we should have

$$A'' \equiv_e C_{A'} \equiv_e C_{\overline{A'}} \equiv_e \overline{A'} \leqslant_e A'.$$

Hence $A'' \leqslant_1 A'$ what is impossible.

We shall say that a set A is pseudo enumerable (**PE**) in case there is a set B such that $A \equiv_e B$ and B is **PR**. Since $A' \equiv_e C_A$ it follows that A' is always **PE**. It is easy to show that regressive sets and almost recursively enumerable sets ([1] and [2]) are also **PE** sets.

Theorem 1 *The following conditions are equivalent:*

(i) *A is* **PE** *set.*
(ii) *There is a set B such that $A \equiv_e B'$.*
(iii) *There is a listing F of A such that $F \leqslant_e A$.*

The proof of this theorem is given by Case in [3].

Examples of sets which are not **PE** have been constructed elsewhere, for instance in [3]. Here we shall give a construction of a set B which is not **PE** but the complement $\overline{B}$ is actually **PR**.

Theorem 2 *Let A be any set. There is a set B such that:*

(i) $A <_T B \leqslant_T A'$.
(ii) $A' <_e B <_e \overline{B} \leqslant_e A''$.

Furthermore if C is **PE** *and $C \leqslant_e B$ then $C \leqslant_e A'$.*

We construct an infinite sequence of pairs of finite disjoints sets: $(B_0^1, B_0^2), (B_1^1, B_1^2), \ldots, (B_n^1, B_n^2), \ldots$ and define $B = \bigcup_n B_n^1$. It will follow that $\overline{B} = \bigcup_n B_n^2$. The construction will be such that $x \in B$ if and only if there is a prime number $p > 3$ such that $p^{x+1} \in \overline{B}$, hence $B \leqslant_e \overline{B}$. Furthermore $x \in A$ if and only if $2^{x+1} \in B$ and $x \in \overline{A}$ if and only if $3^{x+1} \in B$, so we have $A \leqslant_1 B$ and $\overline{A} \leqslant_1 B$. We put $B_0^1 = \{1\}$, $B_0^2 = \{0\}$. Suppose B_n^1 and B_n^2 have been defined. We put $D = B_n^1 \cup B_n^2$ and consider four cases.

(a) $n = 4 \cdot m$. In this case we shall ensure that $B \neq W_m^A$. Let k be the smallest number greater than every number in D, k not a power of 2 and k not a power of 3.

$$\text{If } 5^{k+1} \in W_m^A \text{ we put } B_{n+1}^1 = B_n^1 \cup \{k\},\ B_{n+1}^2 = B_n^2 \cup \{5^{k+1}\}$$
$$\text{If } 5^{k+1} \notin W_m^A \text{ we put } B_{n+1}^1 = B_n^1 \cup \{5^{k+1}\},\ B_{n+1}^2 = B_n^2$$

(b) In this case we shall ensure that every number will be eventually in some pair. At the same time we put in $\overline{B}$ numbers of the form p^{x+1} where $x \in B$ and p is some prime different from 2 or 3. Let p be the smallest prime number different from 2 and 3 and greater than every number in D;

and let k be the smallest number not in D, k not a power of 2, k not a power of 3. We put:

$$B^1_{n+1} = B^1_n \cup \{k\}$$
$$B^2_{n+1} = B_n \cup \{p^{x+1}: x \in B^1_n\}$$

(c) In this case we get $A \leqslant_1 B$ and $\overline{A} \leqslant_1 B$.

If $m \in A$ we put $B^1_{n+1} = B^1_n \cup \{2^{m+1}\}$, $B^2_{n+1} = B^2_n \cup \{3^{m+1}\}$
If $m \notin A$ we put $B^1_{n+1} = B^1_n \cup \{3^{m+1}\}$, $B^2_{n+1} = B^2_n \cup \{2^{m+1}\}$

(d) For this case we introduce the following notation:

$$\Phi_m(X) = \{x: (\exists y)(\langle x, y\rangle \in W_m \ \&\ D_y \subseteq X)\}$$

We shall ensure that whenever F is a total function and $\Phi_m(B) = F$ then $F \leqslant_{\mathrm{T}} A$, hence $F \leqslant_{\mathrm{e}} A'$. We consider two subcases:

(d1) There is a finite extension C of B^1_n such that:

(i) $C \cap B^2_n = \emptyset$
(ii) If $2^{x+1} \in C$ then $x \in A$
(iii) If $3^{x+1} \in C$ then $x \notin A$
(iv) There are numbers x, y, z such that $y \neq z$, $\langle x, y\rangle \in \Phi_m(C)$ and $\langle x, z\rangle \in \Phi_m(C)$.

Note that we can decide whether (d1) holds or not using A'. We assume some effective procedure is given in advance (recursive on A) to generate all C's satisfying conditions (i) to (iv). Let C_1 be the first generated. We put $B^1_{n+1} = C_1$, $B^2_{n+1} = B^2_n$.

(d2) There is no such C. In this case we put $B^1_{n+1} = B^1_n$, $B^2_{n+1} = B^2_n$.

Assume now that F is a total function and that $F = \Phi_m(B)$. Hence case (d1) does not apply and we can compute F by generating the sets C satisfying conditions (i) to (iii) and at the same time generating $\Phi_m(C)$ for each such C. This makes $F \leqslant_{\mathrm{T}} A$.

The whole construction is recursive in A' so both B and $\overline{B}$ are recursively enumerable in A', hence $B \leqslant_{\mathrm{T}} A'$. By construction B is not recursively enumerable in A. Hence we have $A <_{\mathrm{T}} B$. Furthermore since $A \leqslant_1 B$ and $\overline{A} \leqslant_1 B$ we have $A' \leqslant_{\mathrm{e}} B$. But in fact we have $A' <_{\mathrm{e}} B$ because B is not recursively enumerable in A. It follows that B is not **PE**, hence $B <_{\mathrm{e}} \overline{B}$. Since $\overline{B} \leqslant_{\mathrm{T}} A'$, we have $\overline{B} \leqslant_{\mathrm{e}} A''$.

2 *Hyperenumeration reducibility* In this section we define hyperenumeration reducibility and prove some of the fundamental properties of this relation. Let B and A be sets such that for some **RE** set W the following relation holds:

$$x \in B \text{ iff } (\forall f)(\exists v)(\exists y)(\langle \bar{f}(v), x, y\rangle \in W \ \&\ D_y \subseteq A).$$

Then we say that B is hyperenumeration reducible to A and write this relation: $B \leqslant_{\mathrm{he}} A$.

Proposition 4 *The following conditions are equivalent:*

(i) *There is a* Π_1^1*-set* W *such that the following holds:*

$$x \in B \text{ iff } (\forall f)(\exists v)(\exists y)(\langle \bar{f}(v), x, y\rangle \in W \;\&\; D_y \subseteq A)$$

(ii) *There is a primitive recursive predicate* $R(u, x)$ *such that:*

$$x \in B \text{ iff } (\forall f)(\exists v)(R(\bar{f}(v), x) \;\&\; D_{(v)_1} \subseteq A)$$

(iii) $B \leqslant_{\mathrm{he}} A$.

The implication from (i) to (ii) is proved using the standard rules of exportation, permutation, and contraction of quantifiers. The other implications are trivial.

Theorem 3 *If* $B \leqslant_{\mathrm{he}} A$ *and* $C \leqslant_{\mathrm{he}} B$ *then* $C \leqslant_{\mathrm{he}} A$.

Using part (ii) of Proposition 4 we can find primitive recursive predicates R_1 and R_2 such that $x \in C$ iff:

$$(\forall g)(\exists w)\left[R_1(\bar{g}(w), x) \;\&\; (\forall z)(z \in D_{(w)_1} \rightarrow (\forall f)(\exists v)(R_2(\bar{f}(v), z) \;\&\; D_{(v)_1} \subseteq A))\right]$$

By exportation, permutation, and contraction of quantifiers we get primitive recursive predicates R_3 and R_4 such that $x \in C$ iff:

$$(\forall f)(\exists v)\left[R_3(\bar{f}(v), x) \;\&\; (\forall z)(z \in D_{d_1(v)} \rightarrow R_4(\bar{f}(v), z) \;\&\; D_{d_2(v,z)} \subseteq A))\right]$$

where $d_1(v)$ and $d_2(v, z)$ are recursive functions. Define **RE** sets as follows:

$W_1 = \{\langle u, x, y\rangle : R_3(u, x)\}$
$W_2 = \{\langle u, x, y\rangle : (\forall z)(z \in D_{d_1(1(u))} \rightarrow R_4(u, z) \;\&\; D_{d_2(1(u),z)} \subseteq D_y))\}$
$W_3 = W_1 \cap W_2$

and we have

$$x \in C \text{ iff } (\forall f)(\exists v)(\exists y)(\langle \bar{f}(v), x, y\rangle \in W \;\&\; D_y \subseteq A)$$

Theorem 4 *If* $B \leqslant_{\mathrm{e}} A$ *then* $B \leqslant_{\mathrm{he}} A$ *and* $\overline{B} \leqslant_{\mathrm{he}} \overline{A}$.

Assume that

$$x \in B \text{ iff } (\exists y)(\langle x, y\rangle \in W \;\&\; D_y \subseteq A)$$

It follows immediately that $B \leqslant_{\mathrm{he}} A$. To prove $\overline{B} \leqslant_{\mathrm{he}} \overline{A}$ we note that

$$x \in \overline{B} \text{ iff } (\forall y)(\langle x, y\rangle \in W \rightarrow D_y \not\subseteq A)$$

Define arithmetical set W_1 as follows:

$$W_1 = \{\langle u, x, y\rangle : l(u) = l \;\&\; (\langle x, s(u)\rangle \in W \rightarrow y \neq 0 \;\&\; D_y \subseteq D_{s(u)})\}$$

where $s(u)$ is a recursive function with the property that for $v > 0$ $s(\bar{f}(v)) = f(0)$. It follows that

$$x \in \overline{B} \text{ iff } (\forall f)(\exists v)(\exists y)(\langle \bar{f}(v), x, y\rangle \in W_1 \;\&\; D_y \subset \overline{A})$$

hence by Proposition 4, (i) we have $\overline{B} \leqslant_{\mathrm{he}} \overline{A}$.

Remark: The second part of Theorem 4 is actually a special case of the following result: $\overline{B} \leqslant_{\mathrm{he}} \overline{A}$ iff there is a **RE** set W such that

$$x \in B \text{ iff } (\exists f)(\forall v)(\exists y)(\langle \bar{f}(v), x, y\rangle \in W \,\&\, D_y \subseteq A).$$

The equivalence is still true if W is a Σ_1^1-set.

Corollary *Let F be a listing of the set A. Then $C_A \leqslant_{\mathsf{he}} F$.*

Since $A \leqslant_{\mathsf{e}} F$ we have $A \leqslant_{\mathsf{he}} F$ and $\overline{A} \leqslant_{\mathsf{he}} \overline{F} \leqslant_{\mathsf{e}} F$. Hence $C_A \leqslant_{\mathsf{he}} F$.

The degrees induced by the ordering $\leqslant_{\mathsf{he}}$ we shall call he-degrees. Note that $A \vee B$ is again the least upper bound of A and B. If $B \leqslant_{\mathsf{he}} A$ and A is Π_1^1-set then B is also a Π_1^1-set. On the other hand if B is a Π_1^1-set then $B \leqslant_{\mathsf{he}} A$ for every set A. From now on a Π_1^1-set will be called a hyperenumerable (**HE**) set. The hyperjump of the set A will be written $\mathbf{T}(A)$. As usual in place of $\mathbf{T}(\emptyset)$ we write just $\mathbf{T}$.

Theorem 5 *B is Π_1^1 in A iff $B \leqslant_{\mathsf{he}} \mathsf{C}_A$.*

If $B \leqslant_{\mathsf{he}} \mathsf{C}_A$ clearly B is Π_1^1 in A. To prove the converse assume that B is Π_1^1 in A. Then there is a primitive recursive predicate $R(u, z, x)$ such that

$$x \in B \text{ iff } (\forall f)(\exists v)R(\bar{f}(v), \overline{\mathsf{C}}_A(v), x)$$

and we may assume that whenever $R(u, z, x)$ holds then $z = \overline{\mathsf{C}}_A(v)$ for some set A and number v. Let $d(u)$ be a recursive function such that whenever $d(\bar{f}(v)) = y$ then $D_y = \{\langle x, f(x)\rangle : x < v\}$. Define a **RE** set W as follows:

$$W = \{\langle u, x, y\rangle : (\exists z)(R(u, z, x) \,\&\, d(z) = y \,\&\, l(z) = l(u)\}$$

It follows then that

$$x \in B \text{ iff } (\forall f)(\exists v)(\exists y)(\langle \bar{f}(v), x, y\rangle \in W \,\&\, D_y \subseteq \mathsf{C}_A).$$

Corollary

(i) $B \leqslant_1 \mathbf{T}(A)$ *iff* $B \leqslant_{\mathsf{he}} \mathsf{C}_A$
(ii) $\mathsf{C}_A \equiv_{\mathsf{he}} \mathbf{T}(A)$.

Theorem 6 *$B \leqslant_{\mathsf{h}} A$ iff $\mathsf{C}_B \leqslant_{\mathsf{he}} \mathsf{C}_A$.*

$B \leqslant_{\mathsf{h}} A$ iff both B and $\overline{B}$ are Π_1^1 in A, iff $B \leqslant_{\mathsf{he}} \mathsf{C}_A$ and $\overline{B} \leqslant_{\mathsf{he}} \mathsf{C}_A$, iff $\mathsf{C}_B \leqslant_{\mathsf{he}} \mathsf{C}_A$.

Let us recall that $W_0, W_1, \ldots, W_n, \ldots$ is an enumeration of all **RE** sets such that the predicate $x \in W_y$ is also **RE**. If A is any set we define a set $\mathsf{H}(A)$ as follows:

$$\mathsf{H}(A) = \{\langle x, z\rangle : (\forall f)(\exists v)(\exists y)(\langle \bar{f}(v), x, y\rangle \in W_z \,\&\, D_y \subseteq A)\}$$

If $B \leqslant_{\mathsf{he}} A$ there is a number k such that

$$x \in B \text{ iff } \langle x, k\rangle \in \mathsf{H}(A)$$

hence $B \leqslant_1 A$. On the other hand it is very easy to check that $\mathsf{H}(A) \leqslant_{\mathsf{he}} A$.

Proposition 5 *$A \equiv_{\mathsf{he}} B$ iff $\mathsf{H}(A) \equiv_1 \mathsf{H}(B)$.*

Corollary $\mathsf{H}(\mathsf{C}_A) \equiv_1 \mathbf{T}(A)$.

3 *Pseudo hyperarithmetical sets* We shall say a set A is pseudo hyperarithmetical (**PHA**) in case $\overline{A} \leqslant_{he} A$.

Theorem 7 *The following conditions are equivalent:*

(i) $C_A \equiv_{he} A$
(ii) $H(A) \equiv_1 T(A)$
(iii) A *is* **PHA**
(iv) $A \vee \overline{A} \equiv_{he} A$
(v) *There is a listing F of A such that $F \leqslant_{he} A$.*

(i) implies (ii) by Corollary to Proposition 5. (ii) implies (iii) since $\overline{A} \leqslant_1 T(A)$. The implication from (iii) to (iv) is trivial. To prove (v) from (iv) the case A is finite is trivial and case A is infinite it is easy to show that $F_A \leqslant_e A \vee \overline{A}$ where F_A is the strictly increasing listing of A. Now (v) implies (i) by the Corollary to Theorem 4.

Proposition 6

(i) *If A is* **PHA** *and $A \equiv_e B$ then B is* **PHA**.
(ii) *If A is* **PR** *then both A and $\overline{A}$ are* **PHA**.
(iii) *If A is* **PE** *then both A and $\overline{A}$ are* **PHA**.

Let A be **PHA** and $A \equiv_e B$. Then $A \equiv_{he} B$ and $\overline{A} \equiv_{he} \overline{B}$. Hence $B \leqslant_{he} \overline{A} \leqslant_{he} A \leqslant_{he} B$. To prove (ii) note that in case $\overline{A} \leqslant_e A$ then $\overline{A} \leqslant_{he} A$ and $A \leqslant_{he} \overline{A}$. Now (iii) follows from (i) and (ii).

Note that $T(A)$ is not **PHA**; otherwise we should have $T(T(A)) \leqslant_1 T(A)$ and this is impossible. Hence no set of the form $T(A)$ is ever **PE**. On the other hand $\overline{T(A)}$ is **PHA** by the same argument used to show that $\overline{A'}$ is **PR**. But $\overline{T(A)}$ is not **PE**, otherwise $T(A)$ would be **PHA**.

Proposition 7 *If $A \leqslant_h B \leqslant_{he} A$ and B is* **PHA** *then A is* **PHA**.

In this case we have $C_A \leqslant_{he} C_B \leqslant_{he} B \leqslant_{he} A$.

Corollary $H(A)$ *is not* **PHA**.

Assume $H(A)$ is **PHA**. Since $A \leqslant_h H(A) \leqslant_{he} A$ we have that A is also **PHA**. But then $H(A) \equiv_1 T(A)$ so $T(A)$ is **PHA**. Contradiction.

Note that since A' is **PE** it is also **PHA**. It follows from Theorem 7, (i) that A is **PHA** if and only if $A' \leqslant_{he} A$. Hence $A' \equiv_{he} A''$. On the other hand by Theorem 2 there is always some set B such that

$$A' <_e B <_e A''$$

where B is **PHA** but it is not **PE**. If $A \equiv_{he} B$ where A is **PHA** we shall say that B is a pseudo hyperenumerable (**PHE**) set. The next theorem shows that there is exactly two hyperdegrees in any **PHE** he-degree.

Theorem 8 *Let $A \equiv_{he} B$ where A is* **PHA**. *Then:*

(i) *Either $A \equiv_h B$ or $H(A) \equiv_h B$*
(ii) *$A \equiv_h B$ iff B is* **PHA**
(iii) *B is not* **PHA** *iff $H(B) \leqslant_h B$.*

Note that since A is **PHA** then $\mathsf{H}(A) \equiv_1 \mathbf{T}(A)$ and furthermore $A \leq_{\mathsf{h}} B$. If the proof of Theorem 16-XXXII in [5] is relativized to the set A we get from $B \leq_{\mathsf{he}} A$ that either $B \leq_{\mathsf{h}} A$ or $\mathsf{H}(A) \equiv_{\mathsf{h}} B$. This proves (i). To prove (ii) note that $B \leq_{\mathsf{h}} A \leq_{\mathsf{he}} B$ implies B is **PHA** and in case B is **PHA** we have $B \leq_{\mathsf{h}} A$. Now (iii) follows from (i) and (ii) noting that $\mathsf{H}(A) \equiv_1 \mathsf{H}(B)$ and $\mathsf{H}(B)$ is not **PHA**.

In the next section we prove there are sets which are not **PHE**. We have not been able to obtain any information on the hyperdegree structure of the he-degrees of such sets.

4 *Forcing* We prove in this section there are sets which are not **PHE**. The construction uses a forcing technique that was first introduced by Thomason in [6]. We shall construct generic sets A such that whenever $F \leq_{\mathsf{he}} A$ and F is a partial function then there is a Π_1^1 extension of F. Such set A is **PHE** if and only if A is **HE**.

First we note that the set $\mathsf{H}(A)$ can be defined by induction. In fact we can define by induction a set $\mathsf{H}^*(A)$ as follows:

(H1) If $\langle u, x, y\rangle \in W_z$ and $D_y \subseteq A$ then $\langle u, x, z\rangle \in \mathsf{H}^*(A)$.
(H2) If for every number j, $\langle u\cdot j, x, z\rangle \in \mathsf{H}^*(A)$ then $\langle u, x, z\rangle \in \mathsf{H}^*(A)$.

It follows that $\langle x, z\rangle \in \mathsf{H}(A)$ if and only if $\langle 1, x, z\rangle \in \mathsf{H}^*(A)$.

A condition $C = (C_1, C_2)$ is a pair of finite disjoint sets of natural numbers. If $C^1 = (C_1^1, C_2^1)$ is another condition such that $C_1 \subseteq C_1^1$ and $C_2 \subseteq C_2^1$ we shall say that C^1 is an extension of C. Now we define predicate $K_p(C, u, x, z)$ where p is some ordinal, C is a condition, and u, x, and z are numerical variables.

(K1) If $C = (C_1, C_2)$ and there is some y such that $\langle u, x, y\rangle \in W_z$ and $D_y \subseteq C_1$ then $K_p(C, u, x, z)$ holds for any ordinal p.
(K2) If for every number j and extension C^1 of C there is ordinal $q < p$ and extension C^2 of C^1 such that $K_q(C^2, u\cdot j, x, z)$ holds, then $K_p(C, u, x, z)$ holds.

Proposition 8 *If $K_p(C, u, x, z)$ holds and C^1 is an extension of C then $K_p(C^1, u, x, z)$ also holds.*

Let C be the condition (C_1, C_2). Then C^+ denotes the condition $(C_1, \emptyset)$.

Proposition 9 *If $K_p(C, u, x, z)$ holds then $K_p(C^+, u, x, z)$ also holds.*

The proof is by induction in the rules (K1) and (K2). The case in which $K_p(C, u, x, z)$ holds by (K1) is trivial. Assume $K_p(C, u, x, z)$ holds by (K2) and we show $K_p(C^+, u, x, z)$ also holds by (K2). Let $C^1 = (C_1^1, C_2^1)$ be an extension of C^+ and j some number. Since $(C_1^1 - C_2, C_2^1 \cup C_2)$ is an extension of C there is an extension C^2 and ordinal $q < p$ such that $K_q(C^2, u\cdot j, x, z)$ holds. By the induction hypothesis $K_q(C^{2+}, u\cdot j, x, z)$ also holds. If we take $C^3 = (C_1^2 \cup C_1^1, C_2^1)$ we have by Proposition 8 that $K_q(C^3, u\cdot j, x, z)$ holds and C^3 is an extension of C^1. It follows that $K_p(C^+, u, x, z)$ holds.

The predicate $K(C, u, x, z)$ holds in case for some ordinal p $K_p(C, u, x, z)$ holds.

Theorem 9 *The predicate* $K(C, u, x, z)$ *is a* Π_1^1 *predicate.*

Since no upper bound is imposed in the ordinals it is clear that $K(C, u, x, z)$ is the least predicate that satisfies the following rules:

(i) If $C = (C_1, C_2)$ and there is some y such that $\langle u, x, y\rangle \in W_z$ and $D_y \subseteq C_1$ then $K(C, u, x, z)$ holds;
(ii) If for every number j and extension C^1 of C there is extension C^2 of C^1 such that $K(C^2, u, x, z)$ holds then $K(C, u, x, z)$ holds. It follows that $K(C, u, x, z)$ is a Π_1^1 predicate.

Let $C = (C_1, C_2)$ and A some set. We say that C is consistent with A in case $C_1 \subseteq A$ and $C_2 \cap A = \emptyset$. The predicate $K_p(A, u, x, z)(K(A, u, x, z))$ holds in case there is C consistent with A and $K_p(C, u, x, z)(K(C, u, x, z))$ holds.

A set A is quasi-generic in case the following two conditions are satisfied:

(i) If $K_p(A, u, x, z)$ holds then either there is y such that $\langle u, x, y\rangle \in W_z$ and $D_y \subseteq A$ or for every number j there is ordinal $q < p$ such that $K_q(A, u \cdot j, x, z)$ holds.
(ii) If $K(A, u, x, z)$ does not hold then for some number j, $K(A, u \cdot j, x, z)$ does not hold.

Theorem 10 *Let A be a quasi-generic set. Then $\langle u, x, z\rangle \in \mathsf{H}^*(A)$ if and only if $K(A, u, x, z)$ holds.*

First we prove that whenever $K_p(A, u, x, z)$ holds then $\langle u, x, z\rangle \in \mathsf{H}^*(A)$. This follows immediately by transfinite induction on p using condition (i) in the definition of A is quasi-generic. The converse follows by induction in the rules defining $\mathsf{H}^*(A)$ and condition (ii) in the definition.

Next we shall consider a first order language $\mathcal{L}$ for number theory containing negation, conjunction, existential quantifier, constant, functions symbols and predicate symbols with equality included. For each constant, function symbol and predicate symbol some interpretation in the natural numbers is assumed. If n is a number then $n\#$ is a term that denotes n. If t is some constant term then $\#t$ is the numeral denoted by t. So $\#(n\#) = n$. We extend $\mathcal{L}$ to $\mathcal{L}_1$ by adding two new predicate symbols: $S(x)$ and $S^*(u, x, z)$. Whenever S is interpreted as some set A then S^* is interpreted as $\mathsf{H}^*(A)$. If M is a sentence in $\mathcal{L}_1$ then $\models_{S=A} M$ means the M is true if S is interpreted as A.

The forcing relation $C \Vdash M$ between conditions and sentences in $\mathcal{L}_1$ is defined as usual, but in the case M is the sentence $S^*(g, h, t)$ we put $C \Vdash M$ if and only if $K(C, \#g, \#h, \#t)$ holds. If C is consistent with A and $C \Vdash M$ then we write $A \Vdash M$. We shall say that a set A is generic in case it is quasi generic and for every sentence in $\mathcal{L}_1$ either $A \Vdash M$ or $A \Vdash \sim M$.

Proposition 10 *If A is generic and $\models_{S=A} M$ then $A \Vdash M$.*

This property of generic sets is proved as usual, but in the case M is the sentence $S^*(g, h, t)$ we use Theorem 10.

Theorem 11 *There is a generic set A which is not* **HE**.

We proceed as usual by enumerating all sentences in $\mathcal{L}_1$ in the form $M_1, M_2, \ldots$ and defining conditions $C^0, C^1, C^2, \ldots$ where C^0 is an arbitrary condition, C^{n+1} is an extension of C^n and either $C \Vdash M_n$ or $C \Vdash \sim M_n$. The only difference is the case in which M_{n+1} is a sentence of the form $S^*(u\#, x\#, z\#)$ for numbers u, x, z. In case there is an extension C of C^n such that for some ordinal p $K_p(C, u, x, z)$ holds we take one such C for which the ordinal p is the least and put $C^{n+1} = C$. Otherwise there is an extension C of C^n and some number j such that for no extension C' of C $K(C', u\cdot j, x, z)$ holds. Then we put C^{n+1} equal one such C. We put $A = \bigcup_n C_1^n$ and obviously we have to prove only that A is quasi-generic.

Let M_n be $S^*(u\#, x\#, z\#)$ and assume $K_p(A, u, x, z)$ holds for some p. This means $K_p(C^n, u, x, z)$ holds so either there is y such that $\langle u, x, y\rangle \in W_z$ and $D_y \subseteq C_1^n \subseteq A$, or for every number j and extension C' of C^n there is $q < p$ and extension C'' of C' such that $K_q(C'', u\cdot j, x, z)$ holds. But then if M_k is $S^*((u\cdot j)\#, x\#, z\#)$ it follows that $K_q(C^k, u\cdot j, x, z)$ holds also for some $q < p$. Hence $K_q(A, u\cdot j, x, z)$ holds.

Now with the same M_n assume that $K(A, u, x, z)$ does not hold. In this case C^n is taken such that for some number j there is no extension C' of C^n such that $K(C', u\cdot j, x, z)$ holds. Hence $K(A, u\cdot j, x, z)$ does not hold.

Since the collection of **HE** sets is denumerable we can modify the construction of A in such a way that for each **HE** set B there is C^n which is not consistent with B. This makes A different from any **HE** set.

Theorem 12 *Let A be a generic set and F a partial function such that* $F \leq_{\mathrm{he}} A$. *Then there is an extension of F which is a* Π_1^1 *partial function.*

Since $F \leq_{\mathrm{he}} A$ there is a number k such that

$$F(x) = y \text{ iff } \langle 1, \langle x, y\rangle, k\rangle \in \mathsf{H}^*(A) \text{ iff } K(A, 1, \langle x, y\rangle, k).$$

Let M be the sentence:

$$\sim(\exists x)(\exists y)(\exists v)(S^*(1\#, \langle x, y\rangle, k\#) \;\&\; S^*(1\#, \langle x, v\rangle, k\#) \;\&\; y \neq v).$$

Since $\vdash_{S=A} M$ there is a condition $C = (C_1, C_2)$ consistent with A such that $C \Vdash M$. Now let $R(x, y)$ be the following predicate: There is a condition C', C' is an extension of C and $K(C', 1, \langle x, y\rangle, k)$ holds. Since $K(C, u, x, z)$ is a Π_1^1 predicate it follows that $R(x, y)$ is also a Π_1^1 predicate. It is clear that $R(x, y)$ is an extension of $F(x) = y$. We must show that $R(x, y)$ is single valued.

Suppose $R(x, y)$ and $R(x, v)$ both hold with $y \neq v$. By definition there is $C^1 = (C_1^1, C_2^1)$ extension of C and $C^2 = (C_1^2, C_2^2)$ extension of C such that $K(C^1, 1, \langle x, y\rangle, k)$ holds and $K(C^2, 1, \langle x, v\rangle, k)$ holds. Put $C' = (C_1^1 \cup C_1^2, C_2)$. By Propositions 8 and 9 it follows that $K(C', 1, \langle x, y\rangle, k)$ holds and $K(C', 1, \langle x, v\rangle, k)$ also holds. Hence C' is an extension of C such that

$$C' \Vdash S^*(1\#, \langle x\#, y\#\rangle, k\#) \;\&\; S^*(1\#, \langle x\#, v\#\rangle, k\#) \;\&\; y\# \neq v\#)$$

and this is a contradiction with the assumption about C.

Corollary *If A is generic, F is a total function and $F \leq_{he} A$ then F is hyperarithmetical.*

REFERENCES

[1] Appel, K. I. and T. McLaughlin, "On properties of regressive sets," *Transactions of the American Mathematical Society*, vol. 115 (1965), pp. 83-93.

[2] Berry, J. W., "Almost recursively enumerable sets," *Transactions of the American Mathematical Society*, vol. 164 (1972), pp. 241-253.

[3] Case, J., "Enumeration reducibility and partial degrees," *Annals of Mathematical Logic*, vol. 2 (1971), pp. 419-439.

[4] Jockush, C., "Semirecursive sets and positive reducibility," *Transactions of the American Mathematical Society*, vol. 131 (1968), pp. 420-436.

[5] Rogers, H., Jr., *Theory of recursive functions and effective computability*, McGraw-Hill, New York , 1967.

[6] Thomason, S. K., "The forcing method and the upper semilattice of hyperdegrees," *Transactions of the American Mathematical Society*, vol. 129 (1967), pp. 38-57.

[7] Vučković, V., "Almost recursive sets," *Proceedings of the American Mathematical Society*, vol. 23 (1969), pp. 114-119.

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