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SOME OBSERVATIONS ON A METHOD OF McKINSEY

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In an early paper[1] J. C. C. McKinsey proved that no one of the intuitionist connectives $\neg$, $\wedge$, $\vee$, and $\supset$ is definable in terms of the remaining three. Those who are familiar with the details of his argument can have little doubt concerning its soundness. Nonetheless, McKinsey's characterization of that argument is certainly defective.

> More generally, we can see, that if three of the operations we are considering are class-closing on some proper sub-class of the elements of a matrix, while the fourth is not class-closing on this proper subclass, then the fourth operation is not definable in terms of the other three.[2]

Were this account correct, it would be all too easy to show that no connective from a propositional system is definable in terms of other connectives from that system.

Proof: Let $\oplus$ be any n-place connective from an arbitrary propositional system Σ, and let Δ be any set of other connectives from Σ. Consider now the matrix $\mathfrak{N}$. The elements of $\mathfrak{N}$, both of which are designated, are 0 and 1. The operation that $\mathfrak{N}$ associates with $\oplus$ is that n-ary operation on $\{0, 1\}$ that always assumes the value 0. The operations that $\mathfrak{N}$ associates with the members of Δ are operations that always assume the value 1. It is clear that $\mathfrak{N}$ is a matrix *for* Σ (i.e. that each thesis of Σ is a tautology under $\mathfrak{N}$), that the operations $\mathfrak{N}$ assigns to the members of Δ are class closing on $\{1\}$, and that the operation $\mathfrak{N}$ assigns to $\oplus$ is not class closing on $\{1\}$. Thus, if McKinsey's account were correct, $\oplus$ would not be definable in terms of the members of Δ.

A more interesting matrix in this respect, but one that assumes that the rule of substitution holds for Σ, is the Lindenbaum matrix for Σ. A

1. McKinsey, J. C. C., "Proof of the Independence of the Primitive Symbols of Heyting's Calculus of Propositions," *The Journal of Symbolic Logic*, vol. 4 (1939), pp. 155-158.

2. *Ibid.*, p. 156.

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characteristic feature of this matrix is that the value of a formula for any assignment of elements (wffs) to its propositional variables is always a substitution instance of that formula. It follows that the operations assigned to members of Δ will be class closing on that set of elements (wffs) whose main connective belongs to Δ. But, assuming that $\oplus \notin \Delta$, the operation assigned to $\oplus$ is not class closing on that class. Again, if McKinsey's account were correct, $\oplus$ would not be definable in terms of the members of Δ.

The difficulty with McKinsey's account, obviously, is that it is too liberal in the matrices it accepts as supporting claims of nondefinability. Our aim, aside from pointing out this defect, is to provide a needed restriction on permissible matrices. We restrict ourselves to systems having a "suitable" biconditional $\Leftrightarrow$. Consider any propositional system Σ for which the following claim holds: $\oplus$ is definable in terms of Δ iff there are n distinct propositional variables $p_1, \ldots, p_n$ and a formula r containing no connectives other than those from Δ such that

$$\mathbf{D} \quad \oplus(p_1 \ldots p_n) \Leftrightarrow r$$

is a thesis of Σ.

If we can find a matrix $\mathfrak{M}$ fitting McKinsey's account we are assured that there is some assignment a of $\mathfrak{M}$'s elements to the propositional variables of $\mathbf{D}$ for which $V_a[\oplus(p_1 \ldots p_n)] \neq V_a[r]$ (i.e., the value of $\oplus(p_1 \ldots p_n)$ under $a \neq \ldots$). This is exactly where McKinsey's oversight occurred. The distinctness of these values does not in itself guarantee that $V_a[\mathbf{D}]$ is nondesignated and, hence, does not warrant the conclusion that $\oplus$ is not definable in terms of the members of Δ. One easy way to assure that $V_a[\mathbf{D}]$ is not designated is to require of the operation f that $\mathfrak{M}$ assigns to $\Leftrightarrow$ that $f(x, y)$ is nondesignated whenever $x \neq y$. But this requirement is needlessly restrictive, for it is easily verified that this assurance is secured by the following more liberal (albeit more cumbersome) requirement. Where α is a class of $\mathfrak{M}$'s elements that is closed with respect to the operations assigned to the members of Δ, but not closed with respect to the operation g assigned to $\oplus$, a nondefinability claim can be supported if: f sometimes assumes a value $x \notin \alpha$ such that, for each $y \in \alpha$, $f(x, y)$ is nondesignated.

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