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ON CREATIVE DEFINITIONS IN FIRST ORDER FUNCTIONAL CALCULI

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Russell's widely accepted claim that definitions are "theoretically superfluous" is partially vindicated by the

Theorem *In the first order functional calculus definitions of the type*

$$\Theta\alpha_1\alpha_2 \ldots \alpha_n \equiv \omega$$

are not creative, i.e., no new theorems, in primitive notation, are provable using definitions. (See [3], p. 190, for proof).

In this note we point out that this theorem is imprecisely stated in that it is dependent upon the particular axiomatic presentation of the first order functional calculus which is chosen. We do this by giving an axiomatization with respect to which there is a definition of the above type which is creative. As axiom schemata take the following:

(1) $E \equiv F \supset . A \supset . B \supset A$
(2) $E \equiv F \supset . A \supset [B \supset C] \supset . A \supset B \supset . A \supset C$
(3) $E \equiv F \supset . \sim A \supset \sim B \supset . B \supset A$
(4) $E \equiv F \supset . (a)[A \supset B] \supset . A \supset (a)B$, *where a is any individual variable which is not a free variable of A.*
(5) $E \equiv F \supset . (a)A \supset \mathbf{S}_b^a A|$, *where a is an individual variable, b is an individual variable or an individual constant, and no free occurrence of a in A is in a well formed part of A of the form $(b)C$.*
(6) $A \equiv B \supset . A \supset B$
(7) $A \equiv B \supset . B \supset A$
(8) $\sim[A \supset B \supset \sim[B \supset A]] \supset . A \equiv B$

In this presentation $\supset$, $\sim$, and $\equiv$ are primitive.[1] The rules are detachment and generalization.

1. Axioms (6)-(8) are needed since $\equiv$, as well as $\sim$, $\supset$, is primitive. Without them no connection between $\equiv$ and $\sim$, $\supset$ would be provable. One could take $\sim$, $\supset$ as the only primitives. Then read $\supset$ for $\equiv$ in (1)-(5) and drop (6)-(8). But then the rule of definition must allow for the introduction of pairs of implications ($A \supset B$, $B \supset A$) rather than equivalences ($A \equiv B$).

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These axiom schemata are constructed in such a way that the rule of detachment cannot be applied. The only thing that could be detached from (1)-(7) [(8)] is an equivalence [negation], but no instance of any of these axiom schemata is of that form. One can apply the rule of generalization, but no theorem shorter than all of the axiom schemata is obtainable. Hence we do not obtain all of the theorems of the first order functional calculus from (1)-(8) by using the rules of substitution and detachment. In particular, we do not obtain $A \equiv A$, and this is needed to prove the above theorem.

Now let us enrich our system by adding a rule of definition which permits us to add new theses to the system (of a certain prescribed form—see the Appendix of [5] for details). Using this rule add the definition

$$\mathsf{T}(X) .\equiv. \mathsf{F}(X) \supset \mathsf{F}(X),$$

where F is a unary function symbol which is already in our language, and T is the newly defined symbol. Since this is a thesis of our system, and not an abbreviation, it can be detached from (1)-(5) to obtain theorem schemata which are identical to those used in [1] to axiomatize the first order functional calculus. Thus *this definition is creative* as it allows us to prove theorems in primitive notation which were not previously provable. More precisely, this definition is creative with respect to the above axiomatization.

The existence of creative definitions at first seems shocking, but when one reflects on the fact that a new rule has been added to the system, and that new rules usually do permit us to prove more theorems, then the creativity of definitions is seen to be the expected thing. Each definition which is added to the system strengthens it in two ways: a new symbol is added to the language and a new thesis is added to the stock of theorems. So we should expect to prove more theorems, even more theorems in primitive notation.

Admittedly, the usual axiomatizations of the first order functional calculus allow one to prove that no definition is creative. But we wish to stress that this theorem is not independent of the axiomatization. Nonetheless, that situation should be considered the exception. If you go down to the propositional calculus, then creative definitions exist [5]. If you go up to second order logic then there are infinitely many creative definitions (this is one way of construing [2]). Furthermore some systems of arithmetic contain infinitely many creative definitions [4].

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