Notre Dame Journal of Formal Logic
Volume XIX, Number 2, April 1978
NDJFAM

ON THE INDEPENDENCE OF THE BIGOS-KALMÁR AXIOMS FOR SENTENTIAL CALCULUS

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1 *Introduction* Let

$\mathbf{K}_1 = \{CpCqp,\ CCpCqrCCpqCpr\}$,
$\mathbf{K}_2 = \mathbf{K}_1 \cup \{CNpCpq\}$,
$\mathbf{K}_3 = \mathbf{K}_2 \cup \{CpNNp,\ CpCNqNCpq\}$,
$\mathbf{B} = \mathbf{K}_2 \cup \{CCpqCCNpqq\}$,
$\mathbf{M} = \mathbf{K}_3 \cup \{CCpqCCNpqq\}$,
$\mathbf{K} = \mathbf{M} \cup \{CpCqKpq,\ CNqNKpq,\ CNpNKpq,\ CpApq,\ CqApq,\ CNpCNqNApq,$
$CpCqEpq,\ CpCNqNEpq,\ CNpCqNEpq,\ CNpCNqEpq\}$.

By *Kalmár's Lemma* we mean the inference rule denoted as Hilfssatz 3 in Kalmár [2] and as Lemma 1.12 in Mendelson [3]. The proof of Kalmár's Lemma given by Kalmár [2] uses all of the tautologies in **K**. In turn the proof of Kalmár's Lemma given by Mendelson [3] makes use of only the subset **M** of **K**. Following Kalmár and Mendelson, Pogorzelski [4] proves the following theorems.

Lemma 1 *For any sentential calculus* $\mathcal{L}$, *if* $\mathbf{K}_1$ *is a subset of the theorems of* $\mathcal{L}$, *then the Deduction Theorem is a derived inference rule of* $\mathcal{L}$.

Lemma 2 *For any sentential calculus* $\mathcal{L}$, *if* $\mathbf{K}_3$ *is a subset of the theorems of* $\mathcal{L}$, *then Kalmár's Lemma is a derived inference rule of* $\mathcal{L}$.

Theorem 3 *For any sentential calculus* $\mathcal{L}$, *if* **M** *is a subset of the theorems of* $\mathcal{L}$, *then* $\mathcal{L}$ *is complete.*

With respect to Theorem 3, Pogorzelski asked if **M** forms an independent system of axioms for a sentential calculus. In turn Yvonne Bigos [1] gave a proof of the following theorem, which we introduce in section 2.

Theorem I *Tautologies CpNNp and CpCNqNCpq are redundant in axiom system* **M** *for a sentential calculus.*

Lastly, the author proves the following:

Received June 23, 1977

Theorem II *The set* **B** *forms an independent system of axioms for a sentential calculus.*

It follows from Theorem II that $\mathbf{K}_3$ and **M** can be replaced by **B** in the preceding Lemma 2 and Theorem 3.

2 *Theorem of Bigos* By $\mathcal{L}(\mathbf{B})$ we mean the sentential calculus obtained by taking the tautologies of **B** as axioms and using, for inference rules, *modus ponens* and substitution. Within a given proof in $\mathcal{L}(\mathbf{B})$ we will use L_i to denote the wff on the i'th line of the proof. An application of one of the two inference rules is denoted as follows:

(a) *Substitution Rule.* SR(x): $p_1/P_1, \ldots, p_n/P_n \vdash L_k$, where x denotes a theorem of $\mathcal{L}(\mathbf{B})$ or it denotes $L_i (i < k)$, and L_k denotes the wff obtained by replacing in x the sentential variable p_i by the wff P_i ($i = 1, 2, \ldots, n$).
(b) *Modus ponens.* MP(x, y) $\vdash L_k$, where x and y denote theorems of $\mathcal{L}(\mathbf{B})$ or wffs L_i, L_j ($i, j < k$), and y is of the form CxL_k.

An application of the Deduction Theorem, which we will prove in Lemma 4, will be abbreviated as:

$$\mathrm{H} \vdash L_k,\ \mathrm{DT}(L_k, L_j) \vdash CL_kL_j,$$

where $\mathrm{H} \vdash L_k$ denotes that L_k is a hypothesis and L_j is the wff of proof line $j > k$.

We denote the tautologies of **B** by:

B1. *CpCqp*,
B2. *CCpqCCNpqq*,
B3. *CNpCpq*,
B4. *CCpCqrCCpqCpr*.

Lemma 4 *The Deduction Theorem is a derived inference rule of* $\mathcal{L}(\mathbf{B})$.

Proof: Since $\mathbf{K} \subset \mathbf{B}$, Lemma 4 follows immediately from Lemma 1.

Lemma 5 *Cpp is a theorem of* $\mathcal{L}(\mathbf{B})$.

Proof: 1. SR(*B2*): $q/Cpp \vdash CCpCppCCNpCppCpp$. 2. SR(*B1*): $q/p \vdash CpCpp$. 3. MP(L_2, L_1) $\vdash CCNpCCppCpp$. 4. SR(*B3*): $q/p \vdash CNpCpp$. 5. MP(L_4, L_3) $\vdash Cpp$.

Lemma 6 *CpNNp is a theorem of* $\mathcal{L}(\mathbf{B})$.

Proof: 1. $\mathrm{H} \vdash p$. 2. SR(*B1*): $q/Np \vdash CpCNpp$. 3. MP(L_1, L_2) $\vdash CNpp$. 4. SR(*B2*): p/NNp, $q/NNp \vdash CCNNpNNpCCNNNpNNpNNp$. 5. SR(Lemma 5): $p/NNp \vdash CNNpNNp$. 6. MP(L_5, L_4) $\vdash CCNNNpNNpNNp$. 7. SR(L_3): $p/NNp \vdash CNNNpNNp$. 8. MP(L_7, L_6) $\vdash NNp$. 9. DT(L_1, L_8) $\vdash CpNNp$.

Lemma 7 *CCpqCCqrCpr is a theorem of* $\mathcal{L}(\mathbf{B})$.

Proof: 1. $\mathrm{H} \vdash Cpq$. 2. $\mathrm{H} \vdash Cqr$. 3. $\mathrm{H} \vdash p$. 4. MP(L_3, L_1) $\vdash q$. 5. MP(L_4, L_2) $\vdash r$. 6. DT(L_3, L_5) $\vdash Cpr$. 7. DT(L_2, L_6) $\vdash CCqrCpr$. 8. DT(L_1, L_7) $\vdash CCpqCCqrCpr$.

Lemma 8 *CCpqCNqNp is a theorem of* $\mathcal{L}$(**B**).

Proof: 1. H ⊢ *Cpq*. 2. H ⊢ *Nq*. 3. SR(*B3*): *p*/*q*, *q*/*Np* ⊢ *CNqCqNp*. 4. MP(L_2, L_3) ⊢ *CqNp*. 5. SR(Lemma 7): *r*/*Np* ⊢ *CCpqCCqNpCpNp*. 6. MP(L_1, L_5) ⊢ *CCqNpCpNp*. 7. MP(L_4, L_6) ⊢ *CpNp*. 8. SR(*B2*): *q*/*Np* ⊢ *CCpNpCCNpNpNp*. 9. MP(L_7, L_8) ⊢ *CCNpNpNp*. 10. SR(Lemma 5): *p*/*Np* ⊢ *CNpNp*. 11. MP(L_9, L_{10}) ⊢ *Np*. 12. DT(L_2, L_{11}) ⊢ *CNqNp*. 13. DT(L_1, L_{12}) ⊢ *CCpqCNqNp*.

Lemma 9 *CpCNqNCpq is a theorem of* $\mathcal{L}$(**B**).

Proof: 1. H ⊢ *p*. 2. H ⊢ *Cpq*. 3. MP(L_1, L_2) ⊢ *q*. 4. DT(L_2, L_3) ⊢ *CCpqq*. 5. SR(Lemma 8): *p*/*Cpq* ⊢ *CCCpqqCNqNCpq*. 6. MP(L_4, L_5) ⊢ *CNqNCpq*. 7. DT(L_1, L_6) ⊢ *CpCNqNCpq*.

Theorem I *Tautologies CpNNp and CpCNqNCpq are redundant in axiom system* **M** *for a sentential calculus.*

Proof: Theorem I follows immediately from Lemma 6 and 9.

3 *Independence of system B of Bigos-Kalmár Axioms* The following independence proofs will consist of constructing truth-tables for the primitive connectives on the basis of two or more truth-values,

$$0, 1, \ldots, n,$$

with 0 being called the *designated* truth-value.

Definition 1 We say a wff ξ is a *tautology* with respect to the given truth-tables, if for every system of values for the variables of ξ—0, 1, . . ., *n* being admissible values—ξ is reducible to the designated truth-value.

If then the two inference rules have the property of preserving tautologies and every axiom but one is a tautology, it follows that the one axiom that is not a tautology is independent of the remaining. We further note that the inference rule of substitution trivially preserves tautologies regardless of the truth-tables under consideration.

Lemma 10 *B1 is independent of the remaining axioms of* **B**.

Proof: Consider the following tables:

p	*Np*
0	2
1	2
2	0

Cpq	0	1	2
0	0	2	2
1	0	2	2
2	0	0	0.

B2, *B3*, and *B4* are tautologies and *modus ponens* preserves this property. But, with *p* = 1 and *q* = 0, *B1* obtains the value 2, independence follows.

Lemma 11 *B2 is independent of the remaining axioms of* **B**.

Proof: Using the tables

p	*Np*
0	1
1	1

Cpq	0	1
0	0	1
1	0	0,

we find *B1, B3,* and *B4* are tautologies, and *modus ponens* preserves tautologies. Since *B2* obtains the value 1 for $p = q = 1$, it follows that it is independent of the remaining axioms of **B**.

Lemma 12 *B3 is independent of the remaining axioms of* **B**.

Proof: Consider the tables:

p	Np
0	0
1	0

Cpq	0	1
0	0	1
1	0	0.

Simple inspection will reveal that *B1, B2,* and *B4* are tautologies and *modus ponens* preserves tautologies. But, for $p = 0$ and $q = 1$, *B3* obtains the value 1, whence independence follows.

Lemma 13 *B4 is independent of the remaining axioms of* **B**.

Proof: Consider the tables[1]:

p	Np
0	3
1	2
2	1
3	0

Cpq	0	1	2	3
0	0	1	2	3
1	0	0	2	0
2	0	1	0	3
3	0	0	0	0.

B1, B2, and *B3* are tautologies with respect to these tables and *modus ponens* preserves this property. Hence, since *B4* reduces to the value 2 when $p = 1$, $q = 3$, and $r = 2$, *B4* is independent.

Theorem II *The set* **B** *forms an independent system of axioms for a sentential calculus.*

Proof: The theorem follows immediately from Lemmata 9-12.

REFERENCES

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[4] Pogorzelski, H. A., *Mathematical Logic*, 1969, unpublished.

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1. These tables are due to Prof. R. Tredwell.