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AN INDEPENDENCE RESULT CONCERNING INFINITE PRODUCTS OF ALEPHS

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0* We work within **ZFC** (Zermelo-Fraenkel plus Choice) set theory, and for typographical reasons denote (wherever possible) infinite cardinals by lower case script letters instead of employing the usual aleph notation. Infinite ordinals are denoted by lower case Greek letters (with "ω" being reserved for the first such ordinal), and finite ordinals by "k", "m", "n", etc. Ordinals are assumed to be defined in such a manner that each is the set of all smaller ordinals.

Let $\mathfrak{a}$ be an aleph. By "$\mathrm{i}(\mathfrak{a})$" we denote the corresponding initial ordinal, by "$\mathfrak{a}^+$" its successor cardinal, and by "$\mathrm{cf}(\mathfrak{a})$" the cardinality $|\eta|$ of the smallest ordinal η for which there is an increasing η-sequence (σ_ξ) of ordinals $\sigma_\xi < \mathrm{i}(\mathfrak{a})$ such that $\lim_{\xi<\eta}\sigma_\xi = \mathrm{i}(\mathfrak{a})$. We denote by "$\Sigma$" and "$\Pi$" the operations of generalized cardinal addition and generalized cardinal multiplication. Furthermore, if $\Gamma = (\mathfrak{a}_\xi)$ is any α-sequence of alephs, α being a nonzero limit ordinal, we put $\mathcal{S}(\Gamma, \zeta) = \Sigma\{\mathfrak{a}_\xi;\ \xi < \zeta\}$ and $\mathcal{P}(\Gamma, \zeta) = \Pi\{\mathfrak{a}_\xi;\ \xi < \zeta\}$ for each ordinal ζ with $0 < \zeta \leqslant \alpha$; for convenience, we put

$$\mathcal{S}(\Gamma, 0) = 0, \mathcal{P}(\Gamma, 0) = 1, \mathcal{S}(\Gamma) = \mathcal{S}(\Gamma, \alpha), \text{ and } \mathcal{P}(\Gamma) = \mathcal{P}(\Gamma, \alpha).$$

Let α be any nonzero limit ordinal, and $(\mathfrak{a}_\xi)$ any increasing α-sequence of alephs. Then it is well-known that:

(S1) $\lim\{\mathcal{S}(\Gamma, \zeta);\ \zeta < \alpha\} = \mathcal{S}(\Gamma)$,
(S2) $\Sigma\{\mathcal{S}(\Gamma, \zeta);\ \zeta < \alpha\} = \mathcal{S}(\Gamma)$;

where $\Gamma = (\mathfrak{a}_\xi)_{\xi<\alpha}$.

It is natural to ask whether the analogous results hold for multiplication, namely:

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(P1) $\lim\{\mathcal{P}(\Gamma, \zeta);\ \zeta < \alpha\} = \mathcal{P}(\Gamma)$,
(P2) $\Pi\{\mathcal{P}(\Gamma, \zeta);\ \zeta < \alpha\} = \mathcal{P}(\Gamma)$.

Now (P2) does indeed hold; this is proved in [2]. That (P1) does not hold in general is immediate from the following simple result.

Result 1 *Let $\Gamma = (\mathfrak{a}_n)$ be any increasing ω-sequence of alephs. Then* $\lim\{\mathcal{P}(\Gamma, n);\ n < \omega\} < \mathcal{P}(\Gamma)$.

Proof: From the elementary properties of alephs we obtain $\mathcal{P}(\Gamma, n+1) = \mathfrak{a}_n = \mathcal{S}(\Gamma, n+1)$. It now follows from (S1) that $\lim\{\mathcal{P}(\Gamma, n);\ n < \omega\} = \mathcal{S}(\Gamma)$. But König's inequality (the more general theorem of Zermelo is proved in [1], p. 123) tells us that $\mathcal{S}(\Gamma) < \mathcal{P}(\Gamma)$.

1 We have just seen that there exist increasing α-sequences $(\mathfrak{a}_\xi)$ of alephs, α being a nonzero limit ordinal, for which $\lim\{\mathcal{P}(\Gamma, \zeta);\ \zeta < \alpha\} < \mathcal{P}(\Gamma)$, where $\Gamma = (\mathfrak{a}_\xi)$. Now clearly we have $\lim\{\mathcal{P}(\Gamma, \zeta);\ \zeta < \alpha\} \leq \mathcal{P}(\Gamma)$ for every increasing limit sequence Γ of alephs, and so the question naturally arises as to whether the inequality is always strict. Our interest thus lies in the following question. What is the status of the statement (#) in **ZFC** set theory?

For every nonzero limit ordinal α and every increasing α-sequence Γ of alephs, $\lim\{\mathcal{P}(\Gamma, \zeta);\ \zeta < \alpha\} < \mathcal{P}(\Gamma)$.

We shall show that (#) is independent of **ZFC**. The consistency part of our demonstration requires a result of Tarski's; this appeared in [5], but to the best of our knowledge no proof of it has ever been published, and so we take this opportunity of presenting one. The following proof of Tarski's theorem depends, however, on a prior theorem of Tarski's, stated and proved in [4]:

Theorem 1 *Let ρ be an infinite prime component, and let $\Gamma = (\mathfrak{a}_\xi)$ be an increasing ρ-sequence of alephs. Put $\mathfrak{a} = \lim_{\zeta<\rho}\mathfrak{a}_\zeta$. Then $\mathcal{P}(\Gamma) = \mathfrak{a}^{|\rho|}$.*

The theorem in [5] that we require may be stated as follows:

Theorem 2 *Let α be a nonzero limit ordinal, let $\Gamma = (\mathfrak{a}_\xi)$ be an increasing α-sequence of alephs, and put $\mathfrak{a} = \lim_{\xi<\alpha}\mathfrak{a}_\xi$. Then $\mathcal{P}(\Gamma) = \mathfrak{a}^{|\rho|}$ for some positive remainder ρ of α.*

Proof: We let $\alpha = \rho_1 + \ldots + \rho_n$ be the canonical prime component decomposition of α, and for each $i \leq n$ we define the ordinal σ_i as follows:

$$\sigma_0 = 0;$$
$$\sigma_i = \sigma_{i-1} + \rho_i,\ 1 \leq i \leq n.$$

Now for each i with $1 \leq i \leq n$ we define alephs $\mathfrak{b}_i$, $\mathfrak{c}_i$:

$$\mathfrak{b}_i = \Pi\{\mathfrak{a}_{\sigma_{i-1}+\xi};\ \xi < \rho_i\};$$
$$\mathfrak{c}_i = \lim\ \mathfrak{a}_{\sigma_{i-1}+\xi};\ \xi < \rho_i\}.$$

Obviously we have $\mathcal{P}(\Gamma) = \mathfrak{b}_1\mathfrak{b}_2 \ldots \mathfrak{b}_n = \max\{\mathfrak{b}_1, \ldots, \mathfrak{b}_n\}$, and by Theorem 1 $\mathfrak{b}_i = (\mathfrak{c}_i)^{|\rho_i|}$ for each i. Thus $\mathcal{P}(\Gamma) = \max\{(\mathfrak{c}_i)^{|\rho_i|};\ 1 \leq i \leq n\}$; let j

be such that $(c_j)^{|\rho_j|} = \mathcal{P}(\Gamma)$. Now clearly $c_j \leqslant c_n$, and so $(c_j)^{|\rho_j|} \leqslant (c_n)^{|\rho_j|}$. On the other hand we have $|\rho_n| \leqslant |\rho_j|$, and hence $(c_n)^{|\rho_j|} = ((c_n)^{|\rho_n|})^{|\rho_j|} \leqslant ((c_j)^{|\rho_j|})^{|\rho_j|} = (c_j)^{|\rho_j|}$, since all the ρ_i are infinite. Thus we have shown that $\mathcal{P}(\Gamma) = (c_n)^{|\rho_j|}$. But of course $c_n = a$ and $|\rho_j| = |\rho_j + \ldots + \rho_n|$.

We state the Generalized Continuum Hypothesis in the form usually known as the Aleph Hypothesis:

For every aleph a, $2^{a} = a^{+}$.

It is of course well-known that **GCH** is relatively consistent with **ZFC**. It follows that any result derivable from **GCH** is also relatively consistent with **ZFC**.

Result 2 *The statement* (#) *is relatively consistent with* **ZFC**.

Proof: We assume **GCH**, and show that (#) follows from this assumption. Let $\Gamma = (a_\xi)$ be an increasing α-sequence of alephs, α being a nonzero limit ordinal, and put $a = \lim_{\xi<\alpha} a_\xi$. Then by Theorem 2 we have $\mathcal{P}(\Gamma) = a^{|\rho|}$ for some positive remainder ρ of α. Clearly we have $\mathcal{P}(\Gamma) \leqslant a^{|\alpha|}$, and so if τ is the smallest positive remainder of α, it follows that $a^{|\tau|} \leqslant \mathcal{P}(\Gamma) \leqslant a^{|\alpha|}$. Let β be the smallest ordinal such that $\alpha = \beta + \tau$. Then the increasing τ-sequence $(a_{\beta+\xi})_{\xi<\tau}$ of alephs has limit a, and it follows easily that $\mathrm{cf}(a) \leqslant |\tau|$. Hence $a^{\mathrm{cf}(a)} \leqslant \mathcal{P}(\Gamma)$. On the other hand, it is clear that $|\alpha| \leqslant a$, and so $\mathcal{P}(\Gamma) \leqslant a^{a}$. Now it is well-known that under **GCH** we have $d^{\mathrm{cf}(d)} = d^{d} = d^{+}$ for every aleph d. We thus see that under our assumption of **GCH** we have $\mathcal{P}(\Gamma) = a^{+}$.

Exactly the same argument shows that for each nonzero limit ordinal $\gamma < \alpha$, we have $\mathcal{P}(\Gamma, \gamma) = (\lim_{\xi<\gamma} a_\xi)^{+} \leqslant a_\gamma$, whence it is easily seen that $\mathcal{P}(\Gamma, \zeta) \leqslant a_\zeta$ for each $\zeta < \alpha$. However, (a_ξ) is an increasing α-sequence, and so $a_\xi < a$ for each $\xi < \alpha$. Thus we have $\lim \{\mathcal{P}(\Gamma, \zeta);\ \zeta < \alpha\} \leqslant a$. This shows that $\lim \{\mathcal{P}(\Gamma, \zeta);\ \zeta < \alpha\} < \mathcal{P}(\Gamma)$.

2 In this section we show that—assuming **ZFC** consistent—(#) is unprovable in **ZFC**. The obvious method of doing this is to produce a **ZFC**-model in which there is an increasing α-sequence Γ (for some nonzero limit ordinal α) such that $\lim \{\mathcal{P}(\Gamma, \zeta);\ \zeta < \alpha\} = \mathcal{P}(\Gamma)$: and this in essence is what we do. More precisely, we show that the assumption of a certain variant of **GCH** that is known to be relatively consistent with **ZFC** implies the existence of such a sequence.

Our first problem is to find a suitable variant of **GCH**; and our next result provides some clues in this direction.

Result 3 *Assume that there is a nonzero limit ordinal* α *and an increasing* α*-sequence* $\Gamma = (a_\xi)$ *of alephs such that* $\lim \{\mathcal{P}(\Gamma, \zeta);\ \zeta < \alpha\} = \mathcal{P}(\Gamma)$. *Then there exists a nonzero limit ordinal* γ *and an increasing* $\gamma + \omega$*-sequence* $\Delta = (b_\xi)$ *of alephs such that*

(1) $b_{\gamma+\psi+1} = (b_{\gamma+\psi})^{+}$, $\psi < \omega$;
(2) $\mathcal{P}(\Delta, \gamma) = \mathcal{P}(\Delta)$.

Proof: Assume the hypothesis, and let α be the least nonzero limit ordinal for which there is an increasing α-sequence $\Gamma = (\mathfrak{a}_\xi)$ such that $\lim\{\mathcal{P}(\Gamma, \zeta); \zeta < \alpha\} = \mathcal{P}(\Gamma)$. We claim that $\alpha = \gamma + \omega$ for some nonzero limit ordinal γ.

For suppose not; then we must have $\alpha = \omega^2\delta$ for some nonzero ordinal δ. We claim firstly that there is $\theta < \alpha$ such that $\mathcal{P}(\Gamma, \theta) = \mathcal{P}(\Gamma)$. For the α-sequence $(\mathcal{P}(\Gamma, \zeta))_{\zeta<\alpha}$ is certainly nondecreasing, and so if no such θ exists, then $(\mathcal{P}(\Gamma, \zeta))_{\zeta<\alpha}$ must contain a strictly increasing and cofinal subsequence $(\mathcal{P}(\Gamma, \zeta_\chi))_{\chi<\tau}$. Now clearly $\lim\{\mathcal{P}(\Gamma, \zeta_\chi); \chi < \tau\} = \lim\{\mathcal{P}(\Gamma, \zeta); \zeta < \alpha\} = \mathcal{P}(\Gamma)$, and so by (S1) we have $\mathcal{S}((\mathcal{P}(\Gamma, \zeta_\chi))) = \mathcal{P}(\Gamma)$. Moreover, by (P2) we have $\mathcal{P}((\mathcal{P}(\Gamma, \zeta_\chi))) \leqslant \mathcal{P}(\Gamma)$; thus $\mathcal{P}((\mathcal{P}(\Gamma, \zeta_\chi))) \leqslant \mathcal{S}((\mathcal{P}(\Gamma, \zeta_\chi)))$. But Zermelo's Theorem tells us that $\mathcal{S}((\mathcal{P}(\Gamma, \zeta_\chi))) < \mathcal{P}((\mathcal{P}(\Gamma, \zeta_\chi)))$. This contradiction shows that no such subsequence $(\mathcal{P}(\Gamma, \zeta_\chi))$ can exist, and so there must be $\theta < \alpha$ for which $\mathcal{P}(\Gamma, \theta) = \mathcal{P}(\Gamma)$.

Let θ be minimal in this respect. Since $\alpha = \omega^2\delta$, we must have $\theta + \omega < \alpha$. But we know that $\mathcal{P}(\Gamma, \theta) = \mathcal{P}(\Gamma) = \mathcal{P}(\Gamma, \theta + \omega)$. Hence if we put $\Psi = (\mathfrak{a}_\xi)_{\xi<\theta+\omega}$ we have fonnd a $\theta + \omega$-sequence for which $\lim\{\mathcal{P}(\Psi, \zeta); \zeta < \theta\} = \mathcal{P}(\Psi)$. As $\theta + \omega < \alpha$ this contradicts the choice of α. Thus it must be the case that $\alpha = \gamma + \omega$ for some nonzero limit ordinal γ, since the case $\alpha = \omega$ is ruled out by Result 1.

We now define an α-sequence $(\mathfrak{b}_\xi)$ of alephs as follows.

(1) $\mathfrak{b}_\xi = \mathfrak{a}_\xi$ for $\xi < \gamma$;
(2) $\mathfrak{b}_\gamma = \lim\{\mathfrak{b}_\xi, \xi < \gamma\}$,
(3) $\mathfrak{b}_{\gamma+\psi+1} = (\mathfrak{b}_{\gamma+\psi})^+$ for $\psi < \omega$.

We must show that $\lim\{\mathcal{P}(\Delta, \zeta); \zeta < \alpha\} = \mathcal{P}(\Delta)$, where $\Delta = (\mathfrak{b}_\xi)_{\xi<\alpha}$. Now since $\mathfrak{b}_\xi \leqslant \mathfrak{a}_\xi$ for each $\xi < \alpha$, we must have $\mathcal{P}(\Delta) \leqslant \mathcal{P}(\Gamma)$. As we obviously have $\mathcal{P}(\Delta, \gamma) \leqslant \mathcal{P}(\Delta)$, it suffices to show that $\mathcal{P}(\Delta, \gamma) = \mathcal{P}(\Gamma)$.

Assume that $\mathcal{P}(\Gamma, \gamma) < \mathcal{P}(\Gamma)$. We know that the α-sequence $(\mathcal{P}(\Gamma, \zeta))$ has no increasing cofinal subsequence, and so we must have $\mathcal{P}(\Gamma, \gamma + \psi) = \mathcal{P}(\Gamma)$ for some nonzero ordinal $\psi < \omega$. But by the elementary properties of alephs we have $\mathcal{P}(\Gamma, \gamma + \psi) = \mathcal{P}(\Gamma, \gamma)\mathfrak{a}_{\gamma+\psi}$. Thus the assumption $\mathcal{P}(\Gamma, \gamma) < \mathcal{P}(\Gamma)$ implies that $\mathfrak{a}_{\gamma+\psi} = \mathcal{P}(\Gamma)$. This, however, is impossible, since we have $\mathfrak{a}_{\gamma+\psi} < \mathfrak{a}_{\gamma+\psi+1} \leqslant \mathcal{P}(\Gamma)$. Hence we must have $\mathcal{P}(\Gamma, \gamma) = \mathcal{P}(\Gamma)$.

From the definition of Δ we obtain $\mathcal{P}(\Delta, \gamma) = \mathcal{P}(\Gamma, \gamma)$: thus $\mathcal{P}(\Delta, \gamma) = \mathcal{P}(\Gamma)$. As this gives $\mathcal{P}(\Delta, \gamma) = \mathcal{P}(\Delta)$, we have shown that the sequence Δ has the required properties.

In order to show that (#) is independent of **ZFC**, we make use of the fact (see for example Chapter 8 of [3]) that it is relatively consistent with **ZFC** to assume that $2^{\mathfrak{a}} = \aleph_{\gamma+\omega+1}$ for every aleph $\mathfrak{a} \leqslant \aleph_{\gamma+\omega}$, where γ is a previously specified nonzero limit ordinal. We also require the following two theorems of Tarski, given in [4]; the first is his well-known recursion formula.

Theorem 3 *Let* σ, τ *be any two ordinals, and* $\mathfrak{a}$ *an aleph with* $\mathfrak{a} \geqslant |\tau|$. *Then* $(\aleph_{\sigma+\tau})^{\mathfrak{a}} = (\aleph_\sigma)^{\mathfrak{a}} \cdot (\aleph_{\sigma+\tau})^{|\tau|}$.

Theorem 4 *Let* α *be any nonzero limit ordinal, and put* $\Gamma = (\aleph_\xi)_{\xi<\alpha}$. *Then* $\mathcal{P}(\Gamma) = (\aleph_\alpha)^{|\alpha|}$.

Result 4 *If* **ZFC** *is consistent, then* (#) *is unprovable in* **ZFC**.

Proof: Let γ be a specified nonzero limit ordinal, and assume that $2^{\mathfrak{a}} = \aleph_{\gamma+\omega+1}$ for every aleph $\mathfrak{a} \leqslant \aleph_{\gamma+\omega}$. Put $\Gamma = (\aleph_\xi)_{\xi<\gamma+\omega}$. We claim that $\mathcal{P}(\Gamma, \gamma) = \mathcal{P}(\Gamma)$; it follows from this of course that $\lim \{\mathcal{P}(\Gamma, \zeta); \zeta < \gamma + \omega\} = \mathcal{P}(\Gamma)$, and hence that (#) is unprovable in **ZFC**—assuming that **ZFC** is consistent. For by Theorem 4 we have $\mathcal{P}(\Gamma, \gamma) = (\aleph_\gamma)^{\mathfrak{a}}$, where $\mathfrak{a} = |\gamma|$. Since γ is infinite, we have $|\gamma| = |\gamma + \omega|$, and so by Theorem 4 again, $\mathcal{P}(\Gamma) = (\aleph_{\gamma+\omega})^{\mathfrak{a}}$. But of course $\mathfrak{a} \geqslant |\omega|$, and so by Theorem 3 we have $(\aleph_{\gamma+\omega})^{\mathfrak{a}} = (\aleph_\gamma)^{\mathfrak{a}} \cdot (\aleph_{\gamma+\omega})^{\aleph_0}$. Now it is a well-known fact that $\mathfrak{c}^{\mathfrak{c}} = 2^{\mathfrak{c}}$ for every aleph $\mathfrak{c}$. Thus $(\aleph_{\gamma+\omega})^{\aleph_0} \leqslant 2^{\aleph_{\gamma+\omega}} = \aleph_{\gamma+\omega+1} = 2^{\mathfrak{a}} \leqslant (\aleph_\gamma)^{\mathfrak{a}}$. Hence we have $\mathcal{P}(\Gamma) = (\aleph_\gamma)^{\mathfrak{a}} = \mathcal{P}(\Gamma, \gamma)$.

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