Notre Dame Journal of Formal Logic
Volume XIX, Number 2, April 1978
NDJFAM

LOGICS FOR KNOWLEDGE, POSSIBILITY, AND EXISTENCE

RODERIC A. GIRLE

In [2] completeness proofs were set out for several possibility pre-supposition free logics. Use was made of the kind of semantics to be found in Hintikka's work, especially in [3] and in *Knowledge and Belief* [4]. It is of interest to extend the possibility pre-supposition free logics by means of epistemic modalities similar to those in *Knowledge and Belief,* and by means of alethic modalities. In what follows we will be concerned with extensions of $\mathbf{QH}^{2}_{=}$, or systems isomorphic with $\mathbf{QH}^{2}_{=}$. Such extended logics could deal with sentences such as "John knows that the round square is an impossible object," "Everybody knows that Mr. Pickwick is an imaginary character," and "Mr. Pickwick knows who the Queen is."

As in [9] we use the quantifiers π and Σ to range over objects said to be real and objects said to be possible. We will also use the quantifiers U and E, as in [4], to range over objects said to be real or existing. The formula '$(\Sigma x)(x = a)$' would be translated as 'a is a possible object', and '$(\mathsf{E}x)(x = a)$' as 'a exists'. In order to avoid some of the problems which arise in [4] as a result of reading '$K_a p$' as 'a knows that p' and reading '$P_a p$' as 'It is possible, for all that a knows, that p', and holding '$P_a p \equiv \sim K_a \sim p$' we have two epistemic operators, ^{L}P and K. '$K_a p$' is read as above. '$^{L}P_a p$' is read as '$P_a p$' above. Whereas it is indefensible in the logic in [3] to say '$\sim K_a \mathbf{T}$' where $\mathbf{T}$ is a tautology, in the logics set out below we can defensibly say '$\sim K_a \mathbf{T}$' even though it is clearly indefensible to say '$^{L}P_a \sim \mathbf{T}$'.

1 Primitive symbols:

improper symbols $\supset\ \sim\ \pi\ \cup\ K\ ^{L}P\ (\ )\ \diamond$
bound personal variables $x_0, y_0, z_0, x_1, y_1, z_1, x_2, y_2, z_2, \ldots$
free personal variables $a_0, b_0, c_0, a_1, b_1, c_1, a_2, b_2, c_2, \ldots$
bound impersonal variables $i_0, j_0, k_0, i_1, j_1, k_1, i_2, j_2, k_2, \ldots$
free impersonal variables $s_0, t_0, u_0, s_1, t_1, u_1, s_2, t_2, u_2, \ldots$
propositional variables $p_0, q_0, r_0, p_1, q_1, r_1, p_2, q_2, r_2, \ldots$
n-ary predicate variables $(n \geqslant 1)$ $F^n_0, G^n_0, H^n_0, F^n_1, G^n_1, H^n_1, \ldots$
predicate constants $=$, $\mathbf{E}$

Received June 2, 1971

2 Formation Rules:

(i) A propositional variable standing alone is a wff.
(ii) If F^n is any n-ary predicate variable, and if $w_1, \ldots, w_n$ are n free personal or impersonal variables (not necessarily distinct nor of only one kind) then

$$F^n_{w_1 w_2 \ldots w_n} \text{ is a wff.}$$

(iii) If a and b are any free personal variables (not necessarily distinct) then $a = b$ is a wff.
(iv) If s and t are any free impersonal variables (not necessarily distinct) then $s = t$ is a wff.
(v) If w is a free personal or impersonal variable then $\mathsf{E}w$ is a wff.

Wffs according to (i) to (v) are atomic wffs.

(vi) If A is a wff, so is $\sim A$.
(vii) If A and B are wffs, so is $(A \supset B)$.
(viii) If A is a wff and x is any bound personal variable and a is any free personal variable, then both $(\pi x)(A(x//a))$ and $(\mathrm{U}x)(A(x//a))$ are wffs where:

If A is a wff and X is a personal variable free or bound and Y is a personal variable free or bound, then $A(X//Y)$ is the result of substituting X for zero or more occurrences of Y in A, and $A(X/Y)$ is the result of substituting X for every occurrence of Y in A.

'(ix)' is the result of substituting 'impersonal' for every occurrence of 'personal' in (viii), and 'i' for 'x' and 's' for 'a'.
(x) If A is a wff and a is any free personal variable then K_aA is a wff, and ${}^LP_aA$ is a wff, and a will be said to be an *epistemic subscript* in such wffs.
(xi) If A is a wff, then $\Diamond A$ is a wff.

We also adopt the usual definitions of the improper symbols &, v, and ≡ in terms of $\sim$ and $\supset$, and $(\Sigma X)A =_{df} \sim (\pi X) \sim A$, $(\mathsf{E}X)A =_{df} \sim (\mathsf{U}X) \sim A$, $P_aA =_{df} \sim K_a \sim A$, ${}^LK_aA =_{df} \sim {}^LP_a \sim A$, $\Box A =_{df} \sim \Diamond \sim A$, and write $\sim(a = b)$ as $a \neq b$. We also adopt the convention that $X, Y, Z, X_1, Y_1, Z_1, \ldots$ can stand for any free or bound variable permitted by the formation rules. We define "fully ^{K}modalized" as follows:

p is not fully ^{K}modalized when p is any atomic wff. $\sim A$ is fully ^{K}modalized iff A is. $(A \supset B)$ is fully ^{K}modalized iff A and B are both fully ^{K}modalized. ${}^LK_aA$ is fully ^{K}modalized.

3 Axiom Schemata:

1. $A \supset (B \supset A)$
2. $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$
3. $(\sim B \supset \sim A) \supset (A \supset B)$
4. $K_a(A \supset B) \supset (K_aA \supset K_aB)$
5. $(\mathsf{E}x)(x = a) \supset (K_aA \supset A)$
6. $(\mathsf{E}x)(x = a) \supset ((\mathrm{E}x)(x = b) \supset (K_bK_aA \supset K_bA))$

7. $^{L}K_a(A \supset B) \supset (^{L}K_aA \supset {}^{L}K_aB)$
8. $(\mathrm{E}x)(x = a) \supset (^{L}K_aA \supset A)$
9. $^{L}K_aA \supset {}^{L}K_a{}^{L}K_aA$
10. $^{L}K_aA \supset A$, *provided A is fully ^{K}modalized.*
11. $(\mathrm{E}x)(x = a) \supset ((\mathrm{E}x)(x = b) \supset (^{L}K_b{}^{L}K_aA \supset {}^{L}K_bA))$
12. $^{L}K_aA \supset \sim {}^{L}K_a \sim A$
13. $^{L}P_aA \supset P_aA$
14. $K_aA \supset {}^{L}K_aK_aA$
15. $\Box(A \supset B) \supset (\Box A \supset \Box B)$
16. $\Box A \supset A$
17. $\Diamond A \supset \Box\Diamond A$
18. $(\Sigma x)(x = a) \supset (K_aA \supset \Diamond A)$
19. $(\Sigma x)(x = a) \supset (^{L}K_aA \supset \Diamond A)$
20. $\mathbf{E}a \supset (K_aA \supset A)$
21. $\mathbf{E}a \supset (^{L}K_aA \supset A)$
22. $\mathbf{E}a \supset (\mathbf{E}b \supset (K_aK_bA \supset K_aA))$
23. $\mathbf{E}a \supset (\mathbf{E}b \supset (^{L}K_a{}^{L}K_bA \supset {}^{L}K_aA))$
24. $A \supset (\pi X)A$, *provided X does not occur in A.*
25. $(\pi X)(A \supset B) \supset ((\pi X)A \supset (\pi X)B)$
26. $(\Sigma Y)(Y = X) \supset ((\pi Z)A \supset A(X/Z))$ *provided Z occurs in A.*
27. $(\pi X)(\Sigma Y)(Y = X)$
28. $(\mathrm{E}Y)A \supset (\Sigma Y)A$
29. $(\pi Y)A \supset (\mathrm{U}Y)A$
30. $A \supset (\mathrm{U}X)A$ *provided X does not occur in A.*
31. $(\mathrm{U}X)(A \supset B) \supset ((\mathrm{U}X)A \supset (\mathrm{U}X)B)$
32. $(\mathrm{E}X)(X = Y) \supset ((\mathrm{U}X)A \supset A(Y/X))$ *provided X occurs in A.*
33. $(\mathrm{U}X)(\mathrm{E}Y)(Y = X)$
34. $X = X$
35. $X = Y \supset (A \supset A(X//Y))$ *provided A is an atomic wff.*
36. $\mathbf{E}x \supset (\Sigma Y)(Y = X)$

Rules:

R1 $A, A \supset B \rightarrow B$
R2 $A \rightarrow (\pi X)(A(X/Y))$ provided X does not occur in A.
R3 $A \rightarrow {}^{L}K_aA$
R4 $A \rightarrow \Box A$
R5 $A \rightarrow (\mathrm{U}X)(A(X/Y))$ provided X does not occur in A.

4 The Systems **QH$^{\underline{2}}$E**, **QHE**, **QHEM**, **QHK**, and **QHKE** can be axiomatized using sets of axiom schemata and rules as follows:

QHE = {1 - 14, 24 - 35; R1 - R3}
QH$^{\underline{2}}$E = {1 - 4, 7, 9, 10, 12 - 14, 20 - 27, 34 - 36; R1 - R3}
QHEM = {1 - 19, 24 - 35; R1 - R4}
QHK = {1 - 3, 7 - 12, 30 - 35; R1, R3, R5}
QHKE = {1 - 14, 30 - 35; R1, R3, R5}

It will also be clear that $\Diamond$ is a primitive symbol in **QHEM** only, and 2(xi) is a formation rule in **QHEM** only. **E** is a primitive symbol in **QH$^{\underline{2}}$E** only and also 2(v). π is not a primitive symbol in **QHK** nor **QHKE** and 2(viii) must be suitably amended. **E** is not a primitive symbol in **QH$\stackrel{2}{=}$E** and 2(viii) must be suitably amended.

The System **QH$\stackrel{2}{=}$E** is an extension of **QH$\stackrel{2}{=}$** with the epistemic modalities K and ^LP.
QHE is isomorphic with **QH$\stackrel{2}{=}$E**, but uses the quantifiers U and E instead of the predicate constant **E**.
QHEM extends **QHE** by the introduction of $\diamond$ and its axioms.
QHK is a system somewhat like that in [4].
QHKE has both modalities K and ^LP where **QHK** has only P.

In what follows there is a completeness proof for **QHE**. It will also be obvious from the axiom schemata that the logic of $\diamond$ alone is isomorphic with S5 (*cf.* [8]), the logic of ^LP alone is isomorphic with S4 (*cf.* [7]) for existing knowers and with D4 for imaginary knowers, and the logic of K alone is a weak system which could be called E0.5. The logic for both quantification systems is isomorphic with **QH$\stackrel{2}{=}$**. Following the proof of completeness for **QHE** there are appended some remarks concerning various features of **QHE** and the other systems.

5 We define **QHE**-consistency for formulae as in [6], i.e.,

$$\text{QHE-consistent } (A) \;.\!\equiv\!.\; \dashv_{\text{QHE}} \sim A$$

also

a finite set of formulae of **QHE**, $\{A_1, \ldots, A_n\}$ is consistent iff

$$\dashv_{\text{QHE}} \sim (A \;\&\; \ldots \;\&\; A_n),$$

and

an infinite set of formulae of **QHE**, Λ, is consistent iff it contains no not consistent finite subset.

We also define a maximal consistent set of formulae as in [2].

The following can be proved:

L1. *If Δ is maximal consistent relative to* **QHE**, *then for any wff A, A and $\sim A$ are not both in Δ.*

L2. *If Δ is maximal consistent relative to* **QHE**, *then for any wff A, either A or $\sim A$ is in Δ.*

L3. *If Δ is maximal consistent relative to* **QHE**, *then for any wffs A and B, if $A \in \Delta$ and $(A \supset B) \in \Delta$, then $B \in \Delta$.*

L4. *If Δ is maximal consistent relative to* **QHE** *then all the axioms and theorems are in Δ.*

Similarly, if Λ_j is an infinite set of formulae of **QHE** which contain only those free variables that are in some infinite set of free variables, such as $\mathbf{d}_k$, where

$$\mathbf{d}_k = a_0^k,\ b_0^k,\ c_0^k,\ s_0^k,\ t_0^k,\ u_0^k,\ a_1^k,\ \ldots,$$

then $L1^1$, $L2^1$, $L3^1$, and $L4^1$ can be proved. $L1^1$, $L2^1$, $L3^1$, and $L4^1$ are the result of substituting Λ_j for **QHE** in L1, L2, L3, and L4 respectively, and adding "provided that they are themselves in Λ_j".

6 In order to show **QHE** complete, we show that if any formula A is **QHE**-selfsatisfying, then $\vdash_{\mathbf{QHE}} A$; or for every **QHE**-consistent formula there is a satisfying **QHE**-Model.

Procedure: Given a formula, A, which is **QHE**-consistent, we construct, beginning with A, a system of maximal consistent sets, and we construct a **QHE**-Model which satisfies the formulae in the system and therefore satisfies A itself.

Let the system of sets be K, and the sets in K are also members of at least one pair of disjoint sub-sets of K such that for any a and $\Delta_{ij}^a \in K$ $(0 \leqslant i \leqslant 1,\ j \geqslant 1)$ $\langle N_a, N_a^1\rangle$ where $N_a \subseteq K$, $N_a^1 \subseteq K$, $N_a \cap N_a^1 = \emptyset$. Assume that

$$N_a = \{\Delta_{01}^a, \Delta_{02}^a, \ldots, \Delta_{0n}^a, \ldots\}\ (n \geqslant 1),$$

and that

$$N_a^1 = \{\Delta_{11}^a, \Delta_{12}^a, \ldots, \Delta_{1k}^a, \ldots\}\ (k \geqslant 1).$$

K is the smallest such set and has the following features:

(a) $A \in \Delta$ and either $\Delta = \Delta_{01}^a$ or $\Delta = \Delta_{11}^a$ for some a.

(b) For every Δ_{nm}^a $(1 \geqslant n \geqslant 0,\ m \geqslant 1)$ in K and every wff of the form $\sim {}^L K_b B$ in Δ_{nm}^a, there is an ^{L}alternate$_b$ maximal consistent set Δ_{0j}^b $(j \geqslant 1)$ such that

(i) $\Delta_{0j}^b \in N_b$
(ii) $\sim B \in \Delta_{0j}^b$
(iii) for every wff of the form ${}^L K_b C$ in Δ_{nm}^a ${}^L K_b C \in \Delta_{0j}^b$ and $C \in \Delta_{0j}^b$, and for every wff of the form $K_b D$ in Δ_{nm}^a $K_b D \in \Delta_{0j}^b$ and $D \in \Delta_{0j}^b$
(iv) a and b are not necessarily distinct.

(c) For every $\Delta_{0i}^b (i \geqslant 1)$ in $N_b(\Delta_{nm}^a = \Delta_{0i}^b)$ and for every wff of the form $\sim {}^L K_b B$ and every wff of the form $\sim K_b B$ in Δ_{0i}^b, there is an alternate$_b$ maximal consistent set $\Delta_{1k}^b (k \geqslant 1)$ such that

(i) $\Delta_{1k}^b \in N_b^1$
(ii) $\sim B \in \Delta_{1k}^b$
(iii) for every wff of the form $K_b C$ in Δ_{0i}^b $C \in \Delta_{1k}^b$.

(d) For every $\Delta_{1p}^b (p \geqslant 1)$ and for every wff of the form $\sim {}^L K_b B$ and every wff of the form $\sim K_b B$ in Δ_{1p}^b, there is an alternate$_b$ maximal consistent set $\Delta_{1g}^b (g \geqslant 1)$ such that

(i) $\Delta_{1g}^b \in N_b^1$
(ii) $B \in \Delta_{1g}^b$
(iii) for every wff of the form $K_b C$ in Δ_{1p}^b $\sim C \in \Delta_{1g}^b$.

7 Beginning with any **QHE**-consistent (A) we can construct a single maximal consistent set, Δ, of formulae of **QHE**, such that Δ has the $\mathbf{P}_K$-property.

First we define the **P**-property (*cf.* in [2], p. 53, the $\mathbf{P}_2$-property). A set, Λ, is said to have the **P**-property iff for every wff of the form $(\Sigma X)B$ in Λ there is also in Λ a wff of the form $(\Sigma X)(X = Y) \supset B(Y/X)$, provided X occurs in B, for some Y.

To ensure that Δ has the **P**-property, we begin with some definitions:

(i) Any wff of the form

$$(\Sigma X)B \supset ((\Sigma X)(X = Y) \supset B(Y/X)),$$

provided X occurs in B, we shall call a P-formula with respect to Y, or a P^Y-formula.

(ii) All P-formulae which differ only in that each is a P-formula with respect to a different Y (a free variable) will be said to have the same P-form.

Clearly the P-forms are enumerable.

(iii) Let the P-forms be enumerated thus:

$$P_1, P_2, P_3, \ldots, P_n, \ldots$$

and let

$$\mathscr{P} = \{x/(\exists_j)(x = P_j)\},$$

then a set of wffs has the $\mathbf{P}^1$-property iff it is a superset of a selection set for $\mathscr{P}$.

If a maximal consistent set, Λ, of **QHE** has the $\mathbf{P}^1$-property, it also has the **P**-property.

Secondly, we define the ${}^1\mathbf{P}$-property in the same way as the **P**-property above, except that we substitute E for Σ at every point, and ${}^1\mathbf{P}$ for **P**.

If a maximal consistent set, Λ, of **QHE** has the ${}^1\mathbf{P}^1$-property, it also has the ${}^1\mathbf{P}$-property.

Thirdly, we define the $\mathbf{P}_K$-property thus:

(i) Any wff of the form $(\Sigma X)B \supset ((\Sigma X)(X = Y) \supset B(Y/X))$, provided X occurs in B, is a P_K-formula with respect to Y, or a P_K^Y-formula.

(ii) Any wff of the form $(\mathrm{E}X)B \supset ((\mathrm{E}X)(X = Y) \supset B(Y/X))$, provided X occurs in B, is a P_K-formula with respect to Y, or a P_K^Y-formula.

In virtue of (i) and (ii), every P^Y-formula and every ${}^1\mathrm{P}^Y$-formula is also a P_K^Y-formula.

(iii) All P_K-formulae which differ only in that each is a P_K-formula with respect to a different variable will be said to have the same P_K-form.

Clearly the P_K-forms are enumerable.

(iv) A set of wffs has the $\mathbf{P}^1_K$-property iff it contains at least one P_K-formula for each P_K-form.

If a maximal consistent set has the $\mathbf{P}^1_K$-property it also has the **P**-property and the ${}^1\mathbf{P}$-property.

Fourthly, we assume that the wffs on **QHE** are arrayed in some standard ordering, similarly the P_K-forms.

Fifthly, the proof of the following lemma can be retrieved from [2], with suitable modifications:

Lemma (A). *If* Λ *is a consistent set of formulae, none of which contains* Y, *and* G *is a* P^Y_K*-formula, then* $\Lambda \cup \{G\}$ *is consistent with respect to* **QHE**.

8 To construct K:

(a) First we construct Δ, beginning with $\{A\}$ (*cf.* [6], p. 175). Let the free variables in A be $X_1, \ldots, X_n$. Let us now suppose that all the other free variables are arranged in an infinite series of infinite sets, each of which is to be associated, in a way to be set out below, with one of the maximal consistent sets in K. Let us write each free variable with a superscript as follows:

$$\mathbf{d}_1 = \{a_0^1, b_0^1, c_0^1, s_0^1, t_0^1, u_0^1, a_1^1, b_1^1, c_1^1, s_1^1, t_1^1, u_1^1, a_2^1, \ldots\}$$
$$\mathbf{d}_2 = \{a_0^2, b_0^2, c_0^2, s_0^2, t_0^2, u_0^2, \ldots\}$$
$$\vdots$$
$$\mathbf{d}_k = \{a_0^k, b_0^k, c_0^k, s_0^k, t_0^k, u_0^k, \ldots\}$$

Let Λ_1 be the set of all those wffs of **QHE** all of whose free variables either occur in A or are members of $\mathbf{d}_1$. Having begun Δ with $\{A\}$ we then give the set the $\mathbf{P}^1_K$-property by adding for each wff of the form $(\Sigma X)B$ in Λ_1, a wff of the form $(\Sigma X)B \supset ((\Sigma X)(X = Y) \supset B(Y/X))$, provided X occurs in B, where Y does not occur previously in Δ but is drawn from $\mathbf{d}_1$, and also for each wff of the form $(\mathsf{E}X)B$ in Λ_1, a wff of the form $(\mathsf{E}X)B \supset ((\mathsf{E}X)(X = Y) \supset B(Y/X))$, provided X occurs in B, where Y does not occur previously in Δ but is drawn from $\mathbf{d}_1$. The formulae are added to $\{A\}$ alternately. Then we make the set maximal consistent for Λ_1. So Δ is maximal consistent with respect to **QHE** for the free variables in $\mathbf{d}_1$ and A. Let the set of free variables in $\mathbf{d}_k$ and A be $\mathbf{d}'_k$.

(b) Then we show that for some b, *either*

taking $\Delta = \Delta^b_{01}$ $(\Delta^b_{01} \in N_b)$ and some maximal consistent set Δ' taken to be $\Delta' = \Delta^b_{0i} (i \geqslant 1)$, where Δ' has the $\mathbf{P}^1_K$-property and is maximal consistent relative to some Λ_n as set out above, we can construct for any wff of the form $\sim {}^L K_b B$ in Δ^b_{0i} an ^{L}alternate$_b$ maximal consistent set $\Delta^b_{0j} (j \geqslant 1)$ containing $\sim B$ and every wff ${}^L K_b C$ and C such that ${}^L K_b C \in \Delta^b_{0i}$ and every wff $K_b D$ and D such that $K_b D \in \Delta^b_{0i}$; and we can construct for any wff of the form $\sim K_b E$ in Δ^b_{0i} an alternate$_b$ maximal consistent set Δ^b_{1k} $(k \geqslant 1)$ containing $\sim E$ and every wff D such that $K_b D \in \Delta^b_{0i}$.

or

taking $\Delta = \Delta^b_{11}(\Delta^b_{11} \epsilon N^1_b)$ and some maximal consistent set Δ' taken to be $\Delta' = \Delta^b_{1p}(p \geq 1)$, we can construct for any wff of the form $\sim {}^L K_b B$ in Δ^b_{1p} the same ^{L}alternate$_b$ maximal consistent set, Δ^b_{0j}, as above; and we can construct for any wff of the form $\sim {}^L K_b B$ and any wff of the form $\sim K_b B$ an alternate$_b$ maximal consistent set $\Delta^b_{1g}(g \geq 1)$ where Δ^b_{1g} contains B and every wff $\sim D$ such that $K_b D \epsilon \Delta^b_{1p}$.

(c) We begin by taking $\Delta = \Delta^b_{01}$ and for Δ' we construct $\Delta^{b''}_{0j}$ and $\Delta^{b''}_{1k}$. Let $\Delta^{b''}_{0j}$ be a subset of the mth set to be constructed in K, and $\Delta^{b''}_{1k}$ a subset of the 1th set to be constructed in K.

(i) We begin $\Delta^{b''}_{0j}$ with $\sim B$.

(ii) We then add every wff ${}^L K_b C$ and C such that ${}^L K_b C \epsilon \Delta^b_{0i}$, and every wff $K_b D$ and D such that $K_b D \epsilon \Delta^b_{0i}$. The set so constructed is $\Delta^{b''}_{0j}$.

(iii) We begin $\Delta^{b''}_{1k}$ with $\sim E$.

(iv) We then add every wff D such that $K_b D \epsilon \Delta^b_{0i}$. The set so constructed is $\Delta^{b''}_{1k}$.

(d) We then take $\Delta = \Delta^b_{11}$ and for Δ' we construct $\Delta^{b''}_{1g}$. Let $\Delta^{b''}_{1g}$ be a subset of the nth set to be constructed in K.

(i) We begin $\Delta^{b''}_{1g}$ with B.

(ii) We add every wff $\sim D$ such that $K_b D \epsilon \Delta'$. The set so constructed is $\Delta^{b''}_{1g}$.

(e) We now show $\Delta^{b''}_{0j}$ to be consistent. Assume $\Delta^{b''}_{0j}$ is inconsistent. Hence, there are formulae:

$${}^L K_b C_1, {}^L K_b C_2, \ldots, {}^L K_b C_n \text{ such that } {}^L K_b C \epsilon \Delta'$$

and

$$K_b D_1, K_b D_2, \ldots, K_b D_m \text{ such that } K_b D \epsilon \Delta'$$

such that

$\vdash_{\overline{QHE}} \sim ({}^L K_b C_1 \,\&\, \ldots \,\&\, {}^L K_b C_n \,\&\, C_1 \,\&\, \ldots \,\&\, C_n \,\&\, K_b D_1 \,\&\, \ldots \,\&\, K_b D_m \,\&\, D_1 \,\&\, \ldots \,\&\, D_m \,\&\, \sim B)$.

Hence

$\vdash_{\overline{QHE}} {}^L K_b {}^L K_b C_1 \supset (\ldots \supset ({}^L K_b {}^L K_b C_n \supset ({}^L K_b C_1 \supset (\ldots \supset ({}^L K_b C_n \supset ({}^L K_b K_b D_1 \supset (\ldots \supset ({}^L K_b K_b D_m \supset ({}^L K_b D_1 \supset (\ldots \supset ({}^L K_b D_m \supset {}^L K_b B) \ldots))) \ldots))) \ldots))) \ldots)$,

and since Δ' is maximal consistent then ${}^L K_b B \epsilon \Delta'$ but $\sim {}^L K_b B \epsilon \Delta'$ so, by reductio ad absurdum, $\Delta^{b''}_{0j}$ is consistent.

(f) We now show that either $\Delta^{b''}_{1k}$ or $\Delta^{b''}_{1g}$ is consistent. Assume both inconsistent. Therefore:

$\vdash_{\overline{QHE}} \sim (D_1 \,\&\, \ldots \,\&\, D_m \,\&\, \sim E)$

and

$\vdash_{\overline{\mathrm{QHE}}} \sim(\sim D_1 \,\&\, \ldots \,\&\, \sim D_m \,\&\, E)$.

So $\vdash_{\overline{\mathrm{QHE}}} {}^LK_bD_1 \supset (\ldots \supset ({}^LK_bD_m \supset {}^LK_bE) \ldots)$ (R3, Def &)
and $\vdash_{\overline{\mathrm{QHE}}} {}^LK_bE \supset ({}^LK_b \sim(\sim D_1 \,\&\, \ldots \,\&\, \sim D_m))$. (R3, Def &)
Since each ${}^LK_bD \in \Delta'$, then ${}^LK_bE \in \Delta'$
and ${}^LK_b \sim(\sim D_1 \,\&\, \ldots \,\&\, \sim D_m) \in \Delta'$.
Since $\vdash_{\overline{\mathrm{QHE}}} {}^LK_b \sim A \supset \sim K_bA$, (Ax12, Ax13)
then $\sim K_b(\sim D_1 \,\&\, \ldots \,\&\, \sim D_m) \in \Delta'$.
So in at least one case $E = (\sim D_1 \,\&\, \ldots \,\&\, \sim D_m)$,
so $\vdash_{\overline{\mathrm{QHE}}} \sim(D_1 \,\&\, \ldots \,\&\, D_m \,\&\, \sim(\sim D_1 \,\&\, \ldots \,\&\, \sim D_m))$,
so $\vdash_{\overline{\mathrm{QHE}}} \sim(\sim D_1 \,\&\, \ldots \,\&\, \sim D_m) \supset \sim(D_1 \,\&\, \ldots \,\&\, D_m)$
and also $\vdash_{\overline{\mathrm{QHE}}} \sim(\sim D_1 \,\&\, \ldots \,\&\, \sim D_m \,\&\, (\sim D_1 \,\&\, \ldots \,\&\, \sim D_m))$,
so $\vdash_{\overline{\mathrm{QHE}}} \sim(D_1 \,\&\, \ldots \,\&\, D_m)$
so Δ_{0j}^{b}, which contains $D_1, \ldots, D_m$ is inconsistent but Δ_{0j}^{b} is consistent.
Hence, by reductio ad absurdum, one of $\Delta_{1k}^{b''}$ or $\Delta_{1g}^{b''}$ is consistent.

(g) (i) So we construct $\Delta_{0j}^{b'}$ ($\Delta_{0j}^{b'} \subseteq \Delta_{0j}^{b}$) as follows:

We take $\Delta_{0j}^{b''}$ and give the set the $\mathbf{P}_K^1$-property by adding for each wff, in the appropriate Λ_K, of the form $(\mathrm{E}X)B$, and for each wff of the form $(\Sigma X)B$, where X occurs in B, wffs of the forms

$$(\mathrm{E}X)B \supset ((\mathrm{E}X)(X = Y) \supset B(Y/X)) \text{ and } (\Sigma X)B \supset ((\Sigma X)(X = Y) \supset B(Y/X))$$

where Y is a new variable in each case always drawn from the new set $\mathbf{d}_m$. By Lemma (A) $\Delta_{0j}^{b'}$ is consistent if $\Delta_{0j}^{b''}$ is.

(ii) We construct Δ_{0j}^{b} by maximizing $\Delta_{0j}^{b'}$ with respect to Λ_k.

(h) Also we construct on either $\Delta_{1k}^{b''}$ or $\Delta_{1g}^{b''}$, for whichever is consistent, either Δ_{1k}^{b} or Δ_{1g}^{b} respectively by the same method as in (g) above.

9 For a proof of the Completeness of **QHE** we show that we can construct a satisfying **QHE**-Model for A when A is **QHE**-consistent (A). Consider a Hintikka type model as follows:

$\langle \Omega, \Phi, C\rangle$ is a **QHE**-Model where Ω is a model system of maximal model sets (*cf.* [2]) such that the members of Ω are also members of at least one pair of disjoint subsets of Ω such that for any a $\mu_{ij}^{a} \in \Omega$ $(0 \leqslant i \leqslant 1, j \geqslant 1)$ $\langle \Gamma_a, \Gamma_a'\rangle$ where $\Gamma_a \subseteq \Omega$, $\Gamma_a' \subseteq \Omega$, $\Gamma_a \cap \Gamma_a' = \emptyset$,

$$\Gamma_a = \{\mu_{01}^{a}, \mu_{02}^{a}, \ldots, \mu_{0n}^{a}, \ldots\} \; (n \geqslant 1),$$

and

$$\Gamma_a' = \{\mu_{11}^{a}, \mu_{12}^{a}, \ldots, \mu_{1k}^{a}, \ldots\} \; (k \geqslant 1);$$

Φ is a function from $\mathbf{d}$, the set of sets of free variables as set out in 8(a) above: $\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_k, \ldots$, to the members of Ω in their order of construction; and where C is a set of consistency rules for deciding which formulae of **QHE** can be included (or embedded) in any μ_{ij}^{a} (any μ).

We define Φ set (μ): Φ set (μ) is the union of all the sets $\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_r$ together with the set of free variables in the given **QHE**-consistent (A), where $(r \geqslant 1)$ and μ is the rth set to be constructed in the **QHE**-Model for A.

The basic concept is that of Satisfiability:

$$(\exists a)(\exists i)(\exists j)((1 \geqslant i \geqslant 0) \;\&\; (j \geqslant 1) \;\&\; (K_a B) \;\&\; (A \in \mu_{ij}^a)) \;.\!\equiv\!.\; \text{Satisfiable } (A).$$

Also

$$\text{Self-sustaining } (A) \;.\!\equiv\!.\; \sim\text{Satisfiable } (\sim A).$$

The membership of C is as follows for **QHE** Satisfiability:

($C.\sim$) If μ contains an atomic formula it does not contain its negation.
($C.\supset$) If $(A \supset B) \in \mu$, then either $\sim A \in \mu$, or $B \in \mu$, or both.
($C.\sim\pi$) If $\sim(\pi X)A \in \mu$, then $(\Sigma X)\sim A \in \mu$.
($C.\sim\Sigma$) If $\sim(\Sigma X)A \in \mu$, then $(\pi X)\sim A \in \mu$.
($C.\sim\mathsf{U}$) If $\sim(\mathsf{U}X)A \in \mu$, then $(\mathsf{E}X)\sim A \in \mu$.
($C.\sim\mathsf{E}$) If $\sim(\mathsf{E}X)A \in \mu$, then $(\mathsf{U}X)\sim A \in \mu$.
($C.\sim{}^LK$) If $\sim{}^LK_aA \in \mu$, then ${}^LP_a\sim A \in \mu$.
($C.\sim{}^LP$) If $\sim{}^LP_aA \in \mu$, then ${}^LK_a\sim A \in \mu$.
($C.\sim K$) If $\sim K_aA \in \mu$, then $P_a\sim A \in \mu$.
($C.\sim P$) If $\sim P_aA \in \mu$, then $K_a\sim A \in \mu$.
($C.$self $\neq$) μ does not contain any formula of the form $(X \neq X)$.
($C.=$) If $A \in \mu$, $(X = Y) \in \mu$, and A is like B except for the interchange of X and Y at some (or all) of their occurrences, then $B \in \mu$, provided that A and B are atomic formulae.
($C.\mathsf{E}\Sigma$) If $(\mathsf{E}X)A \in \mu$, then $(\Sigma X)A \in \mu$.
($C.\pi\mathsf{U}$) If $(\pi X)A \in \mu$, then $(\mathsf{U}X)A \in \mu$.
($C.\Sigma_0^!$) If $(\Sigma X)A \in \mu$, then if $(\Sigma X)(X = Y) \in \mu$ then $A(Y/X) \in \mu$ for at least one free variable Y, provided that X occurs in A.
($C.\pi_0^!$) If $(\pi X)A \in \mu$, then if $(\Sigma X)(X = Y) \in \mu$, then $A(Y/X) \in \mu$, provided that X occurs in A.
($C.\pi G$) If $A \in \mu$, then $(\pi X)A \in \mu$, provided X does not occur in A.
($C.\Sigma V$) If $(\Sigma X)A \in \mu$ and X does not occur in A, then $A \in \mu$.

There are also the rules ($C.\mathsf{E}_0^!$), ($C.\mathsf{U}_0^!$), ($C.\mathsf{U}G$), and ($C.\mathsf{E}V$) which are parallel to ($C.\Sigma_0^!$), ($C.\pi_0^!$), ($C.\pi G$), and ($C.\Sigma V$) respectively, but with the U and E quantifiers.

($C.\Phi\mu$) μ contains no formulae whose free variables are not in Φ set (μ).
($C.{}^LP*$) If ${}^LP_aA \in \mu_{ij}^b \in \Omega$, then there is at least one ^{L}alternate$_a$ set to μ_{ij}^b, such as μ_{0k}^a where $\mu_{0k}^a \in \Gamma_a$, such that $A \in \mu_{0k}^a$; and if $\mu_{ij}^b = \mu_{1m}^a$ ($\mu_{1m}^a \in \Gamma_a'$) then there is also an alternate$_a$ set to μ_{ij}^a, such as μ_{1g}^a where $\mu_{1g}^a \in \Gamma_a'$, such that $\sim A \in \mu_{1g}^a$.
($C.P\sim$) If $P_aA \in \mu_{0i}^a \in \Omega$, then there is at least one alternate$_a$ set to μ_{0i}^a, such as μ_{1j}^a where $\mu_{1j}^a \in \Gamma_a'$, such that $A \in \mu_{1j}^a$, but if $P_aA \in \mu_{1m}^a \in \Gamma_a'$, then there is at least one alternate$_a$ set to μ_{1m}^a, such as μ_{1p}^a where $\mu_{1p}^a \in \Gamma_a'$, such that $\sim A \in \mu_{1p}^a$.
($C.K\overset{*}{\sim}$) If $K_aA \in \mu_{0i}^a \in \Omega$, then if μ_{1j}^a is an alternate$_a$ set to μ_{0i}^a, then $A \in \mu_{1j}^a$;

but if $K_a A \in \mu^a_{1m} \in \Gamma'_a$, then if μ^a_{1g} is an alternate$_a$ set to μ^a_{1m}, then $\sim A \in \mu^a_{1g}$.

(*C.KK*^{L}) If $K_a A \in \mu^b_{ij} \in \Omega$, then if μ^a_{ok} is an ^{L}alternate$_a$ set to μ^b_{ij}, then $K_a A \in \mu^a_{ok}$ and $A \in \mu^a_{ok}$.

(*C.K*E) If $K_a A \in \mu \in \Omega$ and $(\text{E}X)(X = a) \in \mu$, then $A \in \mu$.

(*C.*^{L}*K*E) If $^L K_a A \in \mu \in \Omega$ and $(\text{E}X)(X = a) \in \mu$, then $A \in \mu$.

(*C.*^{L}*K*^{L}*K*) If $^L K_a A \in \mu^b_{ij} \in \Omega$ and μ^a_{ok} is an ^{L}alternate$_a$ set to μ^b_{ij}, then $^L K_a A \in \mu^a_{ok}$ and $A \in \mu^a_{ok}$.

(*C.KK*tr) If $K_a K_b A \in \mu \in \Omega$ and $(\text{E}X)(X = a) \in \mu$ and $(\text{E}X)(X = b) \in \mu$, then $K_a A \in \mu$.

10 Given **QHE**-consistent (A), we have constructed K such that $A \in \Delta \in K$. We construct Ω as follows:

(a) With each μ^b_{ij} we associate some model set Δ^b_{ij}.

(b) Since each Δ^b_{ij} is maximal consistent with respect to that set of **QHE**-formulae whose free variables either occur in A or in $\mathbf{d}_1 \cup \mathbf{d}_2 \cup \ldots \cup \mathbf{d}_m$ (where Δ^b_{ij} is the mth set constructed in K), we stipulate that in the construction of Ω

$$\mathbf{d}'_1 \cup \mathbf{d}_2 \cup \ldots \cup \mathbf{d}_m \subseteq \Phi \text{ set } (\mu^b_{ij})$$
where Δ^b_{ij} is associated with μ^b_{ij}.

(c) Each atomic wff B is Satisfiable (B) if it is one of the wffs in some Δ^b_{ij}, and is $\sim$Satisfiable (B) if it is not in any Δ^b_{ij}, i.e.,

$$(\forall i)(\forall j)((B \in \Delta^a_{ij}) \equiv (B \in \mu^a_{ij})).$$

(d) Each wff of the form $(\Sigma X)(X = Y)$ is Satisfiable $(\Sigma X)(X = Y)$ if it is one of the wffs in some Δ^b_{ij}, and $\sim$Satisfiable $(\Sigma X)(X = Y)$ if it is not in any Δ^b_{ij}.

(e) Each wff of the form $(\text{E}X)(X = Y)$ is Satisfiable $(\text{E}X)(X = Y)$ if it is one of the wffs in some Δ^b_{ij}, and $\sim$Satisfiable $(\text{E}X)(X = Y)$ if it is not in any Δ^b_{ij}.

(f) For every Y such that $\sim(\Sigma X)(X = Y) \in \Delta^b_{ji}$ and $B(Y/X) \in \Delta^b_{ji}$, then if $(\pi X)B \in \Delta^b_{ji}$ (and X occurs in B), Satisfiable $(\pi X)B$, and if $(\pi X)B \notin \Delta^b_{ji}$ (for every i and j and b) then $\sim$Satisfiable $(\pi X)B$.

'(g)' is as '(f)' but with 'E' for 'Σ' and 'U' for 'π'.

(h) When Δ^b_{ji} contains at least one formula of the form $(\Sigma X)(X = Y)$, and also $\sim B$, each wff of the form $(\pi X)B$ (X does not occur in B) is Satisfiable $(\pi X)B$ if $(\pi X)B \in \Delta^b_{ji}$.

'(i)' is as '(h)' but with 'E' for 'Σ' and 'U' for 'π'.

(j) When Δ^b_{ji} contains no formula of the form $(\Sigma X)(X = Y)$, and also $\sim B$, each wff of the form $(\pi X)B$ (X does not occur in B) is Satisfiable $(\pi X)B$ if $(\pi X)B \in \Delta^b_{ji}$.

'(k)' is as '(j)' but with 'E' for 'Σ' and 'U' for 'π'.

11 Completeness Theorem: *Given Satisfaction as defined above, for every wff B,* Satisfiable (B) *or* $\sim$Satisfiable (B) *according as B is in some Δ^b_{ji}, or B is not in any Δ^b_{ji}, respectively.*

Since by hypothesis our original **QHE**-consistent (A) is in Δ_{01}^{a} or Δ_{11}^{a}, Satisfiable (A). Proof is by induction over the construction of **QHE** formulae. The proof for B, $\sim B$, $(B \supset C)$, and $(\pi X)B$ are as in [2] with suitable modifications. The proof for $(\mathsf{U}X)B$ is as for $(\pi X)B$. So we have K_aB and ${}^LP_aB$:

(a) If the theorem holds for B then it holds for ${}^LP_aB$, i.e., if ${}^LP_aB$ is in some Δ_{ij}^{b} then Satisfiable ${}^LP_aB$, and if ${}^LP_aB$ is in no Δ_{ij}^{b} then $\sim$ Satisfiable ${}^LP_aB$.

Proof: (i) Assume ${}^LP_aB$ is in Δ_{ij}^{b} and ${}^LP_aB$ is in no μ_{mn}^{c}. Let the associate set for Δ_{ij}^{b} be μ_{ij}^{b}. So $\sim {}^LP_aB \in \mu_{ij}^{b}$ by 10b $(C.\Phi\mu)$ and μ_{ij}^{b} maximality. Also ${}^LP_a(p \supset p) \in \mu_{ij}^{b}$ since μ_{ij}^{b} is maximal. So there is an ^{L}alternate$_a$ set to μ_{ij}^{b}. Let it be μ_{0k}^{a}.

$$\sim {}^LP_aB \in \mu_{0k}^{a} \text{ and } \sim B \in \mu_{0k}^{a}.$$

There is also an ^{L}alternate$_a$ set to Δ_{ij}^{b} which will be Δ_{0k}^{a} by 10a. And $B \in \Delta_{0k}^{a}$ by construction of K. But then $B \in \mu_{0k}^{a}$ contrary to $(C.\sim)$. Hence if ${}^LP_aB \in \Delta_{ij}^{b}$ then Satisfiable ${}^LP_aB$.

(ii) Assume ${}^LP_aB$ is in no Δ_{ij}^{b} and ${}^LP_aB$ is in μ_{mn}^{c}. $\sim {}^LP_aB$ will be in all those sets Δ_{ij}^{b} which are maximal consistent relative to the appropriate Λ_j. Let μ_{mn}^{c} be the associate of Δ_{mn}^{c}. So $\sim {}^LP_aB \in \Delta_{mn}^{c}$. Also ${}^LP_a(p \supset p) \in \Delta_{mn}^{c}$ since Δ_{mn} is maximal. So there is an ^{L}alternate$_a$ to Δ_{mn}^{c}. Let it be Δ_{0k}^{a}. $\sim B \in \Delta_{0k}^{a}$. Also there is an ^{L}alternate$_a$ to μ_{mn}^{c} which will be μ_{0k}^{a} and $B \in \mu_{0k}^{a}$, contrary to $(C.\sim)$ and the construction of Ω. Hence if ${}^LP_aB$ is in no Δ_{ij}^{b} then $\sim$ Satisfiable ${}^LP_aB$.

(b) If the theorem holds for B then it holds for K_aB, i.e., if K_aB is in some Δ_{ij}^{b} then Satisfiable K_aB, and if K_aB is in no Δ_{ij}^{b} then $\sim$ Satisfiable K_aB.

Proof: (i) Assume K_aB is in Δ_{ij}^{b} and K_aB is not in any μ_{mn}^{c}. Let the associate set for Δ_{ij}^{b} be μ_{ij}^{b}.

$$\text{Either } \Delta_{ij}^{b} = \Delta_{0k}^{a} \text{ or } \Delta_{ij}^{b} = \Delta_{1p}^{a}.$$

If $K_aB \in \Delta_{0k}^{a}$ then let Δ_{1p}^{a} be the alternate$_a$ set and $B \in \Delta_{1p}^{a}$.

$$\text{Either } \mu_{ij}^{b} = \mu_{0k}^{a} \text{ or } \mu_{ij}^{b} = \mu_{1p}^{a}.$$

Now $\sim K_aB \in \mu_{ij}^{b}$, so $\sim B \in \mu_{1p}^{a}$, contrary to the construction of Ω (10a). If $K_aB \in \Delta_{1p}^{a}$ then Δ_{1q}^{a} is the alternate$_a$ set and $\sim B \in \Delta_{1q}^{a}$ by construction of K. Also $B \in \mu_{1q}^{a}$, contrary to the construction of Ω. So if K_aB is in Δ_{ij}^{b} then Satisfiable K_aB.

(ii) Assume K_aB is in no Δ_{ij}^{b} and $K_aB \in \mu_{mn}^{c}$. $\sim K_aB$ will be in all those sets Δ_{ij}^{b} which are maximal consistent relative to the appropriate Λ_j. Let μ_{mn}^{c} be the associate of Δ_{mn}^{c}. So $\sim K_aB \in \Delta_{mn}^{c}$. So there is an alternate$_a$ set to Δ_{mn}^{c}.

$$\text{Either } \Delta_{mn}^{c} = \Delta_{0k}^{a} \text{ or } \Delta_{mn}^{c} = \Delta_{1p}^{a}.$$

If $\sim K_aB \in \Delta_{0k}^{a}$ then let Δ_{1p}^{a} be the alternate$_a$ set and $\sim B \in \Delta_{1p}^{a}$. Also:

$$\text{Either } \Delta_{mn}^{c} = \Delta_{0k}^{a} \text{ or } \Delta_{mn}^{c} = \Delta_{1p}^{a}.$$

So if $K_aB \epsilon \mu^a_{ok}$ and μ^a_{1p} is the alternate$_a$ set then $B \epsilon \mu^a_{ok}$ contrary to the construction of Ω. Similarly for $\mu^c_{mn} = \mu^a_{1p}$. So the theorem holds for K_aB.

Hence completeness is proved.

REMARKS

1. The Barcan formula and its converse for both quantifiers and both epistemic operators are not theses of **QHE**.

The semantics given do not sustain the commutation of operators and quantifiers in either direction, and so require that the following formulae:

(1) $(\mathsf{U}X)K_aPX$
(2) $(\mathsf{E}X)K_aPX$
(3) $(\mathsf{E}X)K_a(X = b)$,

to be read respectively as

(i) Each and every existing X is such that a knows X is P.

(ii) There is at least one existing X such that a knows that X is P

or as:

One of what a knows to be P, exists.

(iii) What a knows as b, exists

or as:

There is at least one existing X such that a knows that X is b.

Under these readings there is no intuitive credibility to the inferences licensed by either

(4) $K_a(\mathsf{E}X)PX \supset (\mathsf{E}X)K_aPX$, or its converse.

For example, a could know that at least one of the characters in Shakespeares plays existed without knowing who any of the characters were, so it would not be the case that at least one of those whom a knows to be a character in Shakespeares plays actually exists, since there is no one character whom a knows to be in the plays. Conversely, a could know that someone, say Hamlet, was a character in the plays, Hamlet did exist, yet a might well not know that any of the characters in Shakespeare's plays existed.

2. In [1] Follesdal shows how the thesis:

(a) $$(x)(y)(x = y \supset \Box(x = y))$$

can be a problem to modal logic. This is especially so for epistemic logic where the parallel thesis:

(b) $$(x)(y)(x = y \supset K_a(x = y))$$

is problematic. Hintikka has attempted to solve the problem by giving a reading for (b) so that (b) "does not say that all true identities are known to **a**, which would be blatantly false. It only means that all true identities among individuals known to **a** are known by him." ([5], p. 57)

But, unfortunately, there is nothing other than dyadic predicates in the logic in [4] to allow the reading "**a** knows **b**", and hence nothing in (b) on which to base the phrase "individuals known to **a**". Furthermore, it is not clear whether the quantifiers are existence pre-supposition free or not. However, in **QHE**, (b) retains the more intuitively and logically acceptable reading but is not a thesis owing to the failure of the Barcan formula.

3. In **QHE** the operator $\Diamond$ does not occur, and, although there is a theorem:

(a) $$(K_aA \ \& \sim A) \supset \sim (\mathrm{E}X)(X = a),$$

it is not possible to determine whether or not $(\Sigma x)(x = a)$. Clearly, if **T** is a tautology we would want to sustain:

(b) $$K_a \sim \mathbf{T} \supset \sim (\Sigma X)(X = a),$$

but this we cannot do in **QHE** alone. By the addition of the relevant axioms to get **QHEM** and constructing a **QHEM**-Model $\langle \Omega, \Phi, C_{\mathrm{M}} \rangle$ we can sustain:

(c) $$(K_ap \ \& \sim\Diamond p) \supset \sim (\Sigma X)(X = a),$$

which is to much the same effect as (b).

C_{M} would consist of the rules in C above together with:

($C.K\Sigma$) If $K_aA \in \mu_{ij}^b \in \Omega$ and $(\Sigma X)(X = a) \in \mu_{ij}^b$, then $\Diamond A \in \mu_{ij}^b$.
($C.^LK\Sigma$) If ${}^LK_aA \in \mu_{ij}^b \in \Omega$ and $(\Sigma X)(X = a) \in \mu_{ij}^b$, then $\Diamond A \in \mu_{ij}^b$.

($C.\sim\Diamond$) and ($C.\sim\Box$) would be parallel to ($C.\sim P$) and ($C.\sim K$).

($C.\Diamond$*) If $\Diamond A \in \mu_{ij}^b \in \Omega$, then there is in Ω at least one $\Diamond$ alternative to μ_{ij}^b (such as μ_{mn}^a) such that $A \in \mu_{mn}^a$.
($C.\Box\Box$*) If $\Box A \in \mu_{ij}^b \in \Omega$ and μ_{mn}^a is a $\Diamond$ alternative to μ_{ij}^b then $\Box A \in \mu_{mn}^a$
($C.\Box$) If $\Box A \in \mu_{ij}^b \in \Omega$ then $A \in \mu_{ij}^b$.
($C.\Box\Box_+$) If $\Box A \in \mu_{ij}^b \in \Omega$ and μ_{ij}^b is a $\Diamond$ alternative to μ_{mn}^a in Ω, then $\Box A \in \mu_{mn}^a$.

QHEM has what amounts to the logic of two kinds of possibility. The logic of ${}^LP$ gives the logic of what is possible *relative* to what some person knows, the logic of (S5) gives the logic of possibility simpliciter. [8]

REFERENCES

[1] Follesdal, D., "Quine on modality," *Synthese*, vol. 19 (1968), pp. 147-157.

[2] Girle, R. A., "Possibility pre-supposition free logics," *Notre Dame Journal of Formal Logic*, vol. XV (1974), pp. 45-62.

[3] Hintikka, J., "Modality and quantification," *Theoria*, vol. 27 (1961), pp. 119-128.

[4] Hintikka, J., *Knowledge and Belief*: An Introduction to the logic of the two notions, Cornell University Press, Ithaca (1962).

[5] Hintikka, J., "Individuals, possible worlds, and epistemic logic," *Nous*, vol. 1 (1967), pp. 33-61.

[6] Hughes, G. E., and M. J. Cresswell, *An Introduction to Modal Logic*, Methuen, London (1968).

[7] Lemmon, E. J., Review of [4] in *The Philosophical Review*, vol. 74 (1965), pp. 381-384.

[8] Montague, R., "Logical necessity, physical necessity, ethics, and quantifiers," *Inquiry*, vol. 3 (1960), pp. 259-269.

[9] Routley, R., "Some things do not exist," *Notre Dame Journal of Formal Logic*, vol. VII (1966), pp. 251-276.

University of St. Andrews
St. Andrews, Fife, Scotland

and

University of Queensland
Brisbane, Queensland, Australia