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AN AXIOM SYSTEM FOR THREE-VALUED ŁUKASIEWICZ PROPOSITIONAL CALCULUS

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0 *Introduction* In 1920, Łukasiewicz has introduced the notion of three-valued logic. It was not constructed as a formalized axiomatic deductive system but was built up by means of the truth-table method. The matrix defining this logic is the following [3], p. 166:

C	0	$\frac{1}{2}$	1	N
0	1	1	1	1
$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$
1	0	$\frac{1}{2}$	1	0

The three-valued Łukasiewicz logic was later axiomatised by Wajsberg in 1931 (see [3], p. 291). Moisil has given an axiomatisation in order to show that the three-valued Łukasiewicz propositional calculus is an extension of the intuitionistic one. We give here another axiomatisation, different from that of Moisil, showing that the three-valued Łukasiewicz propositional calculus is an extension of a fragment of the three-valued intuitionistic propositional calculus (see [1]; [3], p. 286), answering a problem suggested by A. Monteiro.

Łukasiewicz characteristic matrix can be considered as an algebraic structure. In 1940, Moisil has introduced the notion of three-valued Łukasiewicz algebra as an attempt to give an algebraic approach to the three-valued propositional calculus considered by Łukasiewicz. Following Monteiro [6], we can define a three-valued Łukasiewicz algebra in the following way, where the primitive operations are those chosen by Moisil. Thus an abstract algebra $\langle A, \wedge, \vee, \sim, \nabla, 1\rangle$ is said to be a three-valued Łukasiewicz algebra provided that $\langle A, \wedge, \vee, 1\rangle$ is a distributive lattice where 1, $\sim$, and ∇ are two unary operations on A such that

$$\begin{aligned}
\sim\sim x &= x\\
\sim(x \wedge y) &= \sim x \vee \sim y\\
\sim x \vee \nabla x &= 1\\
x \wedge \sim x &= \sim x \wedge \nabla x\\
\nabla(x \wedge y) &= \nabla x \wedge \nabla y
\end{aligned}$$

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Moisil [4], has shown that three-valued Łukasiewicz algebras are Heyting algebras. Because of the existence of a De Morgan negation they are in reality symmetrical Heyting algebras [7].

In finding our axiom system the definition of a three-valued Łukasiewicz algebra given in [8] has been very useful. Following [8], p. 459, an abstract algebra $\langle A, \wedge, \vee, \Rightarrow, \sim, 1\rangle$ is said to be a three-valued Łukasiewicz algebra if $\langle A, \wedge, \vee, \Rightarrow, 1\rangle$ is a Hilbert-Bernays algebra (or relatively pseudo-complemented lattice), $\sim$ is an unary operation on A and the following equations hold

$$((x \Rightarrow z) \Rightarrow y) \Rightarrow (((y \Rightarrow x) \Rightarrow y) \Rightarrow y) = 1$$
$$\sim\sim x = x$$
$$\sim(x \wedge y) = \sim x \vee \sim y$$
$$(x \wedge \sim x) \wedge (y \vee \sim y) = x \wedge \sim x.$$

In discussing the axiom system we shall use some familiar notions about propositional calculus (see [10]). To save space the following definitions are not as complete as they could be.

Let $\mathcal{L} = \langle A^0, F\rangle$ be a *formalized language* where $A^0 = \{V, \wedge, \vee, \Rightarrow, \sim, (,)\}$ is the *alphabet* and F the set of all *formulas* over A^0. Formation-rules are as usual. Elements p in V are called *propositional variables*; $\wedge, \vee, \Rightarrow, \sim$ propositional *connectives* and the parentheses are auxiliary signs. Let D be the subset of F of *derivables* formulas as it will be defined in section 1 below. A formalized language with a selected subset of derivables formulas make up a *propositional calculus*.

By a *valuation* of $\mathcal{L}$ in a three-valued Łukasiewicz algebra A we shall understand any mapping

$$v\colon\ V \to A,$$

that is, any point $v = \{v_p\}_{p \epsilon V}$ of the Cartesian product A^V. Every formula α in $\mathcal{L}$ uniquely determines a mapping

$$\alpha_A\colon\ A^V \to A$$

defined by induction on the length of α as follows:

$$p_A(v) = v(p)$$
$$(\alpha \wedge \beta)_A(v) = \alpha_A(v) \wedge \beta_A(v)$$
$$(\alpha \vee \beta)_A(v) = \alpha_A(v) \vee \beta_A(v)$$
$$(\alpha \Rightarrow \beta)_A(v) = \alpha_A(v) \Rightarrow \beta_A(v)$$
$$(\sim\alpha)_A(v) = \sim(\alpha_A(v)).$$

If A is a three-valued Łukasiewicz algebra, a formula α of $\mathcal{L}$ will be said *valid in A* provided that $\alpha_A(v) = 1$ for every valuation v of $\mathcal{L}$ in A.

We are going to give an axiom system in such a way that the set of the three-valued Łukasiewicz algebras is characteristic; that is, formula α is derivable in the propositional calculus if and only if α is valid in every three-valued Łukasiewicz algebra.

1 *The axiom system* In the axiom system below $\wedge$ and $\vee$ may be interpreted

as conjunction and disjunction respectively, $\Rightarrow$ as intuitionistic implication and $\sim$ as a negation. To avoid a clumsy statement of the rule of substitution, axiom schemas are considered instead of axioms. The result to be presented here were first announced at the 1964 Meeting of the Unión Matemática Argentina [2].

Axiom schema

(1.1) $x \Rightarrow (y \Rightarrow x)$
(1.2) $(x \Rightarrow (y \Rightarrow z)) \Rightarrow ((x \Rightarrow y) \Rightarrow (x \Rightarrow z))$
(1.3) $(x \wedge y) \Rightarrow x$
(1.4) $(x \wedge y) \Rightarrow y$
(1.5) $(z \Rightarrow x) \Rightarrow ((z \Rightarrow y) \Rightarrow (z \Rightarrow (x \wedge y)))$
(1.6) $x \Rightarrow (x \vee y)$
(1.7) $y \Rightarrow (x \vee y)$
(1.8) $(x \Rightarrow z) \Rightarrow ((y \Rightarrow z) \Rightarrow ((x \vee y) \Rightarrow z))$
(1.9) $((x \Rightarrow z) \Rightarrow y) \Rightarrow (((y \Rightarrow x) \Rightarrow y) \Rightarrow y)$
(1.10) $\sim\sim x \Rightarrow x$
(1.11) $x \Rightarrow \sim\sim x$
(1.12) $(x \wedge \sim x) \Rightarrow (y \vee \sim y)$

Rules of inference

(1.13) $\dfrac{x, x \Rightarrow y}{y}$ Modus Ponens

(1.14) $\dfrac{x \Rightarrow y}{\sim y \Rightarrow \sim x}$ Contraposition rule

Recall that axioms (1.1)-(1.8) and rule (1.13) characterize the positive propositional calculus of Hilbert and Bernays (see [10], p. 236). For references of axiom (1.9) see [11] and [9].

2 *The axiom system is characteristic* Let D be the least set of formulas of $\mathcal{L}$ containing the logical axioms (1.1)-(1.12) and closed under the rules (1.13) and (1.14). The set of formulas F of the formalized language can be considered as an abstract algebra $\mathfrak{F} = \langle F, D, \wedge, \vee, \Rightarrow, \sim\rangle$; V is the set of generators of $\mathfrak{F}$. For $\alpha, \beta \in \mathfrak{F}$ let $\alpha \equiv \beta$ if and only if $\alpha \Rightarrow \beta \in D$ and $\beta \Rightarrow \alpha \in D$. It is well known that, by (1.1)-(1.8), (1.13), and (1.14), $\equiv$ is a congruence on F.

It is possible to show by (1.1)-(1.8) and (1.13) that (see [10], p. 216):

(2.1) $(x \Rightarrow (x \Rightarrow y)) \Rightarrow (x \Rightarrow y)$
(2.2) $(x \Rightarrow y) \Rightarrow ((y \Rightarrow z) \Rightarrow (x \Rightarrow z))$.

The general symmetrical modal logic introduced by Moisil, [5], p. 411, is characterized by axioms and rules of inference (1.1), (2.1), (2.2), (1.3)-(1.8), (1.10), (1.11), (1.13), (1.14), and Moisil has also shown, [5], pp. 412-413, that the more interesting theorems in this logic are those showing that the negation $\sim$ is a duality. That is

(2.3) $(\sim x \vee \sim y) \Rightarrow \sim (x \wedge y)$
(2.4) $\sim (x \vee y) \Rightarrow (\sim x \wedge \sim y)$

(2.5) $(\sim x \wedge \sim y) \Rightarrow \sim(x \vee y)$
(2.6) $\sim(x \wedge y) \Rightarrow (\sim x \vee \sim y)$.

Let $|F|$ be the set of equivalence classes $|\alpha|$ algebraised in a standard way: $|\alpha| \leqslant |\beta|$ if and only if $\alpha \Rightarrow \beta \in D$, $|\alpha| \wedge |\beta| = |\alpha \wedge \beta|$, $|\alpha| \vee |\beta| = |\alpha \vee \beta|$, $|\alpha| \Rightarrow |\beta| = |\alpha \Rightarrow \beta|$, $\sim |\alpha| = |\sim \alpha|$. Further, α is derivable if and only if $|\alpha|$ is the unit element of $|F|$.

Theorem 1 *The Lindenbaum algebra* $\mathfrak{L} = \mathfrak{F}/\equiv = \langle |F|, |D|, \wedge, \vee, \Rightarrow, \sim\rangle$ *is a three-valued Łukasiewicz algebra.*

Proof: This follows immediately from (1.1)-(1.14), (2.3)-(2.6). $|D| = 1$ is the unit element of $\mathfrak{L}$.

Since $\mathfrak{L} = \mathfrak{F}/\equiv$ is a three-valued Łukasiewicz algebra we can interpret formulas of $\mathcal{L}$ as mappings from $\mathfrak{L}^V$ into $\mathfrak{L}$. The valuation v^0: $V \to \mathfrak{L}$ such that

$$v^0(p) = |p| \text{ for every propositional variable } p \text{ of } \mathcal{L}$$

will be called the *canonical valuation* of $\mathcal{L}$ in $\mathfrak{L}$.

Lemma *For every formula* α *of* $\mathcal{L}$

$$\alpha_{\mathfrak{L}}(v^0) = |\alpha|$$

for the canonical valuation v^0.

Proof: In fact, for every propositional variable p

$$p_{\mathfrak{L}}(v^0) = v^0(p) = |p|$$

By induction on the length of α:

$$\begin{aligned}
(\alpha \wedge \beta)_{\mathfrak{L}}(v^0) &= \alpha_{\mathfrak{L}}(v^0) \wedge \beta_{\mathfrak{L}}(v^0) &&= |\alpha| \wedge |\beta| &&= |\alpha \wedge \beta| \\
(\alpha \vee \beta)_{\mathfrak{L}}(v^0) &= \alpha_{\mathfrak{L}}(v^0) \vee \beta_{\mathfrak{L}}(v^0) &&= |\alpha| \vee |\beta| &&= |\alpha \vee \beta| \\
(\alpha \Rightarrow \beta)_{\mathfrak{L}}(v^0) &= \alpha_{\mathfrak{L}}(v^0) \Rightarrow \beta\ (v^0) &&= |\alpha| \Rightarrow |\beta| &&= |\alpha \Rightarrow \beta| \\
(\sim \alpha)_{\mathfrak{L}}(v^0) &= \sim(\alpha_{\mathfrak{L}}(v^0)) = \sim |\alpha| = |\sim \alpha|
\end{aligned}$$

We close with the following result. The method of the proof will be similar to that, which can be found in [10] for other propositional calculus.

Theorem 2 *For every formula* α *of the propositional calculus the following conditions are equivalent:*

(a) α *is derivable in the propositional calculus*
(b) α *is valid in every three-valued Łukasiewicz algebra*

Proof: It is routine to show that a derivable formula in the propositional calculus is valid in every three-valued Łukasiewicz algebra. On the other hand, suppose α is valid in every three-valued Łukasiewicz algebra, so α is valid in $\mathfrak{L}$, that is $\alpha_{\mathfrak{L}}(v) = 1$ for every valuation $v \in \mathfrak{L}^V$. In particular, if v is the canonical valuation $v^0 \in \mathfrak{L}^V$, $\alpha_{\mathfrak{L}}(v^0) = 1$. Because of lemma above, $|\alpha| = 1$ so $\alpha \in D$.

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