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MODAL TREES FOR T AND S5

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1 The simplest decision procedure for the classical sentential calculus is the tree method given by R. Jeffrey in [1]. In this paper we describe an extension of the tree method to give a correspondingly simple decision procedure for the sentential modal logics T and S5. Familiarity with [1] is assumed. We aim just to give enough detail for someone familiar with the method and terminology of [1] to carry out tests for validity and consistency in T and S5. The method described can be adapted to provide decision procedures for B and S4. The rules for these systems are more complex than those for T and S5 and will not be explicitly dealt with here.

2 *Syntax* This is standard except (i) all sentence letters are given superscripts 0, 1, 2, . . .; and (ii) an expression is a wff if and only if it *both* satisfies the usual recursive definitions for modal sentential calculus and all sentence letters have the same superscript. A wff containing a sentence letter with superscript i (and so, only sentence letters with superscript i) is said to be of *degree i*.

3 *Description of the method* We describe the method as applied to a set of wffs (the initial sentences) to test for consistency. The initial sentences are taken to be all of degree 0. The rules of inference for the non-modal logical constants are as in [1] with, of course, the addition of the superscripts. For example, if we use ϕ^i and Ψ^i to range over wffs of degree i, then the rule for ($\supset$) is:

$$(\supset):\quad \phi^i \supset \Psi^i$$
$$\swarrow \qquad \searrow$$
$$\sim\phi^i \qquad \Psi^i$$

and for ($\sim$v) is:

$$(\sim\vee):\quad \sim(\phi^i \vee \Psi^i)$$
$$|$$
$$\sim\phi^i$$
$$\sim\Psi^i$$

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The rules for $(\sim\Diamond)$ and $(\sim\Box)$ are the same for both T and S5:

$$(\sim\Diamond):\quad \begin{array}{c}\sim\Diamond\phi^i\\ |\\ \Box\sim\phi^i\end{array} \qquad\qquad (\sim\Box):\quad \begin{array}{c}\sim\Box\phi^i\\ |\\ \Diamond\sim\phi^i\end{array}$$

The rules for $(\Diamond)$ and $(\Box)$ differ for T and S5. For system T the rules are:

$$(\Diamond\mathrm{T}):\quad \begin{array}{l}\Diamond\phi^i\\ |\\ \phi^j\end{array}$$

where, if ϕ^i occurs as a full line in the path above ϕ^j, $j = i$; and if not $j > i$ and j does not occur previously in the path.

$$(\Box\mathrm{T}):\quad \begin{array}{l}\Box\phi^i\\ |\\ \phi^j\end{array}$$

where $j = i$ or j is the degree of some wff above ϕ^j in the path obtained by an application of $(\Diamond\mathrm{T})$ to a wff of degree i.

The rules for S5 are:

$$(\Diamond\mathrm{S5}):\quad \begin{array}{l}\Diamond\phi^i\\ |\\ \phi^j\end{array}$$

where, if ϕ^k occurs as a full line in the path above ϕ^j for some k, then $j = k$; and if not $j > i$ and j does not occur previously in the path.

$$(\Box\mathrm{S5}):\quad \begin{array}{l}\Box\phi^i\\ |\\ \phi^j\end{array}$$

for any j occurring in the path.

The decision procedure may now be described. Take the initial wffs (all of degree 0) and write them in a list. Apply the rules of inference to these wffs, and then to the resulting wffs, and so on, according to the following specification. If ϕ^i occurs as a line in the tree, apply the relevant rule of inference by writing the list(s) of conclusions of that rule at the bottom of every path of which ϕ^i is a member. Check (✓) ϕ^i, unless the rule applied was $(\Box)$, in which case write the superscript corresponding to the degree of the conclusion next to ϕ^i and check that superscript. A rule of inference may not be applied to an already checked wff; and a wff with a checked superscript may not have $(\Box)$ applied to it to yield a wff with the degree of that superscript. That is, all the rules except $(\Box)$ are applied to a given line just once; and if it is possible to apply $(\Box)$ more than once to a given line (consistent with the provisos in the statement of $(\Box)$), each application must yield a wff of different degree.

Therefore, $(\Box)$ can only be applied to a given line a finite number of times, as the nature of $(\Diamond)$ and the fact the initial sentences are finite in number and length means that there can only be a finite stock of available superscripts. Therefore, there will always be a *finished* tree: a finite tree in which every wff occurring as a full line which is not a sentence letter or the denial of one and which is not of the form $\Box\phi^i$ is checked, and in which every line of the form $\Box \phi^i$ has all superscripts checked which are eligible according to ($\Box$) in T or S5, as the case may be. Moreover, the conclusion(s) of any rule is (are) shorter than the premise; hence, every path in a finished tree contains at least one sentence letter or a denial of a sentence letter occurring as a full line. A path in a finished tree is *closed* iff it contains a sentence letter and its negation (with, of course, the *same* superscript) occurring as full lines, and *open* otherwise. The initial set of wff's is inconsistent iff *every* path of the finished tree is closed. Because there is always a finished tree and every path must be open or closed, we have a *decision procedure*—for T and S5, respectively, depending on whether ($\Diamond$ T) and ($\Box$ T), or ($\Diamond$ S5) and ($\Box$ S5) is used.

The order in which the rules are applied is not crucial, but, generally and obviously, it is simplest to apply non-branching rules before branching ones, non-modal before modal, and ($\Diamond$) before ($\Box$). Also, it is not usually necessary to carry out every possible application of the rules, but it makes the adequacy proof below simpler to suppose this has been done.

4 *An Example* To show $\sim(\Box p \supset \Box\Box p)$ consistent in T, but not in S5.

1. ✓	$\sim(\Box p^0 \supset \Box\Box p^0)$	
2.	✓ ✓ ✓ 0 1 2 $\Box p^0$	from 1 by ($\sim\supset$).
3.	✓ $\sim\Box\Box p^0$	(from 1 by ($\sim\supset$), with 2)
4.	✓ $\Diamond\sim\Box p^0$	from 3 by ($\sim\Box$).
5.	✓ $\sim\Box p^1$	from 4 by ($\Diamond$ T) or ($\Diamond$ S5).
6.	✓ $\Diamond\sim p^1$	from 5 by ($\sim\Box$).
7.	$\sim p^2$	from 6 by ($\Diamond$ T) or ($\Diamond$ S5).
8.	p^0	from 2 by ($\Box$ T) or ($\Box$ S5).
9.	p^1	from 2 by ($\Box$ T) or ($\Box$ S5).
10.	p^2	from 2 by ($\Box$ S5).
	X	from 7 and 10.

The step from 2 to 10 is permitted under ($\Box$ S5), but not under ($\Box$ T); so that in S5, but not T, all paths close.

5 *Adequacy* We outline an adequacy proof of the same general kind as that offered in [1] for non-modal trees. It is given in terms of the possible world semantics for T and S5, as described, for instance, in [2]. When we talk simply of an interpretation (model, valuation), we will mean an interpretation satisfying the requirements for T or S5 (in the terminology of [2], a T-, S5-model), as the case may be.

We define the *base* wff, ϕ, corresponding to a wff, ϕ^i, as the result of deleting all superscripts from ϕ^i: e.g., $\Box(p \supset \Diamond q)$ is the base wff corresponding to $\Box(p^0 \supset \Diamond q^0)$, $\Box(p^1 \supset \Diamond q^1)$, ... (The strategy will be to view a wff, ϕ^i, in terms of ϕ being true in a possible world w_i.) It is sufficient to prove:

(A) There is an interpretation making all base wffs corresponding to the initial sentences true in a (single) world iff there is an open path through a finished tree constructed on these sentences. This can be proved by proving both

(B) The interpretation described by an open path in a finished tree makes all the base wffs corresponding to the initial sentences true in a world.

and

(C) If there is an interpretation making all base wffs corresponding to the initial sentences true in some given world, there is an open path through the tree.

The interpretation, $\langle W, R, V\rangle = \mathbf{I}_0$, *described by an open path* in a tree, is defined as follows:

(i) $W = \{w_i: i$ is the degree of a wff in the path$\}$.
(ii) R = set of ordered pairs satisfying:

for T: (a) for all $w_i \in W$, $\langle w_i, w_i\rangle \in R$.
(b) for all i, j, such that a wff of degree j in the path is obtained from a wff of degree i (as a full line) by ($\Diamond$T), $\langle w_i, w_j\rangle \in R$.

For S5: for all $w_i, w_j \in W$, $\langle w_i, w_j\rangle \in R$.

(iii) Where α^i ranges over sentence letters of degree i and α over the corresponding base wffs (unsuperscripted sentence letters),

(a) $V(\alpha, w_i) = \mathbf{T}$, for all α^i occurring as full lines in the path.
(b) $V(\alpha, w_i) = \mathbf{F}$, for all $\sim\alpha^i$ occurring as full lines in the path.
(c) $V(\alpha, w_i) = \mathbf{F}$ (say), for all α^i such that α^i occurs, but not as a full line, in the path.

We can now prove (B) by induction on the number of lines, n, in an open path of a finished tree *counting from the bottom*. The induction hypothesis is that $\mathbf{I}_0$ is such that for each ϕ^i occurring as one of the last n lines of the open path, the corresponding base wff ϕ is true in w_i. If we describe an interpretation which makes the base wff corresponding to a wff of degree i true in w_i, as making the base wff true in the corresponding world; $\mathbf{I}_0$ makes the base wffs corresponding to the last n lines true in the corresponding worlds.

For $n = 1$, this follows immediately from the definition of $\mathbf{I}_0$ and the fact that any open path in a finished tree must terminate in a sentence letter or its negation. Suppose it is true for $n = k$; and suppose the $(k + 1)$-th line (from the bottom) has ($\Box$S5) applied to it. Then it is of the form $\Box\phi^i$, and ϕ^j must occur below it for every j in the path and so in the last k lines. But the supposition is that the base wffs corresponding to the last k lines are all true in the corresponding worlds on $\mathbf{I}_0$; so ϕ must be true in w_j for all j, on $\mathbf{I}_0$; that is, $\Box\phi$ is true in w_i on $\mathbf{I}_0$. Clearly (if it is consistency in T that is in question), similar considerations apply if ($\Box$T) is the rule applied to the $(k + 1)$-th line; and likewise for all the rules. And if no rule has been applied, the $(k + 1)$-th last line must be a sentence letter or its negation, and the result follows from the definition of $\mathbf{I}_0$.

We prove (C) by showing that every finished tree is such that if after n applications of the rules there is a path, P_n, and an interpretation, $\mathbf{I}$, such that (i) all base wffs corresponding to full lines of P_n are true in some world on $\mathbf{I}$, and (ii) base wffs corresponding to wffs of the same degree are true in the same world on **I**; then there is a path, P_{n+1}, and an interpretation, **I**$'$, such that this is true after $(n + 1)$ applications.

In the case where the $(n + 1)$-th application is of a non-modal rule or of $(\sim\Diamond)$, $(\sim\Box)$, this is obvious. By inspection of these rules, it can be seen that *any* interpretation which makes the base wff corresponding to a premise wff true in a world, makes the base wff(s) corresponding to at least one list of conclusions true in that world. So that we can always set **I** = **I**$'$, and take P_{n+1} to be P_n plus the appropriate list of conclusions.

If the $(n + 1)$-th application is of $(\Box\mathrm{T})$, take P_{n+1} to be obtained by applying it to a (full-line) wff in P_n. If P_n contains no appropriate wff, the result is trivial; if it does, let it be $\Box\phi^i$, and the conclusion wff be ϕ^j. If $j = i$, the result is immediate from the fact that every interpretation which has $\Box\phi$ true in a world must have ϕ true in that world. If $j \neq i$, P_n must have a wff, Ψ^j, obtained from $\Diamond\Psi^i$ by $(\Diamond)$. Hence **I** must assign **T** to $\Diamond\Psi$ and $\Box\phi$ in the same world; and, further, must assign **T** to ϕ in every world accessible to that world, and **T** to Ψ in some world accessible to that world. Hence **I** assigns **T** to ϕ and Ψ in the same world. But ϕ, Ψ are base wffs corresponding to wffs of the *same* degree (j), therefore, we can in this case also set **I** = **I**$'$ to get our result. Similar remarks apply should $(\Box\mathrm{S5})$ be the rule in question.

Finally, if the $(n + 1)$-th application is of $(\Diamond\mathrm{T})$, take P_{n+1} to be got by applying it to a (full-line) wff in P_n. If P_n contains no appropriate wff, the result is trivial; if it does, let it be $\Diamond\phi^i$, and the conclusion be ϕ^j. Take the case where $j = i$: by $(\Diamond\mathrm{T})$, $\phi^j(=\phi^i)$ must be a full-line wff of P_n as well as P_{n+1} and the result is immediate with **I**$'$ = **I**. Take the case where $j \neq i$ and j is new to the path. If **I** assigns **T** to $\Diamond\phi$ in some world, it must assign **T** to ϕ in some world accessible to this world; but j is *new* to the path; so that if **I** assigns **T** to all base wffs in P_n so that base wffs corresponding to wffs of the same degree are assigned **T** in the same world, **I** does likewise for P_{n+1}. So in this final case, we can again set **I** = **I**$'$. Again similar remarks apply if the rule applied is $(\Diamond\mathrm{S5})$.

Hence, by induction, we have shown that if after $n = 0$ applications of the rules, there is a P_0 and an **I** such that (i) all base wffs corresponding to full lines of P_0 are true in some world on **I**, and (ii) base wffs corresponding to wffs of the same degree are true in the same world; this is true when n is the total of possible applications of the rules. Let Q be such a path resulting from all possible applications, then Q cannot contain a sentence letter of a given degree, α^i, and its negation, $\sim\alpha^i$, for no interpretation can assign **T** to the corresponding base wffs, α and $\sim\alpha$, in the same world; so that Q is open. But P_0 is just the initial sentence, so that (C) is proved.

REFERENCES

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