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ON ACKERMANN'S THEORY OF SETS

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Two different proposals to clarify Cantor's intuitive definition of set along axiomatic lines have been made. One is the well-known theory of Zermelo and Von Neumann that a collection is a set only when it is not too large, i.e., the theory of limitation of sizes in Russell's terminology. The other is a recent system of Ackermann, *cf.* [2].

In this system; $x, y, z, \ldots$ are object variables (i.e., class variables). The primary formulae are '$x = y$', '$x \in y$' (x is a member of y) and $\mathbf{M}x$ (x is a set) in which in place of x and y one can use other class variables. Further formulae or expressions can be constructed in the usual fashion with the use of logical connectives: $\neg$(not), $\wedge$(and), $\vee$(or), $\supset$(implies), $\equiv$(equivalent), $(\forall x)$, $(\forall y)$, . . . (for all x, for all y, . . .) and $(\exists x)$, $(\exists y)$, . . . (there is an x, there is an y, . . .). In place of $x = y$, we will use $x \neq y$, $x \subset y$ will be used as an abbreviation for $(\forall z)\,[z \in x \supset z \in y]$, and $\sim x \in y$ will be written as $x \notin y$. We apply predicate calculus of the first degree, inclusive of calculus of equality, to these expressions (which do not contain any predicate variable). $A(x)$ is any expression which contains the free variable x; $A_0(y)$ any expression which contains the free variable y and in which the sign $\mathbf{M}$ does not occur.

The axiom system, based on Cantor's idea, contains four axiom schemata:

(α) $(\forall x)[A(x) \supset \mathbf{M}x] \supset (\exists y)(\forall z)[z \in y \equiv A(z)]$.

(β) $[x \subset y \wedge y \subset x] \supset x = y$.

(γ) $[\mathbf{M}x_1 \wedge \mathbf{M}x_2 \wedge \ldots \wedge \mathbf{M}x_n] \supset [(\forall y)[A_0(y) \supset \mathbf{M}y] \supset (\exists z)$
$[\mathbf{M}z \wedge (\forall u)[u \in z \equiv A_0(u)]]]$.

(δ) $[\mathbf{M}x \wedge [y \in x \vee y \subset x]] \supset \mathbf{M}y$.

(γ) is thereby so comprehended that $x_1, x_2, \ldots, x_n$ are the free variables occurring in addition to y in $A_0(y)$. If these do not occur, $[\mathbf{M}x_1 \wedge \mathbf{M}x_2 \wedge \ldots \wedge \mathbf{M}x_n]$ simply drops out. It embodies the restriction that 'not every class of sets is a set'. Here (α) is for class construction (only classes of sets are sets in some cases), (β) is usual axiom of extensionality

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and (γ) and (δ) are sethood axioms. In order to show that all sets can be built up which are required in Mathematics, Ackermann has proved a series of theorems.

We would like to discuss Ackermann's proof for replacement axiom. The existence of sets built through the replacement axiom of the set theory is proved as follows:

Let $A_0(x, y)$ be an expression which does not contain **M** but only the free variables x and y. Further let $[A_0(x, y) \wedge \mathbf{M}x] \supset \mathbf{M}y$ be provable. Then the following is valid:

(R): $\mathbf{M}z \supset (\exists u)[\mathbf{M}u \wedge (\forall x)[x \epsilon u \equiv (\exists y)[A_0(y, x) \wedge y \epsilon z]]]$

(Axiom of replacement)

The usual assumption made in the case of formulation of replacement axiom, that $A_0(x, y)$ represents a function, i.e., that $[A_0(x, y) \wedge A_0(x, z)] \supset y = z$, is in this case not required. Because here we get a class of all sets in place of set of all sets and hence no contradiction. In usual assumption, if we do not restrict $A_0(x, y)$ to be a function, we have a set of all sets. As regards replacement axiom in (R) which apparently seems to be of lesser strength than that of Fraenkel on the basis of imposed condition on $A_0(y)$ that it should not contain **M**, Ackermann has pointed out that it is not so and for that he has given a proof which follows:

In Fraenkel's system the axiom reads:

"If m is a set and ϕ_x a function, then there exists also the set which results from m in case every element x of m is replaced by the set ϕ_x."

Fraenkel's treatment defines the concept of function regressively. As a function of x is valid:

(a) every solid set.
(b) the set itself.
(c) the negation set of x.
(d) the potential set of x.
(e) in case of ϕ_x and ψ_x being functions of x, the pair $\{\phi_x, \psi_x\}$.
(f) $\phi\psi x$, if ϕ_x and ψ_x are functions of x.

In case one further lets one of the constant sets which are used for the building of a function ϕ to be indefinite or variable and denotes it by y, then one can regard ϕ_x as a function ψxy of two arguments. Further we have:

(g) Let αy, ϕxy, and ψxy are functions, then the set is called of that x for which $x \epsilon \alpha y$ and $\phi xy \epsilon \psi xy$, i.e., a function of y.
(i) If ϕxy and ψxy are functions and if there exists for every set y, the set ϕy of that x for which $\phi xy \epsilon \psi xy$; then ϕy is a function of y.

Ackermann defines a concept of function in the following way:

Let the expression $A_0(x_1, x_2, \ldots, x_n; y)(n \geqslant 1)$ where $x_1, \ldots, x_n$, y are as a whole free variables of A_0. It defines y as a set theory function of $x_1, \ldots, x_n$ where the following are provable:

(1) $[\mathbf{M}x_1 \wedge, \ldots, \wedge \mathbf{M}x_n] \supset (\exists y)[A_0(x_1, \ldots, x_n; y) \wedge \mathbf{M}y]$
(2) $[A_0(x_1, \ldots, x_n; y) \wedge A_0(x_1, \ldots, x_n; z)]$ and $[\mathbf{M}x_1 \wedge \ldots \wedge \mathbf{M}x_n] \supset y = z$

And hence $F_n(A_0)$ is provable. This shows that Fraenkel's replacement axiom is contained in (R) if the above concept of function satisfies (a) to (i). It is because there is a corresponding formula provable in every case. In this connection, Ackermann has proved a formula corresponding to (i) which follows:

(i)* *Let* $A_0(x_1, x_2; y)$ *and* $B_0(x_1, x_2; y)$ *be functions. Further let*

$$\mathbf{M}x \supset (\exists y)[\mathbf{M}y \wedge (\forall u)[u \in y \equiv (\exists v)(\exists w)[A_0(u, x; v) \wedge B_0(u, x; w) \wedge v \in w]]]$$

be provable.

If $(\forall u)[u \in y \equiv (\exists V)(\exists w)[A_0(u, x; v) \wedge B_0(u, x; w) \wedge v \in w]]$ *be denoted by* $E_0(x; y)$*; then* $F_n(E_0)$ *is obviously valid,* i.e., *y is a function of x.*

Azriel Lévy in [4] has pointed out that there is a mistake in the above proof. Instead of functional constant $\mathbf{M}$, he takes the class constant $\mathbf{V}$ denoting the class of all sets, which exists by axiom (α) of Ackermann. In every context, $\mathbf{M}x$ will be replaced by $x \in \mathbf{V}$. He shows that the above proof breaks at the point where Ackermann shows that his notion of function satisfies the requirement (i) and for that he proves a corresponding result in (i)*. But in the view of Lévy, (i)* is not the assumption of (i). The assumption of (i) is:

(i)** $\mathbf{M}x \supset (\exists y)[\mathbf{M}y \wedge (\forall u)[\mathbf{M}u \supset. u \in y \equiv (\exists v)(\exists w)[A_0(u, x; v) \wedge B_0(u, x; w) \wedge v \in w]]]$

which can be written as

$$\mathbf{M}x \supset (\exists y)[\mathbf{M}y \wedge (\forall u)[\mathbf{M}u \wedge u \in y \equiv. \mathbf{M}u \wedge (\exists v)(\exists w)[A_0(u, x; v) \wedge B_0(u, x; w) \wedge v \in w]]].$$

We can drop $\mathbf{M}u$ from left side of $\equiv$ sign, since by axiom (δ): $[\mathbf{M}y \wedge u \in y] \supset \mathbf{M}u$, but there is no reason to drop $\mathbf{M}u$ from the right hand side. Thus $E(x; y)$ given by

$$(\forall u)[u \in y \equiv [\mathbf{M}u \wedge (\exists v)(\exists w)[A_0(u, x; v) \wedge B_0(u, x; w) \wedge v \in w]]]$$

represents a function in the sense of Fraenkel but not in the sense of Ackermann since it contains the functional constant $\mathbf{M}$. Further he has shown that this system is consistent (contradiction free) if Zermelo's system is consistent with the axiom of replacement.

Ackermann, in his review [3] on Lévy's aforesaid paper, has accepted that the Fraenkel's formulation of Ersetzungs axiom with the help of his concept of function cannot be completely proved by the given method. Because the resulting set is not definable without the symbol $\mathbf{M}$ and consequently they do not belong to the sets of the principal system (sets of first kind) rather to the sets of the second kind.

However, for pure set theory, the system of Ackermann seems to be most workable. The above discussion was only for the comparison of strength of the system with that of Fraenkel. Furthermore, if such sets are to be dealt with whose elements are not again sets and not closely described, and which correspond to Cantor's object of intuition'. (I.e., non-sets) Ackermann has changed his formalism somewhat. He uses the primary formulae '**I**x' (x is a basic object, i.e., a set or a non-set). The axioms then take the following form, whereby **O**x is an abbreviation for **M**$x \vee$ **I**x:

(α') $(\forall x)[A(x) \supset \mathbf{O}x] \supset (\exists y)(\forall z)[z \in y \equiv A(z)]$.
(β') $[x \subset y \wedge y \subset x] \supset x = y$.
(γ') $[\mathbf{O}x_i \wedge \ldots \mathbf{O}x_n] \supset (\forall y)[A_0(y) \supset \mathbf{O}y] \supset (\exists z)[\mathbf{M}z \wedge (\forall u)[u \in z \equiv A_0(u)]]]$.
(δ') $[\mathbf{M}x \wedge [y \in x \vee y \subset x]] \supset \mathbf{O}y$.

Ackermann has also remarked that in the case of geometric axioms (i.e., Hilbert's axiomatic for geometry) where non-sets are points ($\mathbf{P}x_i$), straight lines ($\mathbf{G}x_i$) and planes ($\mathbf{E}x_i$); one is to add $[\mathbf{P}x \vee \mathbf{G}x \vee \mathbf{E}x] \supset \mathbf{I}x$ as a further basic formula. Axiom (α) will be changed as $\mathbf{M}x \vee \mathbf{P}x \vee \mathbf{G}x \vee \mathbf{E}x$. Axioms ($\beta$) and ($\delta$) retain their forms. In (γ) every $\mathbf{M}x_i$ will be replaced by $\mathbf{M}x_i \vee \mathbf{P}x_i \vee \mathbf{G}x_i \vee \mathbf{E}x_i$ and $\mathbf{M}y$ by $\mathbf{M}y \vee \mathbf{P}y \vee \mathbf{G}y \vee \mathbf{E}y$, while $\mathbf{M}z$ remains unchanged, because $x \in \mathbf{O}$ is meaningless expression if **O** be a basic object in the present concept.

Now we propose to modify the above axiom system with the help of Quine's [5] idea of an 'Individual'. Quine has remarked that, without any loss of generality, one can take an individual, its unit class, the unit class of that unit class and so on as one and the same thing, i.e., (x is an individual) '**I**x' iff $x = \{x\} = \{\{x\}\} = \ldots$. And hence they are best counted as sets, i.e., every thing comes to be counted as a set, still, individuals remain especially marked off from other sets in being their own sole members. Now, we see that the original axioms are restored.

This modified system of Ackermann is much more general in the sense that we get two different interpretations of Ackermann (one for the sets and the other for individuals) in one form. Thus the two separate spheres of thought in the set theory are redundant and hence Ackermann's separate axiom system for basic objects is redundant. The obnoxious problem regarding the entry of non-sets in the set theory does not stand. Regarding its uniformity, one should note that it was difficult for Ackermann to replace $\mathbf{M}z$ by $\mathbf{O}z$ in (γ') until Quine's interpretation of basic object be taken into account and this breaks the uniformity and hence the spirit. But the condition imposed on $A_0(y)$ that it should not contain **M** is quite natural.

Lastly we pose a problem. Although we are not able to include a model in this paper, we have in our mind a proof to show that if one or the other of Ackermann's system is consistent, it will remain so when an axiom is added providing for the existence of individuals in the sense of Quine.

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