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A NOTE ON SUZUKI'S CHAIN OF HYPERDEGREES

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In his very important work [5] Suzuki found some interesting results about Π_1^1 implicitly definable sets. Precisely he proved that (in the notations of Rogers [4] which we freely use)

1 *If* $\{A\} \in \Pi_1^1$ *and* $\{B\} \in \Pi_1^1$, *then*

1a $A \leq_h B$ *or* $B <_h A$

and

1b $A <_h B$ *iff* $\mathsf{T}^A \leq_h B$ *iff* $\lambda^A < \lambda^B$

2 *If* $\{A\} \in \Pi_1^1$, *then* $\{\mathsf{T}^A\} \in \Pi_1^1$.

Otherwise stated, the hyperdegrees of Π_1^1 implicitly definable sets are well-ordered in a chain $\{a_\alpha\}_{\alpha<\alpha_0}$ such that

3 a_0 *is the hyperdegree of* Δ_1^1 *sets*

and

4 $a_{\alpha+1} = a'_\alpha =$ *the hyperjump of* a_α.

Suzuki left open the characterization of α_0, that we now obtain* using some results of Moschovakis (see [4], p. 416):

Proposition α_0 *is* $\omega(\Delta_2^1)$, *that is the least ordinal which is not a* Δ_2^1-*ordinal.*

Proof: We split it in two parts:

(a) $\alpha_0 \leq \omega(\Delta_2^1)$. Given $\{A\} \in \Pi_1^1$ let w_A be a tree for A ([4], p. 432), that is $w_A \in \mathsf{T}^X$ iff $X = A$. There exist a unique X (viz. A) s.t. $w_A \in \mathsf{T}^X$, so that $\|w_A\|^2 = (\min_{w_A \in T^X} \|w_A\|^X) = \|w_A\|^A$. Then Lemma 1 of [4], p. 432, says that if $\{A\} \in \Pi_1^1$ and $\{B\} \in \Pi_1^1$ we have $\|w_A\|^2 \leq \|w_B\|^2 \Rightarrow A \leq_h B$, and by Suzuki's

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result **1a** quoted at the beginning, $B <_h A \Rightarrow \|w_B\|^2 < \|w_A\|^2$. But the ordinals $\|x\|^2$ for $x \in T^2$ have order type $\omega(\Delta_2^1)$ (see [4], p. 417), so $\alpha_0 \leqslant \omega(\Delta_2^1)$.

(b) $\omega(\Delta_2^1) \leqslant \alpha_0$. Suzuki [5] gives a method to obtain, for every Π_1^1 implicitly definable well-ordering with ordinal β, a subchain of $\{a_\alpha\}_{\alpha < \alpha_0}$ with length β. Because the ordinals $\|x\|^2$ with $x \in T^2$ are cofinal with the Δ_2^1-ordinals ([4], p. 417), and considering trees instead of well-orderings—as usual—, it is sufficient to prove that if $x \in T^2$ then there exist an A s.t. $\{A\} \in \Pi_1^1 \wedge x \in T^A \wedge \|x\|^2 = \|x\|^A$. But by definition $x \in T^2$ iff $(\exists A)(x \in T^A \wedge \|x\|^2 = \|x\|^A)$, and from $\|x\|^2 = \|x\|^A$ iff $(\forall B) \sim (x \in T^B \wedge \|x\|^B < \|x\|^A)$ we have by [4], 16.XXXV and 16.XX, that $x \in T^A \wedge \|x\|^2 = \|x\|^A$ is a $\Pi_1^{1,A}$ expression. So if there exist an A which satisfies it, also there exist such an A with $\{A\} \in \Pi_1^1$ by the Kondo-Addison theorem ([4], 16.XLV).

So we have two important chains:

(α) one chain of Turing degrees, of length $\omega(\Delta_1^1)$, such that every Δ_1^1 set is T-reducible to some element of the chain, and converse (see [4], section 16.8);

(β) one chain of hyperdegrees, of length $\omega(\Delta_2^1)$, such that every Δ_2^1 set is h-reducible to some element of the chain, and converse (see [4], section 16.7).

Of course, the major difference between the two cases is that the first chain is defined from below, and in fact admits degree-theoretic definitions from below (see for example [1]), whether it is unknown if the same holds for the second chain. Partial results on this important problem have been obtained by Richter [3] and Kechris [2].

REFERENCES

[1] Jockusch, C. G., and S. G. Simpson, "A degree-theoretic definition of the ramified analytical hierarchy," *Annals of Mathematical Logic*, vol. 10 (1975), pp. 1-32.

[2] Kechris, A. S., "The theory of countable analytical sets," *Transactions of the American Mathematical Society*, vol. 202 (1975), pp. 259-297.

[3] Richter, W., "Constructive transfinite number classes," *Bulletin of the American Mathematical Society*, vol. 73 (1967), pp. 261-265.

[4] Rogers, H., Jr., *Theory of Recursive Functions and Effective Computability*, McGraw-Hill (1967).

[5] Suzuki, Y., "A complete characterization of the Δ_2^1 functions," *Bulletin of the American Mathematical Society*, vol. 70 (1964), pp. 246-253.

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