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IS THE INTUITIONISTIC BAR-INDUCTION A CONSTRUCTIVE PRINCIPLE?

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Introduction Many metamathematical investigations make use of the Bar-Induction. The following remarks* clarify that even a simple version of this proof-principle is not constructive in two important senses: It is not constructive (i) in the sense of Lorenzen's game-theoretical approach and (ii) in the sense of Kleene's constructive ordinals.

1 *A game-theoretical interpretation of intuitionistic logics*

1.1 *Rules of the game* We introduce a 2-persons game in which number-theoretical assertions are attacked and defended to win the game. An attack to an assertion A is marked by ?. If A is a connected assertion $A_1 * A_2$, an attack to A_i is marked by A_i ?.

1.11 *Attacking and defending rules of connectives and quantifiers*

Assertion	Attack	Defense
$A \wedge B$	A ? B ?	A B
$A \vee B$	? ?	A B
$A \rightarrow B$	A ?	B
$A \leftarrow B$	B ?	A

Definition $\neg A \Leftrightarrow A \rightarrow \mathbf{f}$; $\mathbf{f}$ is a non-defensible sentence.

It is a definite rule that the person who asserts $\wedge x A(x)$ must defend

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$A(n)$ for each number n demanded by the opponent. To defend $\bigvee xA(x)$, one may choose some number m to defend $A(m)$:

$\bigwedge xA(x)$	n ?	$A(n)$
$\bigvee xA(x)$	?	$A(m)$

1.12 *Rule of constructive dialogues* Proponent is the person who begins by asserting a thesis. Opponent is the person who attacks the thesis of the proponent.

Opponent: He may *attack* the sentence asserted by the proponent in the preceding move. He may *defend* himself against the attack of the proponent in the preceding move.

Proponent: He may *attack* a sentence asserted by the opponent. He may *defend* himself against the attack of the opponent in the preceding move.

1.13 *Winning rule of the proponent*

1st case: The proponent defends a prime-sentence.
2nd case: The opponent cannot defend a prime-sentence.
3rd case: The proponent asserts a prime-sentence which was already asserted by the opponent.

1.2 *Winning strategies*

a step of a strategy	is an application of one of the rules of **1.11**
a strategy	is a series of steps regulated by the rule **1.12**
a winning strategy	is a strategy of the proponent with the last step (closed) by rule **1.13**.

1.21 *The system of the winning strategies* A strategy is written down in a tableau . . . $\|$ . . . On the right side, the proponent notes his assertions, attacks, and defenses. On the left side, the opponent does the same. There is a complete system of all steps to construct the winning strategies (see K. Lorenz [6]). A logical constructive true $\leftrightharpoons$ every strategy of the proponent to defend A is a winning strategy.

$$A_1, \ldots, A_n \text{ implies } A \quad \leftrightharpoons \begin{array}{c} A_1 \\ \vdots \\ A_n \end{array} \Bigg\| \; A \qquad \text{defensible}$$

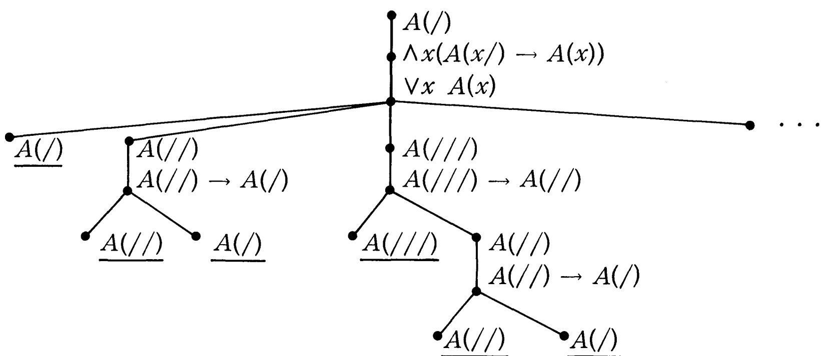
The system of winning strategies is an interpretation of Gentzen's intuitionistic calculus G3. G3 is equivalent to Heyting's calculus.

1.22 *Proof of consistency* (Proof by induction on all steps of winning strategies—see Lorenzen [8].

2 *The arithmetic principle of induction* Natural numbers can be represented by stroke numbers /, //, ///, . . ., which can be constructed by the rules $\Rightarrow /$, $n \Rightarrow n/$ with n as an 'eigenvariable'. We can now construct the winning strategies to defend a version of the arithmetical induction: (The tableaus with underlined assertions are closed by repeating a thesis which was already asserted by the opponent. The number of the attacked line is noted in brackets.)

1)	$\bigvee x A(x)$	
2)	$\bigwedge x(A(x/) \rightarrow A(x))$	$A(/)$
3)	$?(2)$	$?(1)$
4)	$A(///)$	$//?(2)$
5)	$A(///) \rightarrow A(//)$	$\underline{A(///)}$
	$A(//)$	$/ \quad ?(2)$
	$A(//) \rightarrow A(/)$	$\underline{A(//)}$
	$\underline{A(/)}$	

We can illustrate the construction of winning strategies in a tree-diagram to mark the splitting tableaus:



The length of every winning strategy (i.e., the number of steps) depends on the example of the opponent to defend $\bigvee x A(x)$. So, the principle of induction is arithmetical constructive-true by the construction of /, //, ///, . . .

3 *An analytical principle of induction* (*Bar-Induction*)

3.1 *Universal spread* (Brouwer [3], Kleene [4]) A function $f(0), f(1), \ldots$ is intuitionistically only given by a finite sequence of effectively determined choices $f(0), f(1), \ldots, f(y-1)$. Codification of finite sequences: Marked by $\bar{f}(y)$ (see Kleene [4]). Prolongation of finite sequences: Marked by $\bar{f}(y) * [s]$ (Kleene [4]). The constructing rule of the universal spread provides all number-theoretical functions as sequences of choices by $\Rightarrow \bar{f}(0)$ and $\bar{f}(y) \Rightarrow \bar{f}(y) * [s]$. $\bar{f}(0)$ is often abbreviated by [ ] ('blank').

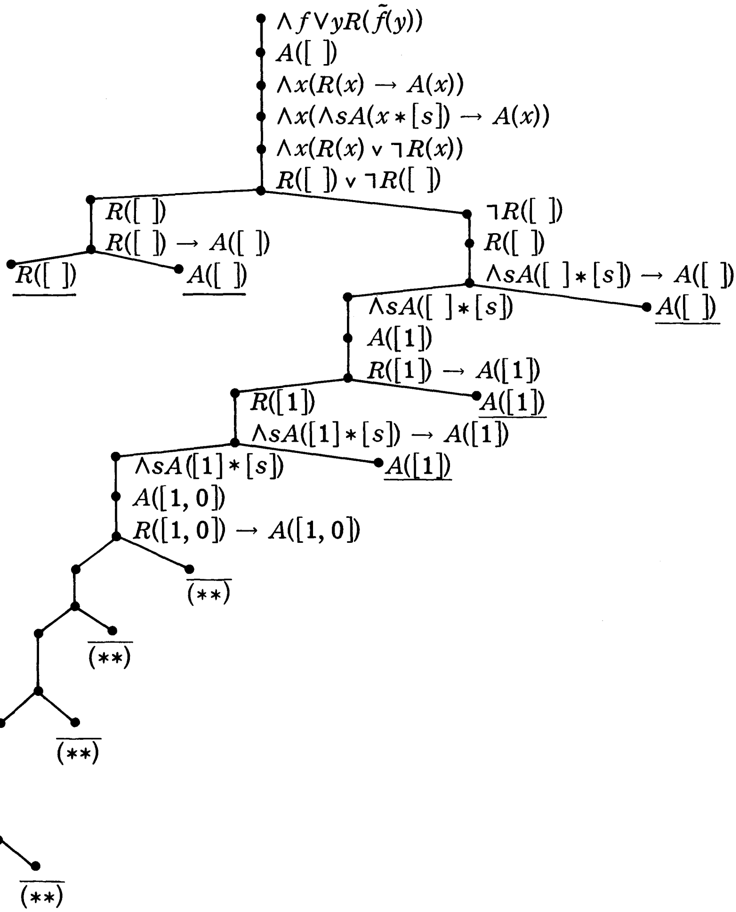
3.2 *Bar-Induction*

Lemma *R decidable equivalent to* $\bigwedge x(R(x) \vee \neg R(x))$ *defensible.*

To prove a property A by induction on the universal spread, we must intuitionistically restrict us to the finite sequences which can be codified by numbers of sequences (see the property 'Seq' in Kleene) and can be tested by a process R of decision.

$\wedge x(\mathrm{Seq}(x) \rightarrow R(x) \vee \neg R(x))$	
$\wedge x(\mathrm{Seq}(x) \wedge R(x) \rightarrow A(x))$	
$\wedge f \vee y R(\tilde{f}(y))$	
$\wedge x(\mathrm{Seq}(x) \wedge \wedge s A(x * [s]) \rightarrow A(x))$	$A([\ ])$ defensible,

because *every winning strategy* on a sequence f of choices (on a branch f of the universal spread) *is closed by the hypothesis* (the BAR) $\wedge f \vee y R(\tilde{f}(y))$. We can illustrate the construction of winning strategies in a tree-diagram in the case of $s = 0$ or 1, but, BAR-Induction is not defensible in the definite sense of Lorenzen's strategies, because a *free* choice sequence f cannot be represented by a *definite term*. (See Mainzer [10].)



$(*)$ *is closed* by the hypothesis (BAR) $\wedge f \vee y R(\tilde{f}(y))$; we *do not know* for which vertex of a chosen branch f in the universal spread the property R is right; but we *do know* (by the BAR) that there is such a vertex for each branch.

$(**)$ *is closed by repeating a thesis* of the property A which was already asserted by the opponent.

3.3 *A classical classification of the winning strategies* From the classical standpoint, a winning strategy on a branch f of the universal spread can be classified by the *complexity of ordinals*. We shall see that the class **HA** of all hyperarithmetical functions which can be classified by constructive ordinals does not suffice to exhaust the full force of the BAR-hypothesis. If we remember Kleene's index-system O of constructive ordinals and Kleene's H_x-predicates with $x \in O$, we can define

$$\mathbf{HA}: = \{f \mid \vee y \in O: \ f \text{ recursive in } H_y\}.$$

Lemma (Kleene [5]) *For all Q recursive, there is a P recursive with:*

$$\wedge a(\vee f \in \mathbf{HA}: \wedge x: Q(a,f,x) \leftrightarrow \wedge f: \vee x: P(a,f,x))$$

Remark *There is a predicate R decidable with:*

$$\wedge f \in \mathbf{HA}: \vee x R(\tilde{f}(x)) \leftrightarrow \wedge f \vee x R(\tilde{f}(x)) \text{ is false}$$

Proof: By the lemma, we may choose a relation $Q(a,f,x): \leftrightarrow \neg T(\tilde{f}(x), a, a)$ with Kleene's *T-predicate*. So, by the enumeration-theorem, we get a Gödel-number e with

$$\wedge a(\vee f \in \mathbf{HA}: \wedge x \ \neg T(\tilde{f}(x), a, a) \leftrightarrow \wedge f: \vee x: T(\tilde{f}(x), e, a))$$

Finally, we get by classical logic and a special $a: = e$:

$$\neg \wedge f \in \mathbf{HA}: \vee x: T(\tilde{f}(x), e, e) \leftrightarrow \wedge f: \vee x: T(\tilde{f}(x), e, e).$$

So, by this remark, the winning strategies of the BAR-Induction may be—classically spoken—more complex than all hyperarithmetical processes.

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