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DEGREES OF UNSOLVABILITY AND STRONG FORMS OF $\mathbf{\Lambda}_{\mathrm{R}} + \mathbf{\Lambda}_{\mathrm{R}} \nsubseteq \mathbf{\Lambda}_{\mathrm{R}}$

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1 *Introduction* In [2], J. C. E. Dekker showed that the class Λ_{R} of *regressive isols* is not closed under addition (equivalently, in view of [2], Proposition P 18, Λ_{R} is not closed under multiplication) within the ring Λ^* ([4]) of isolic integers. In [1], Barback gave a different proof of additive non-closure which, in contrast to Dekker's proof, does not appeal to the notion of *degree of unsolvability* (of a regressive isol; see [2]). Since Barback's proof makes no use of degrees, one is naturally led to wonder whether the failure of additive closure for Λ_{R} is *totally independent of degree*, in the sense that if $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ are any given nonzero degrees of unsolvability then there exist retraceable sets α and β such that $\alpha \in \boldsymbol{d}_1$, $\beta \in \boldsymbol{d}_2$, and the respective isols A and B determined by α and β fail to have a regressive sum. In view of [5], Theorem T2 and [2], Proposition P 17, this is always the case when $\boldsymbol{d}_1 \neq \boldsymbol{d}_2$; so we need only concern ourselves with the realization of additive non-closure *within* a given degree. Barback's proof of additive non-closure can easily be embellished with enough auxiliary coding to produce one particular well-behaved class of degrees $\boldsymbol{d}$ for which the additive non-closure of Λ_{R} can be realized within $\boldsymbol{d}$; the class we here have in mind is $\{\boldsymbol{d} | \boldsymbol{d} \geqq \boldsymbol{\phi}''\}$. As an immediate corollary to the main result of section **3**, we shall conclude that each element of the larger class $\{\boldsymbol{d} | \boldsymbol{d} \geqq \boldsymbol{\phi}'\}$ bears internal witness to the additive non-closure of Λ_{R} ($\{\boldsymbol{d} | \boldsymbol{d} \geqq \boldsymbol{\phi}'\}$ is well known to have the interesting property of being co-extensive with the range of the jump operator ([8])); thus, we have total independence of degree for additive non-closure of Λ_{R} *within the class* $\{\boldsymbol{d} | \boldsymbol{d} \geqq \boldsymbol{\phi}'\}$. (This much, in fact, is very easy to prove without any appeal to the limiting procedures employed in our proof of Theorem 3.1. Theorem 3.1, however, goes a bit further with respect to the form of the summands and the extent of their illrelatedness.) In section **4** we shall prove that the one-sidedness of condition (iii) in the statement of Theorem 3.1 is inescapable, at least for $n_0 = 0$. In section **5**, we restrict our attention to the subsemilattice of *recursively enumerable* degrees; there we shall establish, by means of a suitable elementary priority construction, that, for the entire class $\mathcal{J}_r - \{\boldsymbol{\phi}\}$

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of nonzero recursively enumerable degrees, the additive non-closure of Λ_R is indeed a totally degree-independent phenomenon.

2 *Preliminaries* We shall, as far as possible, follow the notational and terminological conventions of [1] and [3]. However, we shall employ a number of concepts and notations not mentioned in [1] or [3]; these additional concepts and notations (with the exception of "movable marker" notations, which are exploited in sections **3** and **5**) will be either explicitly explained or appropriately referenced in the present section. To begin with, we find it convenient to supplement the notation of [1] with an alternative convention regarding isols: Barback in [1] uses upper-case Roman letters to denote isols; we shall do the same, but shall occasionally also employ $[\alpha]$ as a notation for the isol determined by the set α. Next, we recall that by Kleene's Normal Form Theorem there exist a fixed recursive predicate $\mathsf{T}(n, x, y)$ and a fixed recursive function U such that $\{\mathsf{U}((\mu y)\mathsf{T}(n, x, y)) \mid n = 0, 1, 2, \ldots\}$ = the class of all partial recursive functions of one variable; here, as usual, μ denotes the least number operator. We shall denote the n-th partial recursive function, $\mathsf{U}((\mu y)\mathsf{T}(n, x, y))$, by φ_n; and for each n and s we set

$$\varphi_n^{(s)} = \{\mathsf{j}(x, y) \mid (\exists z)[z \leqq s \ \&\ \mathsf{T}(n, x, z) \ \&\ (\forall w)(w < z \Longrightarrow \neg\mathsf{T}(n, x, w)) \ \&\ y = \mathsf{U}(z)]\},$$

where j is the recursive pairing function used in [1]. Note that $\varphi_n^{(s)}$ is a coding of a finite subfunction of φ_n, that the graph of φ_n is equal to $\bigcup_{s=0}^{\infty} \{\langle x, y\rangle \mid \mathsf{j}(x, y) \in \varphi_n^{(s)}\}$, and that both the characteristic function and the cardinality of the finite set $\varphi_n^{(s)}$ can be effectively determined from n and s. As in [1], ε denotes the set $\{0, 1, 2, \ldots\}$ of all natural numbers. A set $\alpha \subseteqq \varepsilon$ is said to be *recursively enumerable* (abbr.: *r.e.*) $\Longleftrightarrow_{df} \alpha$ = the domain, $\delta\varphi_n$, of some partial recursive function φ_n. These notions can (as is very well known; see [13]) be relativized in a uniform way: there exists a fixed recursive predicate $\mathsf{T}(X, n, x, y)$ of one set-theoretic (i.e., second-order) variable and three number-theoretic variables, such that $(\forall\alpha)\ [\alpha \subseteqq \varepsilon \Longrightarrow (\{\mathsf{U}((\mu y)\mathsf{T}(\alpha, n, x, y)) \mid n = 0, 1, 2, \ldots\}$ = the class of all one-place functions partial recursive *in* α)]; then, setting $\varphi_n^{\alpha} =_{df} \mathsf{U}((\mu y)\mathsf{T}(\alpha, n, x, y))$, we may define $\varphi_n^{\alpha,(s)}$ from φ_n^{α} exactly as in the previous ("unrelativized") case, and then observe that the characteristic function and the cardinality of $\varphi_n^{\alpha,(s)}$ can be effectively determined from n, s, and α. (The classes $\{\varphi_n \mid n \in \varepsilon\}$ and $\{\varphi_n^{\phi} \mid n \in \varepsilon\}$ are equal, though it is not necessarily the case that $\varphi_n = \varphi_n^{\phi}$ holds for a given n.) Since we shall use them in section **5**, we call attention to the two-variable counterparts, φ_n^2 and $\varphi_n^{2,(s)}$, of φ_n and $\varphi_n^{(s)}$; for explicit definition of φ_n^2 and (uncoded) $\varphi_n^{2,(s)}$, see [12]. Often, especially in section **5**, we shall *identify* $\varphi_n^{(s)}$ or $\varphi_n^{2,(s)}$ with the finite function which it codes. The classes $\Sigma_n^{(s)}$ and $\Pi_n^{(s)}$ of second-order sets and relations, involving in their definitions both free number-theoretic variables and free set-theoretic variables but with quantification restricted to number-theoretic variables, are as in [17], section **15.1**. If an element of the class

$\Sigma_n^{(s)}$ involves exactly one set-theoretic variable in its definition and if we substitute the particular set α for the set-theoretic variable in question, then we say that the resulting set or predicate of natural numbers belongs to Σ_n^α; a similar meaning attaches to the notation Π_n^α. We shall assume familiarity on the reader's part with the concept of *degree of unsolvability* of a set of natural numbers ([18]) and with the more elementary properties of the collection of all such degrees; in particular, we assume familiarity with the notion of the *jump*, $\boldsymbol{d}'$, of a given degree $\boldsymbol{d}$ ([18], section 1). We always use boldface, lower-case, Greek or Roman letters to denote degrees of unsolvability, with the exception that $\boldsymbol{\phi}$ is used to denote the zero degree (i.e., the degree of the empty set $\emptyset$). For each $\alpha \subseteqq \varepsilon$, $\boldsymbol{\alpha}$ denotes the degree of unsolvability of α; symbols such as $\boldsymbol{d}$ simply denote degrees, without reference to specific representatives. If $\boldsymbol{d}$ is given degree, then $\boldsymbol{d}'$, $\boldsymbol{d}''$, $\boldsymbol{d}'''$, . . . is the sequence of finite-order jumps of $\boldsymbol{d}$; we also denote by $\boldsymbol{d}^{(n)}$ the n-th jump of $\boldsymbol{d}$. (For a precise definition of the jump operator, see [18], section 1, p. 2.) We recall from the strong form of the Hierarchy Theorem ([17], section **14.5**) that the sets r.e. in $\boldsymbol{\alpha}^{(n)}$ (i.e., the sets of form $\delta\varphi_m^\beta$ for some m and some β such that $\boldsymbol{\beta} = \boldsymbol{\alpha}^{(n)}$) are just the sets belonging to the class Σ_{n+1}^α; the latter class, of course, contains sets of degree $\boldsymbol{\alpha}^{(n+1)}$. A degree $\boldsymbol{d}$ is said to be *complete* $\Longleftrightarrow_{df} (\exists\beta)[\beta \subseteqq \varepsilon \;\&\; \boldsymbol{d} = \boldsymbol{\beta}']$. Friedberg's famous characterization of the complete degrees ([8]) states that $\boldsymbol{d}$ is complete $\Longleftrightarrow \boldsymbol{d} \geqq \boldsymbol{\phi}'$.

In section **5** we shall make use of the notion of a *semirecursive set* ([10], [11]). For an extensive list of useful properties of semirecursive sets, the reader is referred to [11]. As to the definition, we remark that one characterization of semirecursiveness reads as follows ([11], Theorem 4.1): α is semirecursive $\Longleftrightarrow \alpha$ is a lower cut in a recursive linear ordering of ε; this characterization is often useful for proving "positive" results about semirecursive sets (for illustrations of this utility, see [11]). On the other hand, in proving Theorem 5.1 (which has a "negative" flavor) we find it convenient to make use of the original definition ([10], [11]), which reads as follows: α is semirecursive $\Longleftrightarrow$ there is a total recursive function $f(x, y)$ such that $(\forall x)(\forall y)[f(x, y) \in \{x, y\} \;\&\; (\{x, y\} \cap \alpha \neq \emptyset \Rightarrow f(x, y) \in \alpha)]$. We shall also have occasion (in section **4**) to consider the notion of *point-decomposability*. An infinite set α of natural numbers is said to be *point-decomposable* ([16]) if there is a recursive function f such that (i) $n \neq m \Rightarrow \delta\varphi_{f(n)} \cap \delta\varphi_{f(m)} = \emptyset$, (ii) $\alpha \subseteqq \bigcup_{n=0}^{\infty} \delta\varphi_{f(n)}$, and (iii) $(\forall n)[\alpha \cap \delta\varphi_{f(n)}$ is a singleton].

By a *tree* we shall mean a function $\mathbf{T}: \zeta \to \varepsilon$, $\zeta \subseteqq \varepsilon$, having the property that $\rho\mathbf{T} \subseteqq \delta\mathbf{T} \;\&\; (\forall x)\,[x \in \delta\mathbf{T} \Rightarrow \{x, \mathbf{T}(x), \mathbf{T}(\mathbf{T}(x)), \ldots\}$ is a finite set] & $(\forall x)(\forall y)\,[(x \in \delta\mathbf{T} \;\&\; y \in \delta\mathbf{T} \;\&\; y \in \{x, \mathbf{T}(x), \mathbf{T}(\mathbf{T}(x)), \ldots\} \;\&\; x \neq y) \Rightarrow x \notin \{y, \mathbf{T}(y), \mathbf{T}(\mathbf{T}(y)), \ldots\}]$. (The last conjunct asserts that the graph of $\mathbf{T}$ *has no proper loops*.) In the foregoing, $\rho\mathbf{T}$ denotes the range of $\mathbf{T}$ and $\delta\mathbf{T}$ denotes the domain of $\mathbf{T}$ (as in [1]). If $\mathbf{T}$ is a tree, and if $x \in \delta\mathbf{T}$, then by the $\mathbf{T}$-*height*, $\mathbf{T}^*(x)$, of x we mean the number $(\mu y)[\mathbf{T}^y(x) = \mathbf{T}^{y+1}(x)]$. ($f^y(x)$ is defined inductively by the equations $f^0(x) = x$ and $f^{n+1}(x) = f(f^n(x))$.) Let f be a function from ζ into ε, where $\zeta \subseteqq \varepsilon$; and suppose $\rho f \subseteqq \delta f$. Then if $x \in \delta f$, we denote by $\hat{f}(x)$ the set $\{x, f(x), f(f(x)), \ldots\}$. A tree $\mathbf{T}$ is said to be *recursive* if $\{ j(x, y) \mid \mathbf{T}(x)$ is defined and $= y\}$ is a recursive set; and $\mathbf{T}$ is said to be *well-branched* if $\delta f \subseteqq \rho f$.

Finally, we wish to explicitly recall Barback's definition of the relation $\mathring{\mathbf{V}}$ between regressive isols. If $\langle a_n \rangle_{n=0}^{\infty}$, $\langle b_n \rangle_{n=0}^{\infty}$ are regressive sequences of natural numbers (Barback uses simply a_n and b_n as notation for such sequences), and if α and β are the respective *ranges* of the sequences $\langle a_n \rangle_{n=0}^{\infty}$ and $\langle b_n \rangle_{n=0}^{\infty}$, then $\langle a_n \rangle_{n=0}^{\infty} \mathring{\mathbf{V}} \langle b_n \rangle_{n=0}^{\infty}$ is defined to mean that $\alpha | \beta$ & $(\exists p)$ [p is a partial recursive function & $(\forall n)$ [either $a_n \in \delta p$ & $p(a_n) = b_n$ or $b_n \in \delta p$ & $p(b_n) = a_n$]]; here, as in [1], $\alpha | \beta$ means that $(\exists n)(\exists m)[\alpha \subseteqq \delta\varphi_n$ & $\beta \subseteqq \delta\varphi_m$ & $\delta\varphi_n \cap \delta\varphi_m = \emptyset]$. As Barback has noted in (1), if $\langle a_n \rangle_{n=0}^{\infty}$, $\langle b_n \rangle_{n=0}^{\infty}$, $\langle c_n \rangle_{n=0}^{\infty}$, and $\langle d_n \rangle_{n=0}^{\infty}$ are regressive sequencings, respectively, of the infinite regressive sets α, β, γ, and δ, and if $\alpha \simeq \gamma$ & $\beta \simeq \delta$ & $\alpha | \beta$ & $\gamma | \delta$, then $\langle a_n \rangle_{n=0}^{\infty} \mathring{\mathbf{V}} \langle b_n \rangle_{n=0}^{\infty} \Longleftrightarrow \langle c_n \rangle_{n=0}^{\infty} \mathring{\mathbf{V}} \langle d_n \rangle_{n=0}^{\infty}$; hence, the relation $\mathring{\mathbf{V}}$ can be defined for pairs of regressive *isols* without any restrictions on choice of (separated) representatives or on choice of regressive sequencings of (separated) representatives. Theorem 1.2 of [1] states that $[A + B \in \Lambda_{\mathrm{R}}$ & A infinite & B infinite$] \Longrightarrow A \mathring{\mathbf{V}} B$. Therefore, in order to produce infinite regressive isols A and B whose sum does not belong to Λ_{R}, it is enough to arrange that $A \in \Lambda_{\mathrm{R}}^{\infty}$ & $B \in \Lambda_{\mathrm{R}}^{\infty}$ & $\neg(A \mathring{\mathbf{V}} B)$; here we are using $\Lambda_{\mathrm{R}}^{\infty}$ to denote the class of *infinite* regressive isols. The existence of such isols A and B is precisely the content of [1], Theorem 1.3; while the impossibility of having $\neg(A \mathring{\mathbf{V}} B)$ with *both* A and B co-simple is the content of our Corollary 4.2.

3 *Complete Degrees*

Theorem 3.1 *Let $\boldsymbol{d}$ be a degree of unsolvability satisfying $\boldsymbol{d} \geqq \boldsymbol{\phi}'$. Then there exist retraceable sets β_1 and β_2 such that* (i) $\boldsymbol{\beta}_1 = \boldsymbol{\beta}_2 = \boldsymbol{d}$ *and* (ii) $\neg([\beta_1] \mathring{\mathbf{V}} [\beta_2])$. *If, in addition, $\boldsymbol{d}$ satisfies the equation $\boldsymbol{d} = \boldsymbol{\phi}^{(n_0+1)}$ (here n_0 can be any fixed natural number) then we can further require:* (iii) $\varepsilon - \beta_2$ *is* r.e. *in* $\boldsymbol{\phi}^{(n_0)}$*; i.e.,* $\varepsilon - \beta_2 \in \sum_{n_0+1}^{\phi}$.

Proof: We start by defining a certain binary-branching, well-branched recursive tree $\mathbf{T}$. (By "binary-branching" we mean that exactly two branches of $\mathbf{T}$ emanate from each node of $\mathbf{T}$, i.e., that $\mathbf{T}^{-1}(x)$ has exactly two (exactly three) elements for every $x \in \delta\mathbf{T}$ such that $\mathbf{T}(x) \neq x$ (such that $\mathbf{T}(x) = x$). For $i \in \{0, 1, 2\}$, let $\tau_i =_{df} \{n | n \equiv i (\mathrm{mod}\ 3)\}$. We take $\mathbf{T}$ to be the tree determined by the following equations:

$$\delta\mathbf{T} = \rho\mathbf{T} = \{0\} \cup (\varepsilon - \tau_0);$$
$$\mathbf{T}^{-1}(0) = \{0, 1, 2\};$$
$$\mathbf{T}^{-1}(n + 1) = \{3(n + 1) + 1, 3(n + 1) + 2\}, \text{ for } n + 1 \in \varepsilon - \tau_0.$$

Obviously $\mathbf{T}$ is recursive, well-branched and binary-branching. Next, by means of an elementary priority scheme, we shall construct a co-r.e. retraceable set β such that $\boldsymbol{\beta} = \boldsymbol{\phi}'$ & no infinite branch $\{t(n) | n \in \varepsilon\}$ of $\mathbf{T}$ is $\mathring{\mathbf{V}}$-related to an infinite retraceable subset γ of β unless $\gamma \leqq^* \rho t$. (For the definition of $\leqq^*$, see [1]. When we say that $\{t(n) \mid n \in \varepsilon\}$ is a *branch* of $\mathbf{T}$, we mean that $(\forall n)\ [t(n) \in \delta\mathbf{T}\ \&\ t(n+1) \in \delta\mathbf{T}\ \&\ \mathbf{T}(t(n+1)) = t(n)\ \&\ \mathbf{T}(t(0)) = t(0)]$.) β will be a subset of $\tau_0 - \{0\}$; hence, we automatically have $\beta \mid \rho t$ for every infinite branch $\{t(n) \mid n \in \varepsilon\}$ of $\mathbf{T}$. The construction of β (more precisely, the construction of $\varepsilon - \beta$), together with that of a partial recursive function p such that p retraces β, is carried out in stages according to the following prescriptions.

Stage 0 Set $\overline{\beta}^{(0)} =_{df} (\varepsilon - \tau_0) \cup \{0\}$; attach the marker Λ_0 to the number 3; define $p^{(0)} = \{\langle 3, 3\rangle\}$; then proceed to Stage 1.

Stage $s+1$ In the description which we are about to give, we understand λ_e^s as denoting 0 if the marker Λ_e is not attached at the conclusion of Stage s; otherwise, $\lambda_e^s =_{df}$ the number to which Λ_e is attached (sometimes referred to as the *position of* Λ_e) at the conclusion of Stage s. No marker will ever be attached to 0. The procedure as a whole will be such that there exists a recursive function $m(x)$ with the property that $(\forall s)(\forall e)[(e \leqq m(s) \Rightarrow \lambda_e^s > 0)\ \&\ (e > m(s) \Rightarrow \lambda_e^s = 0)]$; this function will be used in describing Stage $s+1$, to the extent that at Stage $s+1$ we take the value of $m(s)$ as known (which assumption will readily be seen to be justified once the construction has been fully described). We are now ready to set forth our procedure at Stage $s+1$. There are two principal cases.

Case I s is even. In this case, we attack the usual "domination requirements" relevant to securing the relation $\beta = \boldsymbol{\phi}'$, as in [21], proof of Theorem 3. First, suppose that there exist numbers k, l, r, and y such that $0 < k \leqq m(s)\ \&\ l \leqq k\ \&\ r \leqq k\ \&\ \mathrm{j}(l, y) \in \varphi_r^{(s)}\ \&\ y \geqq \lambda_k^s$. Let k_0 be the smallest k for which such l, r, and y exist. Detach all markers Λ_q such that $\Lambda_q^s > 0\ \&\ q \geqq k_0$. Define $u_0 = (\mu u)[u \notin \delta p^{(s)} \cup \overline{\beta}^{(s)}]$. Attach Λ_{k_0} to u_0. Set $\overline{\beta}^{(s+1)} =_{df} \overline{\beta}^{(s)} \cup \{w \mid \lambda_{k_0}^s \leqq w < u_0\}$, and define $p^{(s+1)} = p^{(s)} \cup \{\langle u_0, \lambda_{k_0-1}^s\rangle\}$; then proceed to Stage $s+2$. (Note that $m(s+1) = k_0$ in this instance.) If, on the other hand, no such numbers k, l, r, and y exist, then we define u_0 as before, set $\overline{\beta}^{(s+1)} =_{df} \overline{\beta}^{(s)} \cup \{w \mid \lambda_{m(s)}^s < w < u_0\}$, define $p^{(s+1)} = p^{(s)} \cup \{\langle u_0, \lambda_{m(s)}^s\rangle\}$, attach $\Lambda_{m(s)+1}$ to u_0, and proceed to Stage $s+2$. (Here, $m(s+1) = m(s)+1$.) We remark that it will be clear, when our description of the construction is complete, that in this second subcase (i.e., when numbers k, l, r, and y of the indicated type do not exist) we can equivalently define $\overline{\beta}^{(s+1)} = \overline{\beta}^{(s)}$.

Case II s is odd. Here we shall attack requirements relevant to insuring that if $\{t(n) \mid n \in \varepsilon\}$ is a branch of $\mathbf{T}$ and γ is an infinite retraceable subset of β then $[\rho t]\ \mathbf{\check{V}}\ [\gamma] \Rightarrow \gamma \leqq^* \rho t$. First, suppose that there are numbers w, k, l, and r such that $0 < k \leqq m(s)\ \&\ l \leqq k\ \&\ r \in \delta\mathbf{T}\ \&\ \mathbf{T}^*(r) = k\ \&\ \mathrm{j}(r, w) \in \varphi_l^{(s)}\ \&\ w \geqq \lambda_k^s$. (Recall that $\mathbf{T}^*(r)$ is our notation for the $\mathbf{T}$-height of r.) Then we take k_0 to be the least k for which such numbers w, l, and r exist and proceed exactly as in the first subcase of Case I. If no such numbers w, k, l, and r exist, then we proceed exactly as in the second subcase of Case I.

That completes our description of Stage $s+1$. We set $\alpha =_{df} \bigcup_{s=0}^{\infty} \overline{\beta}^{(s)}$ and

$p =_{df} \bigcup_{s=0}^{\infty} p^{(s)}$. Clearly, α is an r.e. set and p is a partial recursive function. As usual with such constructions our first concern is to verify that all markers attain final positions, i.e., that $(\forall n)[s \overset{\lim}{\to} \infty\ \lambda_n^s$ exists and is $>0]$. This is accomplished by a straightforward inductive argument. Clearly, $\lambda_0^s > 0$ for all s; moreover, it is obvious that Λ_0 is not detached at any point in the construction subsequent to the end of Stage 0. Thus $s \overset{\lim}{\to} \infty\ \lambda_0^s$ exists and is >0. Now suppose that $s \overset{\lim}{\to} \infty\ \lambda_n^s$ exists and is >0 for all $n \leqq k$. Let s_0 be the smallest number s such that $(\forall w)[w \geqq s \Rightarrow (\forall n)(n \leqq k \Rightarrow \lambda_n^w = t \overset{\lim}{\to} \infty\ \lambda_n^t)]$. Then, in view of the manner in which markers are attached and detached in our construction, we see that $\lambda_{k+1}^s > 0$ for all $s \geqq s_0 + 1$. In addition, it is clear that Λ_{k+1} can be moved during Stage s, $s \geqq s_0 + 1$, only if *either* s is even & $(\exists l)(\exists r)(\exists y)[l \leqq k+1$ & $r \leqq k+1$ & $\mathrm{j}(l, y) \in \varphi_r^{(s)}$ & $y \geqq \lambda_{k+1}^{s-1}]$ *or* s is odd & $(\exists w)(\exists l)(\exists r)[\mathrm{j}(r, w) \in \varphi_l^{(s)}$ & $l \leqq k+1$ & $r \in \delta \mathbf{T}$ & $\mathbf{T}^*(r) = k+1$ & $w \geqq \lambda_{k+1}^s]$; in either case, since markers move only from smaller to larger numbers, it is evident that only finitely many such moves can occur. Hence $s \overset{\lim}{\to} \infty\ \lambda_{k+1}^s$ exists and is >0. By induction, $s \overset{\lim}{\to} \infty\ \lambda_n^s$ exists and is >0 for all values of n. From here on, we denote $\lim_{s\to\infty} \lambda_n^s$ by λ_n. It is plain from our description of the construction that $(\forall n)[\lambda_n < \lambda_{n+1}]$ and that p retraces $\{\lambda_n | n \in \varepsilon\}$. We define $\beta = \{\lambda_n | n \in \varepsilon\}$; then, it is easy to see that $\alpha = \varepsilon - \beta$. Let g be an arbitrarily given partial recursive function of one variable; and let n_0 be the least positive number n such that $g = \varphi_n$. Then it is clear from the description of Stage $s+1$ for even values of s that $(\forall m)[m > n_0$ & $\varphi_{n_0}(m)$ defined $\Rightarrow \lambda_m > \varphi_{n_0}(m)]$. Thus, each partial recursive function of one variable is eventually majorized by the sequence $\langle \lambda_k \rangle_{k=0}^{\infty}$; hence ([21], p. 465), $\boldsymbol{\beta} \geqq \boldsymbol{\phi}'$. Since β has r.e. complement, $\boldsymbol{\beta} = \boldsymbol{\phi}'$. Next, suppose that $\{t(n) | n \in \varepsilon\}$ is a branch of $\mathbf{T}$ and that $[\rho t] \mathbin{\nabla} [\beta]$. Let h be a partial recursive function of one variable such that $(\forall n)[(t(n) \in \delta h$ & $h(t(n)) = \lambda_n)$ or $(\lambda_n \in \delta h$ & $h(\lambda_n) = t(n))]$. Let m_0 be the least positive number m such that $h = \varphi_m$. We see from the description of Stage $s+1$ for odd values of s that $(\forall n)[n > m_0 \Rightarrow (\lambda_n \in \delta h$ & $h(\lambda_n) = t(n))]$, since for all $n > m_0$ we have $\lambda_n > h(t(n))$. From this last statement it readily follows that $\beta \leqq^* \rho t$ is witnessed by some finite modification of the function h. But *a fortiori*, if γ is an infinite retraceable subset of β and the relation $[\rho t] \mathbin{\nabla} [\gamma]$ is witnessed by the partial recursive function $\tilde{h}$ then $\gamma \leqq^* \rho t$ holds via some finite modification of $\tilde{h}$; for, given any number k, if $\lambda_k \in \gamma$ then the k-th member of γ (in order of magnitude) is $\geqq \lambda_k$. Thus, β has the desired properties. (At this point we would like to interject the remark that the existence of a co-r.e. retraceable set β of the type just constructed is a trivial corollary to a general "thinning theorem" contained in the author's paper [15]. We have here included a detailed construction of β, rather than a mere reference to [15], because the priorities needed to obtain β from scratch are completely straightforward.)

Next we shall select from $\mathbf{T}$ a branch $\{t(n) | n \in \varepsilon\}$ such that $\boldsymbol{\rho t} = \boldsymbol{\phi}'$ & $\neg([\rho t] \mathbin{\nabla} [\beta])$. Because of the properties with which we have endowed β, it will be sufficient to arrange that $\beta \not\leqq^* \rho t$ in order to insure that $[\rho t]$ and $[\beta]$ are not ∇-related. We define t as follows:

$$t(0) = 0;$$
$$t(n+1) = \begin{cases} (\mu k)[k \in \mathbf{T}^{-1}(t(n)) \mathbin{\&} (\varphi_{\frac{n+1}{2}}(\lambda_{n+1}) \text{ is defined } \Longrightarrow \\ \varphi_{\frac{n+1}{2}}(\lambda_{n+1}) \neq k)] \text{ if } n+1 \text{ is even}, \\ (\mu k)[k \in \mathbf{T}^{-1}(y(n)) \mathbin{\&} k \equiv 1(\mathrm{mod}\ 3)] \text{ if} \\ n+1 \text{ is odd and } \frac{n}{2} \in \beta, \\ (\mu k)[k \in \mathbf{T}^{-1}(t(n)) \mathbin{\&} k \equiv 2(\mathrm{mod}\ 3)] \text{ if} \\ n+1 \text{ is odd and } \frac{n}{2} \notin \beta. \end{cases}$$

Clearly, $\rho t \leqq \phi'$. Conversely, it is easily shown that $\phi' \leqq \rho t$. For, given any number n, we can check whether n belongs to β by determining whether $t(2n+1) \equiv 1(\mathrm{mod}\ 3)$; so, since $\beta = \phi'$, we obtain $\phi' \leqq \rho t$. Finally, suppose $\beta \leqq^* \rho t$. Then by [2], Proposition P11 there must exist a partial recursive function q such that $\beta \subseteqq \delta q$ & $(\forall n)[q(\lambda_n) = t(n)]$. Let m_0 be the least nonzero number m such that $q = \varphi_m$. Since we have defined t in such a way that $t(2m_0) \neq \varphi_{m_0}(\lambda_{2m_0})$, we obtain a contradiction. Therefore, we conclude that $\beta \nleqq^* \rho t$ and hence that $[\rho t]$ is not $\mathring{\nabla}$-related to $[\beta]$. Thus the theorem is proved, for the particular case $d = \phi'$. We shall extend the result to all $d > \phi'$ with the aid of [5], Propositions P2 and P4 and [12], Theorem 4.14(2). Let d be a degree strictly greater than ϕ'; and let τ be a retraceable set belonging to d. As in [5], let p_τ denote the uniquely determined function h: $\varepsilon \to \varepsilon$ such that h is strictly increasing and $\rho h = \tau$. By [5], Proposition P4, the strictly increasing sequence $\langle \lambda_{p_\tau(n)} \rangle_{n=0}^{\infty}$ is retraceable. Let $\delta = \rho(\langle \lambda_{p_\tau(n)} \rangle_{n=0}^{\infty})$. Since $\rho p_\tau = d > \beta$ & $\delta \subseteqq \beta$, we have $\delta \leqq d$. On the other hand, since $\delta \geqq \beta$ holds by [5], Proposition P2, we can compute ρp_τ from δ; hence $d \leqq \delta$ and so $d = \delta$. By [12], Theorem 4.14(2), if $d = {}_{,}\phi^{(n+1)}$ for some $n \geqq 1$ then the set τ can be chosen from the class Π_{n+1}^{ϕ}. Since β belongs to the class Π_1^{ϕ}, it follows that we may assume $\delta \in \Pi_{n+1}^{\phi}$ in case $(\exists n)[n \geqq 1$ & $d = \phi^{(n+1)}]$. But then $\varepsilon - \delta \in \Sigma_{n+1}^{\phi}$ (equivalently, $\varepsilon - \delta$ is r.e. in $\phi^{(n)}$). It remains to choose from $\mathbf{T}$ a branch $\{u(n) \mid n \in \varepsilon\}$ such that $\rho u = d$ & $\neg([\rho u] \mathring{\nabla} [\delta])$. Since δ is an infinite retraceable subset of β, it suffices (in view of the properties of β relative to $\mathbf{T}$) to arrange that $\delta \nleqq^* \rho u$ in order to insure that $[\rho u]$ and $[\delta]$ are not $\mathring{\nabla}$-related. Letting p_δ be the function which enumerates δ in increasing order of magnitude, we proceed with the definition of μ just as we did with the definition of t in the case $d = \phi'$:

$$u(0) = 0;$$
$$u(n+1) = \begin{cases} (\mu k)[k \in \mathbf{T}^{-1}(u(n)) \mathbin{\&} (\varphi_{\frac{n+1}{2}}(p_\delta(n+1)) \text{ is defined } \Longrightarrow \\ \varphi_{\frac{n+1}{2}}(p_\delta(n+1)) \neq k)] \text{ if } n+1 \text{ is even}, \\ (\mu k)[k \in \mathbf{T}^{-1}(u(n)) \mathbin{\&} k \equiv 1(\mathrm{mod}\ 3)] \text{ if} \\ n+1 \text{ is odd } \& \frac{n}{2} \in \delta, \\ (\mu k)[k \in \mathbf{T}^{-1}(u(n)) \mathbin{\&} k \equiv 2(\mathrm{mod}\ 3)] \text{ if} \\ n+1 \text{ is odd } \& \frac{n}{2} \notin \delta. \end{cases}$$

Exactly as in our treatment of the case $\boldsymbol{d} = \boldsymbol{\phi}'$, we verify that $\boldsymbol{\rho u} = \boldsymbol{d}$ & $\delta \not\leq^* \rho u$; this shows that if $\beta_1 = \rho u$ and $\beta_2 = \delta$ then β_1, β_2, satisfy conditions (i), (ii), and (subject to the indicated provision regarding $\boldsymbol{d}$) (iii), and concludes our proof of Theorem 3.1.

Corollary 3.2 *If $\boldsymbol{d}$ is a degree such that $\boldsymbol{d} \geqq \boldsymbol{\phi}'$, then there exist retraceable sets $\beta_1, \beta_2 \in \boldsymbol{d}$ such that $\beta_1 | \beta_2$ & $\beta_1 \cup \beta_2$ is not regressive.*

Proof: Theorem 3.1 and [1], Theorem 1.2. (As mentioned in section 1, Corollary 3.2 admits an easy proof not involving limiting considerations. Most of the maneuvering in our proof of Theorem 3.1 relates to condition (iii).)

Corollary 3.3 *If $\boldsymbol{d}$ is restricted to vary over the class of all complete degrees, then the failure of additive closure for $\boldsymbol{\Lambda}_{\mathrm{R}}$ is totally independent of $\boldsymbol{d}$ (in the sense explained in section 1).*

Proof: Corollary 3.3 follows immediately from Corollary 3.2 and [2], Proposition P 17(d).

Before leaving section 3 we remark that, degrees of summands aside, the following very simple result covers a number of examples of $\Lambda_{\mathrm{R}} + \Lambda_{\mathrm{R}} \not\subseteq \Lambda_{\mathrm{R}}$:

Theorem 3.4 *Let $\mathfrak{F}$ be a subclass of 2^{ε} such that $\mathfrak{F}$ is closed under relative recursive enumerability* (i.e., $[\alpha \in \mathfrak{F}$ & β r.e. in $\alpha] \Longrightarrow \beta \in \mathfrak{F}$); *and let C be a nonempty countable collection of infinite, regressive, non-*r.e. *subsets of* ε, *such that $C \subseteqq \mathfrak{F}$. Let $\Gamma(C) = \{A | A \in \Lambda_{\mathrm{R}}^{\infty}$ & $(\exists\gamma)[\gamma \in C$ & $A + [\gamma] \in \Lambda_{\mathrm{R}}]\}$. Then $\Gamma(C)$ is a countably infinite subclass of the $\mathfrak{F}$-isols.*

(By an $\mathfrak{F}$-*isol* we mean an isol A such that $A \cap \mathfrak{F} \neq \emptyset$.)

Proof: Let $C^* = \{\delta | (\exists\gamma)[\gamma \in C$ & $\gamma \simeq \delta]\}$. If $\mathfrak{A}$ is the collection of all sets α such that $\boldsymbol{\alpha} \leqq \boldsymbol{\delta}'$ holds for some $\delta \in C^*$, then $\mathfrak{A}$ contains a representative of each isol A such that $A \in \Gamma(C)$. For if $A \in \Lambda_{\mathrm{R}}^{\infty}$ & $\alpha \in A$ & $\delta \in C^*$ & $\alpha | \delta$ & $\alpha \cup \delta$ is regressive, then, since $\alpha \cup \delta$ is r.e. in δ and α is recursive in $\alpha \cup \delta$, we see that $\boldsymbol{\alpha} \leqq \boldsymbol{\delta}'$. Since $\mathfrak{A}$ is countably infinite, $\Gamma(C)$ is countable. Moreover, it is easily seen that $\{[\gamma] | \gamma \in C\} \subseteqq \Gamma(C)$; hence $\Gamma(C) \neq \emptyset$. By a routine deletion argument, $\Gamma(C) \neq \emptyset \Longrightarrow \Gamma(C)$ is infinite; thus, we have $\Gamma(C)$ countably infinite. Finally, if $A \in \Gamma(C)$ then $(\exists\alpha)[\alpha \in A \cap \mathfrak{F}]$. For, by definition of $\Gamma(C)$, we have $[\alpha] + [\gamma] \in \Lambda_{\mathrm{R}}$, with $\alpha | \gamma$, for some $\alpha \in A$ and $\gamma \in C$. Since $C \subseteqq \mathfrak{F}$, $\gamma \in \mathfrak{F}$. But $\alpha \cup \gamma$ is r.e. in γ; so, since $\alpha | \gamma$, α is r.e. in γ. Thus $\alpha \in \mathfrak{F}$, since $\mathfrak{F}$ is closed under relative recursive enumerability. That completes the proof.

As one rather obvious instance of Theorem 3.4, let $C_0 = \{\gamma | \gamma$ is non-r.e. & γ is regressed by a partial recursive function p such that p regresses at most $\aleph_0$ infinite sets$\}$. A partial recursive function p is a *countable regressing function* (*cf.* [12]) if p regresses a total of K infinite sets, where $1 \leqq K \leqq \aleph_0$. It is easily seen that C_0 is a nonempty subclass of the class HYP of all *hyperarithmetical* sets; and, of course, HYP is countable and is closed under relative recursive enumerability.

In connection with the class C_0, one might very naturally inquire whether $\Gamma(C_0) \subseteqq \{[\gamma] \mid \gamma \in C_0\}$ holds as well as $\{[\gamma] \mid \gamma \in C_0\} \subseteqq \Gamma(C_0)$. We conjecture this to be the case. It is easy to give examples of countable classes C for which the identity in question does (does not) hold. For instance, let C_k be any countable collection of non-r.e. regressive sets such that $(\forall\beta)(\forall\gamma)[(\beta \in C_k \;\&\; \gamma \text{ r.e. in } \beta \;\&\; \gamma \text{ non-r.e. and regressive}) \Longrightarrow \gamma \in C_k]$; the class of all non-r.e., arithmetical regressive sets and the class of all non-r.e., hyperarithmetical regressive sets are two particular classes which satisfy this condition. Since $(\beta \mid \gamma \;\&\; \beta \cup \gamma$ is regressive & β is infinite) $\Longrightarrow \gamma$ is r.e. in β, we have $\Gamma(C_k) \subseteqq \{[\gamma] \mid \gamma \in C_k\}$. For an example of $\Gamma(C) \nsubseteq \{[\gamma] \mid \gamma \in C\}$, we call upon a theorem of Hassett's concerning universal elements of Λ_R^∞. (The notion of a *universal* isol was first defined and studied in [6], where it was shown that such isols are abundant in the sense of Baire Category.) Hassett showed (in an as-yet-unpublished manuscript) that there exists a universal regressive isol A such that $A = 2B + C$ for certain isols B and C with B infinite. (I.e., he showed that universal regressive isols need not be *multiple-free* in the sense of [4].) By a theorem of Ellentuck's, the isol C in the above equation cannot be of the form $2D + E$ where E is finite. Hence, if we take $A = 2B + C$ with A a universal element of Λ_R^∞ and with B infinite, then the collection $C = \{2nB \mid n \in \varepsilon,\ n \geqq 1\}$ (consisting of the non-trivial *even* multiples of B) is such that $\{[\gamma] \mid \gamma \in C\} \nsubseteq \Gamma(C)$. (In particular, we have $C \in \Gamma(C) - \{[\gamma] \mid \gamma \in C\}$.)

4 *Any two co-simple regressive isols are* $\overset{*}{\nabla}$*-related* It is natural to inquire whether condition (iii) in the statement of Theorem 3.1 can be made to apply simultaneously to the sets β_1 and β_2. The answer is in the negative, at least at the bottom level $n_0 = 0$.

Theorem 4.1 *Let α and β be infinite, point-decomposable sets such that $\alpha \mid \beta$ & $\varepsilon - \alpha$ is* r.e. & $\varepsilon - \beta$ *is* r.e. *Let f and h be recursive functions such that* (a) $\bigcup_{n=0}^{\infty} \delta\varphi_{f(n)}$ *is disjoint from* $\bigcup_{n=0}^{\infty} \delta\varphi_{h(n)}$, (b) $\langle \delta\varphi_{f(n)} \rangle_{n=0}^{\infty}$ *witnesses point-decomposability of α, and* (c) $\langle \delta\varphi_{h(n)} \rangle_{n=0}^{\infty}$ *witnesses point-decomposability of β. Then there exists a partial recursive function p such that* $(\forall x)(\forall y)(\forall z)$ $[(x \in \alpha \cap \delta\varphi_{f(z)} \;\&\; y \in \beta \cap \delta\varphi_{h(z)}) \Longrightarrow [(x \in \delta p \;\&\; p(x) = y)$ *or* $(y \in \delta p \;\&\; p(y) = x)]]$.

Proof: Let k_0 and l_0 be numbers such that $\delta\varphi_{k_0} = \bigcup_{n=0}^{\infty} \delta\varphi_{f(n)}$ and $\delta\varphi_{l_0} = \bigcup_{n=0}^{\infty} \delta\varphi_{h(n)}$. Let g_1, g_2 be recursive functions such that $\rho g_1 = \delta\varphi_{k_0} - \alpha$ and $\rho g_2 = \delta\varphi_{l_0} - \beta$. (Such functions exist because α and β are co-r.e. sets with $\alpha \subseteqq \delta\varphi_{k_0}$ & $\beta \subseteqq \delta\varphi_{l_0}$.) Given a natural number n, we shall make a stage-by-stage construction of a partial recursive function r_n such that $\delta r_n \cup \rho r_n \subseteqq \delta\varphi_{f(n)} \cup \delta\varphi_{h(n)}$ & $(r_n(a) = b$ or $r_n(b) = a)$, where a is the unique element of $\alpha \cap \delta\varphi_{f(n)}$ and b is the unique element of $\beta \cap \delta\varphi_{h(n)}$; it will be clear that the construction of r_n is uniform in n, so that $\bigcup_{n=0}^{\infty} r_n$ will be a partial recursive function having the property required by the theorem. At Stage s of the construction, we shall add either one new pair or no new pairs to r_n; each pair put into r_n during the construction will be of the form $\langle x, y\rangle$ where

$$(\exists n)[(x \in \delta\varphi_{f(n)} \ \&\ y \in \delta\varphi_{h(n)}) \text{ or } (x \in \delta\varphi_{h(n)} \ \&\ y \in \delta\varphi_{f(n)})].$$

Immediately after a pair $\langle x, y\rangle$ has been added to r_n, x will bear the tag **D** (for "domain") and y will bear the tag **R** (for "range"); if either x or y subsequently shows up in $\rho g_1 \cup \rho g_2$, it will then lose whatever tag (or tags) it has acquired prior to its entry into $\rho g_1 \cup \rho g_2$. Let ψ_1, ψ_2 be recursive functions such that $\rho\psi_1 = \delta\varphi_{f(n)}$ & $\rho\psi_2 = \delta\varphi_{h(n)}$. Our exact procedure is as follows:

Stage 0 Set $r_n^{(0)} =_{df} \{\langle\psi_1(0), \psi_2(0)\rangle\}$; give $\psi_1(0)$ a tag **D** and give $\psi_2(0)$ a tag **R**; then go to Stage 1.

Stage $s+1$ There are two main cases.

Case I s is even. Subcase IA Some pair $\langle x, y\rangle$ in $r_n^{(s)}$ has the property that x currently bears a **D**, y currently bears an **R** and $g_1(s) \in \{x, y\}$. As will be clear when our description of the construction is complete, x and y are uniquely determined by these conditions. If $g_1(s) = x$, proceed as follows. Remove **D** from x and set $r_n^{(s+1)} =_{df} r_n^{(s)} \cup \{\langle\psi_1(z_0), y\rangle\}$, where $z_0 = (\mu z)[\psi_1(z) \notin \delta r_n^{(s)} \cup \{g_1(t) \mid t \leqq s+1\}]$. (Such a number z_0 must exist, since $\alpha \cap \delta\varphi_{f(n)} \neq \emptyset$ and since, as will be clear when our description of the construction is finished, no number m ever loses a tag **D** at Stage s, $s > 0$, unless *all* the elements of $\delta r_n^{(s-1)}$ have already been enumerated in $\rho g_1 \cup \rho g_2$.) Give $\psi_1(z_0)$ a tag **D**. Then go to Stage $s+2$. If, on the other hand, $g_1(s) = y$, proceed in the following way. Remove **R** from y and set $r_n^{(s+1)} =_{df} r_n^{(s)} \cup \{\langle\psi_1(w_0), x\rangle\}$, where $w_0 = (\mu w)[\psi_1(w) \notin \delta r_n^{(s)} \cup \{g_1(t) \mid t \leqq s+1\}]$. (Such a number w_0 must exist, for reasons parallel to those given in support of the existence of z_0 in the case $g_1(s) = x$.) Give $\psi_1(w_0)$ a tag **D** and give x a tag **R**. Then go to Stage $s+2$.

Subcase IB There is no pair $\langle x, y\rangle \in r_n^{(s)}$ such that x currently bears a **D**, y currently bears an **R** and $g_1(s) \in \{x, y\}$. Set $r_n^{(s+1)} =_{df} r_n^{(s)}$; then proceed to Stage $s+2$.

Case II s is odd. Subcase IIA Some pair $\langle x, y\rangle$ in $r_n^{(s)}$ has the property that x currently bears a **D**, y currently bears an **R** and $g_2(s) \in \{x, y\}$. As in Subcase IA, these conditions uniquely determine x and y. If $g_2(s) = x$, proceed exactly as in the $g_1(s) = x$ alternative under Subcase IA, but with g_1 replaced by g_2 and ψ_1 by ψ_2. If $g_2^{(s)} = y$, proceed exactly as in the $g_1(s) = y$ alternative under Subcase IA, but with g_1 replaced by g_2 and ψ_1 by ψ_2.

Subcase IIB There is no pair $\langle x, y\rangle \in r_n^{(s)}$ such that x currently bears a **D**, y currently bears an **R** and $g_2(s) \in \{x, y\}$. Set $r_n^{(s+1)} =_{df} r_n^{(s)}$; then go to Stage $s+2$.

That completes the description of the construction; clearly, the function r_n defined by $r_n = \bigcup_{s=0}^{\infty} r_n^{(s)}$ is partial recursive and has both its range and its domain included in $\delta\varphi_{f(n)} \cup \delta\varphi_{h(n)}$. Moreover, it is plain that $\langle x, y\rangle \in r_n \Rightarrow [(x \in \delta\varphi_{f(n)} \;\&\; y \in \delta\varphi_{h(n)})$ or $(x \in \delta\varphi_{h(n)} \;\&\; y \in \delta\varphi_{f(n)})]$. By induction on s, we readily establish the following two assertions: (1) at the end of Stage s, exactly one pair $\langle x, y\rangle \in r_n$ has the property that neither x nor y has yet been ruled out of $\alpha \cup \beta$ by virtue of appearing in $\rho g_1 \cup \rho g_2$, and (2) a new pair $\langle x, y\rangle$ enters r_n at Stage $s + 1$ if and only if this is necessary in order to preserve (1) from Stage s to Stage $s + 1$. Since $\alpha \cap \delta\varphi_{f(n)}$ and $\beta \cap \delta\varphi_{h(n)}$ are singletons, and since ψ_1, ψ_2 respectively enumerate $\delta\varphi_{f(n)}$, $\delta\varphi_{h(n)}$, it is therefore clear that there exists a stage s_0 such that

$$\begin{aligned}&(\forall g)[s \geqq s_0 \Rightarrow r_n^{(s)} = r_n^{(s_0)}] \;\&\\ &(\exists x)(\exists y)[\langle x, y\rangle \in r_n^{(s_0)} \;\&\; [(\{x\} = \alpha \cap \delta\varphi_{f(n)} \;\&\; \{y\} = \beta \cap \delta\varphi_{h(n)})\\ &\text{or } (\{x\} = \beta \cap \delta\varphi_{h(n)} \;\&\; \{y\} = \alpha \cap \delta\varphi_{f(n)})]].\end{aligned}$$

Hence $\bigcup_{n=0}^{\infty} r_n$ is a partial recursive function p as required for the theorem.

Corollary 4.2 *If* $[\alpha] \in \Lambda_R^\infty \;\&\; [\beta] \in \Lambda_R^\infty \;\&\; \varepsilon - \alpha$ *is* r.e. & $\varepsilon - \beta$ *is* r.e., *then* $[\alpha] \overset{*}{\mathbf{V}} [\beta]$.

Proof: We may suppose, with no loss of generality, that $\alpha | \beta$. Let $\alpha \subseteqq \delta\varphi_{k_0} \;\&\; \beta \subseteqq \delta\varphi_{l_0}$ where $\delta\varphi_{k_0} \cap \delta\varphi_{l_0} = \emptyset$. Let p be a partial recursive function such that p regresses $\alpha \;\&\; \rho p \subseteqq \delta p \subseteqq \delta\varphi_{k_0} \;\&\; (\forall x)[x \in \delta p \Rightarrow (\exists y)(p^{y+1}(x) = p^y(x))]$; and let q be a partial recursive function such that q regresses $\beta \;\&\; \rho q \subseteqq \delta q \subseteqq \delta\varphi_{l_0} \;\&\; (\forall x)[x \in \delta q \Rightarrow (\exists y)(q^{y+1}(x) = q^y(x))]$. Let p^*, q^* be related to p, q, respectively, as indicated in section 2. Then there exist recursive functions f and h such that $(\forall n)[\delta\varphi_{f(n)} = \{x | x \in \delta p \;\&\; p^*(x) = n\} \;\&\; \delta\varphi_{h(n)} = \{x | x \in \delta q \;\&\; q^*(x) = n\}]$. Corollary 4.2 now follows from Theorem 4.1, using f and h.

Remark 4.3: The class $\mathcal{P}$ of co-r.e., point-decomposable sets is more extensive than the class $\mathcal{R}$ of co-r.e. infinite regressive sets; for, the isols determined by elements of $\mathcal{P}$ are closed under addition.

Remark 4.4: It would appear difficult to extend Theorem 4.1 in any very significant way. For, by means of an easy priority argument of the classical "finite injury" type, one can establish that if α is *any* infinite co-r.e. set then there exists an infinite co-r.e. set β for which no partial recursive function p exists with the property that $(\forall x)[(p\,(x) \in \delta p \;\&\; p(p_\alpha(x)) = p_\beta(x))$ or $(p_\beta(x) \in \delta p \;\&\; p(p_\beta(x)) = p_\alpha(x))]$. (Here, of course, p_α and p_β are the functions which enumerate α, β, respectively, in order of magnitude.)

5 *Recursively enumerable degrees* Our principal result, to be established in this section, is that for any non-zero recursively enumerable degree $\boldsymbol{d}$ there exist two co-r.e. retraceable sets α and β, both in $\boldsymbol{d}$, such that $\alpha | \beta \;\&\; \alpha \cup \beta$ is not regressive. In view of Corollary 4.2 above, we cannot hope to arrive at this result by forcing $\neg([\alpha] \overset{*}{\mathbf{V}} [\beta])$, as we did in the case of Theorem 3.1. With the $\overset{*}{\mathbf{V}}$ relation no longer available for spoilage, we could simply make a direct attack on regressiveness; this would involve the spoiling of obvious and easily-handled threats. However, with no more effort we can spoil *semirecursiveness*, which, as was demonstrated in [11], is a weaker property than regressiveness for the sets now under consideration. Our proof will be a "finite-injury priority argument" which combines the following three features: (a) a procedure employed by Jockusch in [10] is used to secure non-semirecursiveness of $\alpha \cup \beta$; (b) the Friedberg-Yates "permitting" technique ([21], proof of Theorem 2; see also [19]) is applied to insure $\boldsymbol{\alpha} \leqq \boldsymbol{d}$ & $\boldsymbol{\beta} \leqq \boldsymbol{d}$; and (c) special "marker coding" is used, in order to insure $\boldsymbol{d} \leqq \boldsymbol{\alpha}$ & $\boldsymbol{d} \leqq \boldsymbol{\beta}$. The entire construction is undoubtedly an instance of the type of priority procedure guaranteed to succeed in virtue of the general considerations in Soare's paper [19]; we think, however, that the proof will be more readable if we do not attempt to cast it in the very general form discussed by Soare.

Theorem 5.1 *Let* $\boldsymbol{d}$ *be a non-zero degree of unsolvability containing a recursively enumerable set. Then there exist two recursively enumerable sets* α, β *such that*

(1) $\varepsilon - \alpha \,|\, \varepsilon - \beta$,
(2) $\boldsymbol{\alpha} = \boldsymbol{\beta} = \boldsymbol{d}$,
(3) $\varepsilon - \alpha$ *and* $\varepsilon - \beta$ *are retraceable,*

and

(4) $(\varepsilon - \alpha) \cup (\varepsilon - \beta)$ *is not semirecursive.*

Proof: We shall construct α and β in stages, together with partial recursive functions p and q which retrace $\varepsilon - \alpha$ and $\varepsilon - \beta$ respectively. We shall need a specific enumeration of an r.e. set of degree $\boldsymbol{d}$; so let h be a one-to-one recursive function such that $\boldsymbol{\rho h} = \boldsymbol{d}$ & $0 \notin \rho h$. As in our proof of Theorem 3.1, we make use of movable markers; this time, however, we require *two* sequences of such markers, one for $\varepsilon - \alpha$ and the other for $\varepsilon - \beta$. We shall use markers $\Lambda_0, \Lambda_1, \Lambda_2, \ldots$ to keep track of the "approximate complement", $\varepsilon - \alpha^{(s)}$, of α; and we shall employ markers $\Sigma_0, \Sigma_1, \Sigma_2, \ldots$ to perform a corresponding service relative to β. In addition, we shall define a two-place partial recursive function g, with $\delta g = \{2n \,|\, n \in \varepsilon\} \times \varepsilon$, whose intuitive significance will be as follows: for each s, $g(2x, s) = z$ means that the markers Λ_{2x} and Σ_{2x} carry responsibility for the z-th requirement at the conclusion of Stage s. (By "the z-th requirement", we here mean the statement that φ_z^2 is *not* a total recursive function witnessing semirecursivity of $(\varepsilon - \alpha) \cup (\varepsilon - \beta)$.) Finally, we shall define, along with the construction of α, β, p, and q, a recursive function $m(s)$ with the property that $(\forall s)$ [exactly the markers in $\{\Lambda_i \,|\, i \leqq m(s)\} \cup \{\Sigma_i \,|\, i \leqq m(s)\}$ are attached to numbers at the end of Stage s]. As in our proof of Theorem 3.1, we denote by λ_i^s the number to which Λ_i is attached at the end of Stage s, if $i \leqq m(s)$; similarly, σ_i^s denotes the position of Σ_i at the end of Stage s, provided $i \leqq m(s)$. Since odd-numbered stages are divided into two "steps" each, we shall denote by $\lambda_i^{A,s}$ the position, *at the end of Step A*, of Λ_i, provided Λ_i is attached at the end of Step A of Stage s, s an odd number; similarly for the notation $\sigma_i^{A,s}$ relative to Σ_i. The construction will be so arranged that, for all s and all $i \leqq m(s)$, $\min\{\lambda_i^s, \sigma_i^s\} > i$. Because we intend to exploit the definition of semirecursiveness given in terms of two-place recursive functions, we shall need to work with the arrays $\langle \varphi_n^2 \rangle_{n=0}^{\infty}$ and $\langle \varphi_n^{2,(s)} \rangle_{n=0, s=0}^{\infty, \infty}$ in the construction and proof which follow. Our procedure for insuring property (4) was employed by Jockusch in his thesis [10]. (The procedure in question was not subsequently published by Jockusch, presumably because he did not require it for the presentation of the material in [11].) The device used to insure $\boldsymbol{d} \leqq \boldsymbol{\alpha}$ & $\boldsymbol{d} \leqq \boldsymbol{\beta}$ is a straightforward coding procedure (via marker-displacements) of a type which has been used (for example) by A. H. Lachlan in an unpublished simplification of Martin's proof ([14]) that maximal r.e. sets inhabit all r.e. degrees having jump $\mathbf{0''}$. (We remark in passing, in this connection, that Leonard Sasso has obtained a fairly general theorem concerning "trapping" constructions, in which degrees are constrained precisely. In the present instance, however, our specific coding device would appear to be easier to digest than is the framework of Sasso's general theorem; hence, we do not attempt to place Theorem 5.1 explicitly within the scope of Sasso's result (though in all likelihood it does fall there).) Our stage-by-stage construction proceeds, in detail, as follows.

Stage 0 Attach Λ_0 to 2 and Σ_0 to 1, and give 0 a *. Set $p^{(0)} = \{\langle 2, 2\rangle\}$, $q^{(0)} = \{\langle 1, 1\rangle\}$, $\alpha^{(0)} = \beta^{(0)} = \{0\}$, $m(0) = 0$, and $g(2x, 0) = x$ for all x. Then go to Stage 1.

Stage $s + 1$, $s + 1 \equiv 1 \pmod 2$ We divide our procedure into two steps.

Step A There are three principal cases to be considered. Case A1 $(\exists k)(\exists e)[0 < k \leqq m(s)$ & k is even & $e \leqq k$ & $g(k, s) = e$ & $(\forall l)[(0 < l \leqq k$ & l even & $g(l, s) = e) \Rightarrow (\varphi_e^{2,(s)}(\lambda_l^s, \sigma_l^s)$ is defined and belongs to $\{\lambda_l^s, \sigma_l^s\})]$ & $h(s + 1) \leqq \min\{\lambda_k^s, \sigma_k^s\}$ & e does not bear a * at the end of Stage $s]$. Among all such k, let $k_1, \ldots, k_t$ be those for which the corresponding e has minimal value; let $k_0 = \min\{k_1, \ldots, k_t\}$; let $k_{t+1} = \max\{k_1, \ldots, k_t\}$; and let e_0 be the e corresponding to k_0 (equivalently, to k_{t+1}).

Subcase A1(i) $\varphi_{e_0}^{2,(s)}(\lambda_{k_0}^s, \sigma_{k_0}^s) = \lambda_{k_0}^s$. Here we begin by detaching all markers Λ_k such that $k_0 \leqq k \leqq m(s)$ and all markers Σ_k such that $k_0 < k \leqq m(s)$; at the same time, we remove * from all those numbers $e > 0$ (if indeed such e can be found) for which $(\exists k)[k$ is even & $k_0 < k \leqq m(s)$ & $g(k, s) = e$ & e bears a * at the conclusion of Stage $s]$. We then give e_0 a * and attach Λ_{k_0} to $2j_0$, where $j_0 = (\mu j)[j > \max\{x \mid x \in \delta p^{(s)} \cup \delta q^{(s)}\}]$. We say that e_0 *has its * on account of* k_0. Next, we set $p_0^{(s+1)} = p^{(s)} \cup \{\langle 2j_0, \lambda_{k_0-1}^s\rangle\}$, $q_0^{(s+1)} = q^{(s)}$. Finally, we define $m_0(s + 1) = k_0$ and $g(2x, s + 1) = g(2x, s)$ for all x; then we proceed to Step B.

Subcase A1(ii) $\varphi_{e_0}^{2,(s)}(\lambda_{k_0}^s, \sigma_{k_0}^s) = \sigma_{k_0}^s$. We detach all markers Λ_k such that $k_0 < k \leqq m(s)$ and all markers Σ_k such that $k_0 \leqq k \leqq m(s)$; we remove * from all numbers $e > 0$ (if any such e exist) for which $(\exists k)[k$ is even & $k_0 < k \leqq m(s)$ & $g(k, s) = e$ & e bears a * at the conclusion of Stage $s]$; we give e_0 a * and attach Σ_{k_0} to $2l_0 + 1$, where $l_0 = (\mu l)[l > \max\{x \mid x \in \delta p^{(s)} \cup \delta q^{(s)}\}]$, and we stipulate that e_0 *has its* * *on account of* k_0; we set $p_0^{(s+1)} = p^{(s)}$ and $q_0^{(s+1)} = q^{(s)} \cup \{\langle 2l_0 + 1, \sigma_{k_0-1}^s\rangle\}$; we define $m_0(s+1) = k_0$ and $g(2x, s+1) = g(2x, s)$ for all x; then we go on to Step B.

Case A2 Case A1 does not hold as stated, but *does* hold if we delete the conjunct "$h(s+1) \leqq \min\{\lambda_k^s, \sigma_k^s\}$". Let k_0, e_0, and k_{t+1} be defined exactly as in Case A1. If $(\forall l)\,[2l > k_{t+1} \Rightarrow g(2l, s) \neq e_0]$, we define: $p_0^{(s+1)} = p^{(s)}$, $q_0^{(s+1)} = q^{(s)}$, $m_0(s+1) = m(s)$, $g(2x, s+1) = g(2x, s)$ for $x \leqq \frac{k_{t+1}}{2}$, and $g\left(2\left(\frac{k_{t+1}}{2} + x + 1\right), s+1\right) = e_0 + x$ for all x; we remove * from any number $e > 0$ which has a * on account of a number $2k$ satisfying $2k > k_{t+1}$; then we go to Step B. Otherwise, we set $p_0^{(s+1)} = p^{(s)}$, $q_0^{(s+1)} = q^{(s)}$, $m_0(s+1) = m(s)$, and $g(2x, s+1) = g(2x, s)$ for all x; then we proceed to Step B.

Case A3 Neither Case A1 nor Case A2 holds. In this event, we set $p_0^{(s+1)} = p^{(s)}$, $q_0^{(s+1)} = q^{(s)}$, $m_0(s+1) = m(s)$, and $g(2x, s+1) = g(2x, s)$ for all x; then we go to Step B.

Step B Here, there are two cases.

Case B1 $2h(s+1) + 1 \leqq m_0(s+1)$. There are two subcases.

Subcase B1(i) $\neg(\exists j)\,[2j > 2h(s+1) + 1\ \&\ g(2h(s+1), s+1)$ currently bears a * on account of $2j\ \&\ \varphi^{2,(s)}_{g(2h(s+1), s+1)}(\lambda^{A,s+1}_{2h(s+1)}, \sigma^{A,s+1}_{2h(s+1)})$ is defined and belongs to $\{\lambda^{A,s+1}_{2h(s+1)}, \sigma^{A,s+1}_{2h(s+1)}\}]$. In this event, we proceed as follows. First, we remove all markers Λ_k and Σ_k such that $2h(s+1) + 1 \leqq k \leqq m_0(s+1)$; then we remove * from those numbers $e > 0$ (if any) such that e bears a * at the end of Step A and $\neg(\exists k)$ [k is even & $k < 2h(s+1) + 1$ & e has its * on account of k]. We then set $\alpha^{(s+1)} = \alpha^{(s)} \cup \{x \mid x \in \delta p_0^{(s+1)} - \widehat{p_0^{(s+1)}}(\lambda^{A,s+1}_{2h(s+1)})\}$ and $\beta^{(s+1)} = \beta^{(s)} \cup \{x \mid x \in \delta q_0^{(s+1)} - \widehat{q_0^{(s+1)}}(\sigma^{A,s+1}_{2h(s+1)})\}$. (When our description of the construction is complete, it will be clear that at the end of Step A we have $\Lambda_{2h(s+1)}$ and $\Sigma_{2h(s+1)}$ attached to elements of $\delta p_0^{(s+1)}$, $\delta q_0^{(s+1)}$, respectively, provided $2h(s+1) + 1 \leqq m_0(s+1)$; it will, moreover, be evident that $\rho p_0^{(s+1)} \subseteqq \delta p_0^{(s+1)}$ & $\rho q_0^{(s+1)} \subseteqq \delta q_0^{(s+1)}$ & $(\forall x)[(x \in \delta p_0^{(s+1)} \Rightarrow p_0^{(s+1)}(x) \leqq x)\ \&\ (x \in \delta q_0^{(s+1)} \Rightarrow q_0^{(s+1)}(x) \leqq x)]$. Thus, our definitions of $\alpha^{(s+1)}$ and $\beta^{(s+1)}$ make sense.) We attach $\Lambda_{2h(s+1)+1}$ to $2m_0$ and $\Sigma_{2h(s+1)+1}$ to $2m_0 + 1$, where $m_0 = (\mu m)[m > \max\{x \mid x \in \delta p_0^{(s+1)} \cup \delta q_0^{(s+1)}\}]$; and we define $p^{(s+1)} = p_0^{(s+1)} \cup \{\langle 2m_0, \lambda^{A,s+1}_{2h(s+1)}\rangle\}$, $q^{(s+1)} = q_0^{(s+1)} \cup \{\langle 2m_0 + 1, \sigma^{A,s+1}_{2h(s+1)}\rangle\}$. Finally, we set $m(s+1) = 2h(s+1) + 1$; then we proceed to Stage $s+2$.

Subcase B1(ii) Otherwise. First, suppose $\varphi^{2,(s)}_{g(2h(s+1), s+1)}(\lambda^{A,s+1}_{2h(s+1)}, \sigma^{A,s+1}_{2h(s+1)}) = \lambda^{A,s+1}_{2h(s+1)}$. Remove all markers Λ_k such that $2h(s+1) \leqq k \leqq m_0(s+1)$ and all markers Σ_k such that $2h(s+1) < k \leqq m_0(s+1)$; remove * from all numbers e (including $g(2h(s+1), s+1)$) such that e bears a * at the end of Step A & $\neg(\exists k)$ [k is even & $k < 2h(s+1) + 1$ & e has its * on account of k]; give $g(2h(s+1), s+1)$ a fresh *, with the stipulation that $g(2h(s+1), s+1)$ now bears * on account of $2h(s+1)$; attach $\Lambda_{2h(s+1)}$ to $2m_0$ where m_0 is defined as in Subcase B1(i); define $\alpha^{(s+1)} = \alpha^{(s)} \cup \{x \mid \delta p_0^{(s+1)} - \widehat{p_0^{(s+1)}}(\lambda^{A,s+1}_{2h(s+1)-1})\}$ and

$\beta^{(s+1)} = \beta^{(s)} \cup \{x \mid x \in \delta q_0^{(s+1)} - \widehat{q_0^{(s+1)}}(\sigma_{2h(s+1)}^{A,s+1})\}$; set $p^{(s+1)} = p_0^{(s+1)} \cup \{\langle 2m_0, \lambda_{2h(s+1)-1}^{A,s+1}\rangle\}$ and $q^{(s+1)} = q_0^{(s+1)}$; define $m(s+1) = 2h(s+1)$; then go on to Stage $s+2$. Next, suppose $\varphi_{g(2h(s+1),\, s+1)}^{2,(s)}(\lambda_{2h(s+1)}^{A,s+1}, \sigma_{2h(s+1)}^{A,s+1}) = \sigma_{2h(s+1)}^{A,s+1}$. Remove all markers Λ_k such that $2h(s+1) < k \leqq m_0(s+1)$ and all markers Σ_k such that $2h(s+1) \leqq k \leqq m_0(s+1)$; remove * from numbers under the same conditions as in the case $\varphi_{g(2h(s+1),s+1)}^{2,(s)}(\lambda_{2h(s+1)}^{A,s+1}, \sigma_{2h(s+1)}^{A,s+1}) = \lambda_{2h(s+1)}^{A,s+1}$; give $g(2h(s+1), s+1)$ a fresh *, *on account of* $2h(s+1)$; attach $\Sigma_{2h(s+1)}$ to $2m_0+1$, where m_0 is defined as in Subcase B1(i); define $\alpha^{(s+1)} = \alpha^{(s)} \cup \{x \mid x \in \delta p_0^{(s+1)} - \widehat{p_0^{(s+1)}}(\lambda_{2h(s+1)}^{A,s+1})\}$ and $\beta^{(s+1)} = \beta^{(s)} \cup \{x \mid x \in \delta q_0^{(s+1)} - \widehat{q_0^{(s+1)}}(\sigma_{2h(s+1)-1}^{A,s+1})\}$; set $p^{(s+1)} = p_0^{(s+1)}$ and $q^{(s+1)} = q_0^{(s+1)} \cup \{\langle 2m_0+1, \sigma_{2h(s+1)-1}^{A,s+1}\rangle\}$; define $m(s+1) = 2h(s+1)$; then proceed to Stage $s+2$.

Case B2 $2h(s+1)+1 > m_0(s+1)$. Here, we define: $\alpha^{(s+1)} = \alpha^{(s)} \cup \{x \mid x \in \delta p_0^{(s+1)} - \widehat{p_0^{(s+1)}}(\lambda_{m_0(s+1)}^{A,s+1})\}$; $\beta^{(s+1)} = \beta^{(s)} \cup \{x \mid x \in \delta q_0^{(s+1)} - \widehat{q_0^{(s+1)}}(\sigma_{m_0(s+1)}^{A,s+1})\}$; $p^{(s+1)} = p_0^{(s+1)}$; $q^{(s+1)} = q_0^{(s+1)}$; $m(s+1) = m_0(s+1)$. (Again, once our description of the construction is complete, it will be clear that this definition makes good sense: $m_0(s+1)$ is so specified, in Step A, that precisely the markers Λ_i and Σ_i with $i \leqq m_0(s+1)$ are attached to numbers at the conclusion of Step A.) We then proceed to Stage $s+2$.

Stage $s+1, s+1 \equiv 0 \pmod 2$ Let $r_0 = (\mu r)[r > \max\{x \mid x \in \delta p^{(s)} \cup \delta q^{(s)}\}]$. We attach $\Lambda_{m(s)+1}$ to $2r_0$ and $\Sigma_{m(s)+1}$ to $2r_0+1$; we define $p^{(s+1)} = p^{(s)} \cup \{\langle 2r_0, \lambda_{m(s)}^{s}\rangle\}$ and $q^{(s+1)} = q^{(s)} \cup \{\langle 2r_0+1, \sigma_{m(s)}^{s}\rangle\}$; we set $\alpha^{(s+1)} = \alpha^{(s)} \cup \{x \mid x \leqq 2r_0 \;\&\; x \notin \widehat{p^{(s+1)}}(2r_0)\}$ and $\beta^{(s+1)} = \beta^{(s)} \cup \{x \mid x \leqq 2r_0+1 \;\&\; x \notin \widehat{q^{(s+1)}}(2r_0+1)\}|$; and we let $m(s+1) = m(s)+1$, $g(2x, s+1) = g(2x, s)$ for all x. Then we proceed to Stage $s+2$.

That completes our description of the construction. We define $\alpha = \bigcup_{s=0}^{\infty} \alpha^{(s)}$, $\beta = \bigcup_{s=0}^{\infty} \beta^{(s)}$, $p = \bigcup_{s=0}^{\infty} p^{(s)}$, and $q = \bigcup_{s=0}^{\infty} q^{(s)}$. It is obvious that α, β, p, and q, thus defined, are recursively enumerable, and that p and q are functions. It is, moreover, trivial to verify by induction on s that $(\forall x)[(x \in \delta p \Rightarrow p(x) \leqq x) \;\&\; (x \in \delta q \Rightarrow q(x) \leqq x)]$, that $\delta p = \{y \mid (\exists s)(\exists i)[i \leqq m(s) \;\&\; (y = \lambda_i^s \text{ or } y = \lambda_i^{A,s})]\} \;\&\; \delta q = \{y \mid (\exists s)(\exists i)[i \leqq m(s) \;\&\; (y = \sigma_i^s \text{ or } y = \sigma_i^{A,s})]\}$, and that $\delta p \subseteq \{x \mid x \text{ is even}\} \;\&\; \delta q \subseteq \{x \mid x \text{ is odd}\}$ (whence, automatically, $\varepsilon - \alpha \mid \varepsilon - \beta$ holds provided that p retraces $\varepsilon - \alpha$ and q retraces $\varepsilon - \beta$). Next, we observe (the formal proof by induction on s is trivial) that $g(2x, s)$ is non-increasing as a function of s for fixed x, that $g(2x, s)$ is non-decreasing as a function of x for fixed s, and that for each fixed value of s the function $g(2x, s)$ is *strictly increasing with x for all sufficiently large x*; moreover, it is plain that $\rho g(2x, s) = \varepsilon$ for each fixed value of s.

As the first step in establishing that the construction "settles down" in a suitable way, we shall verify that $\lim_{s\to\infty} \lambda_i^s$ and $\lim_{s\to\infty} \sigma_i^s$ exist for every i (which assertion is understood to entail that Λ_i and Σ_i are attached at the end of Stage s for all sufficiently large s), that, similarly, $\lim_{s\to\infty} \lambda_i^{A,s}$ and $\lim_{s\to\infty} \sigma_i^{A,s}$ exist for each $i > 0$, and that

$$i > 0 \Rightarrow (\lim_{s\to\infty} \lambda_i^{A,s} = \lim_{s\to\infty} \lambda_i^s \;\&\; \lim_{s\to\infty} \sigma_i^{A,s} = \lim_{s\to\infty} \sigma_i^s).$$

It is evident from our statement of the construction that Λ_0 and Σ_0 remain attached to 2 and to 1, respectively, from the end of Stage 0 onward. To proceed by induction, assume that s_0 is an even number such that

$$(\forall t)[(0 < s_0 \leqq t \,\&\, 0 < i \leqq i_0) \Longrightarrow (i \leqq \min\{m_0(t), m(t)\} \\ \&\, \lambda_i^t = \lambda_i^{A,t} = \lambda_i^{s_0} \,\&\, \sigma_i^t = \sigma_i^{A,t} = \sigma_i^{s_0})].$$

In view of our choice of s_0 and our procedure at even positive stages of the construction, the markers Λ_{i_0+1} and Σ_{i_0+1} must be found attached to numbers at the end of some stage t satisfying $s_0 \leqq t \leqq s_0 + 2$; let t_0 be the least such t. Now, if Λ_{i_0+1} or Σ_{i_0+1} is detached subsequent to the end of Stage t_0, it is immediately reattached elsewhere; moreover, clearly, neither Λ_{i_0+1} nor Σ_{i_0+1} can be detached more than once as a result of Case B1 holding at Step B of an odd stage $> s_0$ (recall that h is one-to-one). Let t_1 be the least number greater than t_0 such that for no odd stage $t \geqq t_1$ does Case B1 of Step B induce detachment of Λ_{i_0+1} or Σ_{i_0+1}; then if either Λ_{i_0+1} or Σ_{i_0+1} moves subsequent to Stage $t_1 - 1$, it does so via Case A1 of Step A (with $k_0 = i_0 + 1$). But it is clear that any move of Λ_{i_0+1} or Σ_{i_0+1} via Case A1 of Step A at an odd stage $t \geqq t_1$, with $k_0 = i_0 + 1$, results in a permanent location of that marker, *unless* $(\exists u)[u > t \,\&\, g(i_0 + 1, u) < g(i_0 + 1, t)]$. (For, the move in question causes $i_0 + 1$ to endow $g(i_0 + 1, t)$ with a *, which, since $t > \max\{t_1, s_0\}$ and $g(2x, s)$ is non-increasing in s for fixed x, is thereafter a barrier to the movement of Λ_{i_0+1} or Σ_{i_0+1} *unless* $i_0 + 1$ lowers its "g-associate" at some stage later than t.) Hence, if we let t_2 be a number $\geqq t_1$ such that $(\forall t)[t \geqq t_2 \Longrightarrow g(i_0 + 1, t) = g(i_0 + 1, t_2)]$ then neither Λ_{i_0+1} nor Σ_{i_0+1} can move more than once subsequent to Stage t_2. So, since it is clear from the construction that $\lambda_{i_0+1}^{A,s} = \lambda_{i_0+1}^{s}$ and $\sigma_{i_0+1}^{A,s} = \sigma_{i_0+1}^{s}$ provided both Λ_{i_0+1} and Σ_{i_0+1} remain attached, *with no movement*, throughout Stage s, we see that

$$(t \geqq t_3 \,\&\, 0 < i \leqq i_0 + 1) \Longrightarrow (i \leqq \min\{m_0(t), m(t)\} \,\&\, \lambda_i^t = \lambda_i^{A,t} = \lambda_i^{t_3} \,\&\, \sigma_i^{A,t} = \sigma_i^{t_3})$$

where t_3 is a number greater than t_2 such that neither Λ_{i_0+1} nor Σ_{i_0+1} moves subsequent to the end of Stage $t_3 - 1$. That completes the induction step from i_0 to $i_0 + 1$. From now on, we use λ_i to denote $\lim_{s\to\infty} \lambda_i^s$ and σ_i to denote $\lim_{s\to\infty} \sigma_i^s$; and we shall denote $\lim_{s\to\infty} g(2x, s)$ by $g(2x)$. It is obvious from the construction that

$$(\forall s)(\forall i)[0 < i \leqq m(s) \Longrightarrow (\langle \lambda_i^s, \lambda_{i-1}^s\rangle \in p^{(s)} \,\&\, \langle \sigma_i^s, \sigma_{i-1}^s\rangle \in q^{(s)})];$$

so, since $p = \bigcup_{s=0}^{\infty} p^{(s)}$ & $q = \bigcup_{s=0}^{\infty} q^{(s)}$ and since $\lambda_0^s = 2$ for all s and $\sigma_0^s = 1$ for all s, we see that p retraces the set $\{\lambda_i | i \in \varepsilon\}$ and q retraces the set $\{\sigma_i | i \in \varepsilon\}$. But, in view of the construction of the sequences $\langle \alpha^{(s)}\rangle_{s=0}^{\infty}$ and $\langle \beta^{(s)}\rangle_{s=0}^{\infty}$ and the definitions of α and β as $\bigcup_{s=0}^{\infty} \alpha^{(s)}$, $\bigcup_{s=0}^{\infty} \beta^{(s)}$, respectively, we have: $\varepsilon - \alpha = \{\lambda_i | i \in \varepsilon\}$, $\varepsilon - \beta = \{\sigma_i | i \in \varepsilon\}$. Thus, p retraces $\varepsilon - \alpha$ and q retraces $\varepsilon - \beta$. Since $\delta p | \delta q$, we conclude that $\varepsilon - \alpha | \varepsilon - \beta$; thus (1) and (3) are established.

To prove (4), we must first show that $(\forall e)[\{2j | (\exists s)[g(2j, s) = e]\}$ is finite]. We shall verify this by induction on e, noting that since no attempt is ever made to move Λ_0 (to move Σ_0) from 2 (from 1) subsequent to Stage 0, we have $(\forall s)(\forall j)[g(2j, s) = 0 \Rightarrow j = 0]$. Assume that $(\forall e)\,[e \leqq e_0 \Rightarrow \{2j \mid (\exists s)[g(2j, s) = e]\}$ is finite], and let $j_0 = (\mu j)[(\forall s)(\forall k)[k \geqq j \Rightarrow g(2k, s) > e_0]]$. Suppose, for an argument by contradiction, that $\{j \mid (\exists s)[g(2j, s) = e_0 + 1\}$ is infinite. Let $s_0 = (\mu s)[(\forall j)(\forall t)[(t \geqq s \;\&\; j \leqq j_0) \Rightarrow (h(t) > j_0 \;\&\; g(2j, t) = g(2j, s) \;\&\; 2j \leqq m_0(t) \;\&\; \lambda_{2j}^{t} = \lambda_{2j}^{s} \;\&\; \sigma_{2j}^{t} = \sigma_{2j}^{s})]]$. Observe that since $\rho g(2x, s) = \varepsilon$ for each fixed value of s, and since g is non-decreasing in x for each fixed s and non-increasing in s for each fixed x, we must have $(\forall s)[s \geqq s_0 \Rightarrow g(2j_0, s) = g(2j_0) = e_0 + 1]$; moreover, in view of our requirements for changing (some of) the values of g during Step A of an odd stage, we see that there must be an initial stage $s_1 \geqq s_0$ such that $2(j_0 + 1) \leqq m(s_1)$ & $g(2(j_0 + 1), s_1) = e_0 + 1 \;\&\; (\forall l)[(0 < l \leqq j_0 + 1 \;\&\; g(2l, s_1) = e_0 + 1) \Rightarrow \varphi_{e_0+1}^{2,(s_1)}(\lambda_{2l}^{s_1}, \sigma_{2l}^{s_1})$ is defined and belongs to $\{\lambda_{2l}^{s_1}, \sigma_{2l}^{s_1}\}]$. We now claim that $\lambda_{2(j_0+1)}^{s_1} = \lambda_{2(j_0+1)}$, that $\sigma_{2(j_0+1)}^{s_1} = \sigma_{2(j_0+1)}$, and that $(\forall t)[t > s_1 \Rightarrow h(t) > \min\{\lambda_{2(j_0+1)}^{s_1}\ \sigma_{2(j_0+1)}^{s_1}\}]$. If $\lambda_{2(j_0+1)}^{s_1} \neq \lambda_{2(j_0+1)}$ or $\sigma_{2(j_0+1)}^{s_1} \neq \sigma_{2(j_0+1)}$, then there is a *smallest* number t, say t_0, such that t is odd and $t > s_1$ and either $\Lambda_{2(j_0+1)}$ or $\Sigma_{2(j_0+1)}$ is detached during Stage t. Suppose that $\Lambda_{2(j_0+1)}$ is detached during Step A of Stage t_0. Then $h(t_0) \leqq \min\{\lambda_{2(j_0+1)}^{t_0-1}, \sigma_{2(j_0+1)}^{t_0-1}\} = \min\{\lambda_{2(j_0+1)}^{s_1}, \sigma_{2(j_0+1)}^{s_1}\}$, and (since $t_0 > s_1 \geqq s_0$) $e_0 + 1$ receives a * *on account of* $2(j_0 + 1)$; this * is never subsequently lost by $e_0 + 1$, since such a loss could only occur if $h(u) = j_0$ for some $u \geqq t_0$ (which equation is impossible in view of the fact that $t_0 > s_0$). Since a permanently-held * on $e_0 + 1$ contradicts the infinitude of $\{j \mid (\exists s)[g(2j, s) = e_0 + 1]\}$, we conclude that $\Lambda_{2(j_0+1)}$ *cannot*, in fact, be detached during Step A of Stage t_0. For precisely parallel reasons, $\Sigma_{2(j_0+1)}$ cannot be detached during Step A of Stage t_0. Suppose, on the other hand, that $\Lambda_{2(j_0+1)}$ or $\Sigma_{2(j_0+1)}$ is detached during Step B of Stage t_0. Since $t_0 > s_0$, this cannot happen because Subcase B1(i) causes Λ_{2j_0+1} and Σ_{2j_0+1} to move; hence, it must occur via Subcase B1(ii). But, then, $e_0 + 1$ receives a * *on account of* $2(j_0 + 1)$; since this * is never subsequently lost, we again have a contradiction to the infinitude of $\{j \mid (\exists s)[g(2j, s) = e_0 + 1]\}$. Thus we are forced to conclude that, in point of fact, $\lambda_{2(j_0+1)}^{s_1} = \lambda_{2(j_0+1)}$ and $\sigma_{2(j_0+1)}^{s_1} = \sigma_{2(j_0+1)}$. It now follows that *for every* $t > s_1$ we have: $2(j_0 + 1) \leqq m(t) \;\&\; (\forall l)[(0 < l \leqq j_0 + 1 \;\&\; g(2l, t) = e_0 + 1) \Rightarrow \varphi_{e_0+1}^{2,(t)}(\lambda_{2l}^{t}, \sigma_{2l}^{t})$ is defined and is a member of $\{\lambda_{2l}^{t}, \sigma_{2l}^{t}\} = \{\lambda_{2l}, \sigma_{2l}\}]$. Hence, further, $t > s_1 \Rightarrow h(t) > \min\{\lambda_{2(j_0+1)}^{s_1}, \sigma_{2(j_0+1)}^{s_1}\}$; for otherwise, one of the markers $\Lambda_{2(j_0+1)}$, $\Sigma_{2(j_0+1)}$ would be obliged to move at a stage later than s_1. Now let $s_2 = (\mu s)[s > s_1 \;\&\; g(2(j_0 + 2), s) = e_0 + 1 \;\&\; 2(j_0 + 2) = m(s) \;\&\; \varphi_{e_0+1}^{2,(s)}(\lambda_{2(j_0+2)}^{s}, \sigma_{2(j_0+2)}^{s})$ is defined and belongs to $\{\lambda_{2(j_0+2)}^{s}, \sigma_{2(j_0+2)}^{s}\}]$. We claim that $\lambda_{2(j_0+2)}^{s_2} = \lambda_{2(j_0+2)}$, $\sigma_{2(j_0+2)}^{s_2} = \sigma_{2(j_0+2)}$, and $(\forall t)[t > s_2 \Rightarrow h(t) > \min\{\lambda_{2(j_0+2)}^{s_2}, \sigma_{2(j_0+2)}^{s_2}\}]$. If $\lambda_{2(j_0+2)}^{s_2} \neq \lambda_{2(j_0+2)}$ or $\sigma_{2(j_0+2)}^{s_2} \neq \sigma_{2(j_0+2)}$, then there is a *first* stage t, say t_1, such that $t > s_2$ and one of the markers $\Lambda_{2(j_0+2)}$, $\Sigma_{2(j_0+2)}$ is detached during Stage t. Suppose, e.g., that $\Lambda_{2(j_0+2)}$ is detached during Step A of Stage t_1. Then $\min\{\lambda_{2(j_0+1)}^{t_1-1}, \sigma_{2(j_0+1)}^{t_1-1}\} = \min\{\lambda_{2(j_0+1)}, \sigma_{2(j_0+1)}\} < h(t_1) \leqq \min\{\lambda_{2(j_0+2)}^{t_1-1}, \sigma_{2(j_0+2)}^{t_1-1}\} = \min\{\lambda_{2(j_0+2)}^{s_2}, \sigma_{2(j_0+2)}^{s_2}\}$ and $e_0 + 1$ receives a * *on account of* $2(j_0 + 2)$; since, as an obvious consequence of the construction, we have $2(j_0 + 1) + 1 \leqq \min\{\lambda_{2(j_0+1)}^{t_1-1}, \sigma_{2(j_0+1)}^{t_1-1}\}$, and since $g(2(j_0 + 2), t_1 - 1) = g(2(j_0 + 2))$, this * remains attached to $e_0 + 1$ forever, in contradiction to the assumed infinitude of $\{j \mid (\exists s)[g(2j, s) = e_0 + 1]\}$. We conclude that, in fact, $\Lambda_{2(j_0+2)}$ cannot be detached during Step A of Stage t_1. By a precisely parallel argument, we see that $\Sigma_{2(j_0+2)}$ is likewise restrained from detachment during Step A of Stage t_1. If, on the other hand, $\Lambda_{2(j_0+2)}$ or $\Sigma_{2(j_0+2)}$ is detached during Step B of Stage t_1, then, since $2(j_0 + 1) + 1 \leqq \min\{\lambda_{2(j_0+1)}^{t_1-1}, \sigma_{2(j_0+1)}^{t_1-1}\}$, this must occur under Subcase B1(ii). But then $e_0 + 1$ receives a * *on account of* $2(j_0 + 2)$; since this * cannot subsequently be lost by $e_0 + 1$, we have a contradiction, once again, to the infinitude of $\{j \mid (\exists s)[g(2j, s) = e_0 + 1]\}$. So, we must conclude that $\lambda_{2(j_0+2)}^{t_1} = \lambda_{2(j_0+2)}$ & $\sigma_{2(j_0+2)}^{t_1} = \sigma_{2(j_0+2)}$. It follows that $(\forall t)[t > t_1 \Rightarrow \varphi_{e_0+1}^{2(t)}(\lambda_{2(j_0+2)}^{t}, \sigma_{2(j_0+2)}^{t})]$ is defined and belongs to $\{\lambda_{2(j_0+2)}^{t}, \sigma_{2(j_0+2)}^{t}\} = \{\lambda_{2(j_0+2)}, \sigma_{2(j_0+2)}\}$; whence, if $e_0 + 1$ is not eventually to receive a permanently-held *, we must conclude that

$$(\forall t)[t > t_1 \Rightarrow h(t) > \min\{\lambda_{2(j_0+2)}^{t_1}, \sigma_{2(j_0+2)}^{t_1}\} = \min\{\lambda_{2(j_0+2)}, \sigma_{2(j_0+2)}\}].$$

Now replace "s_2" by "s_3", "s_1" by "s_2", and "$j_0 + 2$" by "$j_0 + 3$" in the above definition of s_2, and repeat the argument; if this procedure is iterated to infinity then, since $\min\{\lambda_k, \sigma_k\} < \min\{\lambda_{k+1}, \sigma_{k+1}\}$ holds for all k, we obtain effectively computable sequences $\langle u_i \rangle_{i=0}^{\infty}$ and $\langle w_i \rangle_{i=0}^{\infty}$ such that

$$(\forall i)[u_i < u_{i+1}] \ \& \ (\forall x)[x \leqq u_i \Rightarrow (x \in \rho h \Leftrightarrow x \in \{h(w) \mid w \leqq w_i\})].$$

But this means that ρh is recursive: contradiction. Hence, $\{j \mid (\exists s)[g(2j, s) = e_0 + 1]\}$ is finite. By induction, then, $\{j \mid (\exists s)[g(2j, s) = e]\}$ is finite for all e. We are now in a position to verify (4). If (4) is false, then there is a *smallest* number e, say e_0, such that

$$\begin{aligned}&\varphi_e^2 \text{ is total } \& \ (\forall x)(\forall y)[\varphi_e^2(x, y) \in \{x, y\} \ \& \ [(x \in (\varepsilon - \alpha) \cup (\varepsilon - \beta) \\ &\vee y \in (\varepsilon - \alpha) \cup (\varepsilon - \beta)) \Rightarrow \varphi_e^2(x, y) \in (\varepsilon - \alpha) \cup (\varepsilon - \beta)]].\end{aligned}$$

We may safely assume $e_0 > 0$, either by *requiring* $e > 0$ in the preceding statement, or else by insisting upon the "usual" enumeration $\langle \varphi_e^2 \rangle_{e=0}^{\infty}$ (i.e., that of [13]), in which $\varphi_0^2 = \emptyset$. Let $j_0 = \max\{j \mid \alpha(2j) = e_0\}$; and let

$$\begin{aligned}u_0 = (\mu u)[(\forall s)[s \geqq u \Rightarrow (2j_0 \leqq m(s) \ \& \ \lambda_{2j_0}^s = \lambda_{2j_0} \ \& \ \sigma_{2j_0}^s = \sigma_{2j_0} \ \& \ g(2j_0, s) = e_0 \ \& \\ (\forall l)[(0 < l \leqq j_0 \ \& \ g(2l, u) = e_0) \Rightarrow [g(2l, u) = g(2l) \ \& \ (\varphi_{e_0}^{2,(s)}(\lambda_{2l}^s, \sigma_{2l}^s) = \\ \varphi_{e_0}^{2,(s)}(\lambda_{2l}, \sigma_{2l}) \text{ is defined and belongs to } \{\lambda_{2l}^s, \sigma_{2l}^s\} = \{\lambda_{2l}, \sigma_{2l}\})]])]].\end{aligned}$$

If e_0 does not already bear a * at the beginning of Stage $u_0 + 1$, then it (if not some even *smaller* e) must receive one at Step A of the first odd stage u, say u_1, such that $u \geqq u_0 + 1$ (since otherwise it would be the case that $g(2(j_0 + 1)) = e_0$); the reception of this *, in view of our specifications of j_0 and u_0, *must be on account of some number* $\leqq 2j_0$, and so either Λ_{2j_0} or Σ_{2j_0} (or both) must be detached during Stage u_1: contradiction (to the choice of u_0). So, in fact, e_0 must bear a * at the outset of Stage $u_0 + 1$. Let w_0 be that stage prior to Stage $u_0 + 1$ during which *the particular * in question* became attached to e_0. (Possibly, $w_0 = u_0$.) Since $e_0 > 0$, we have $w_0 > 0$, and the attachment in question had to be *on account of* some number $2l_0$ where $l_0 \leqq j_0$ & $g(2l_0, w_0) = e_0$. But then $e_0 = \lim_{s \to \infty} g(2l_0, s) = g(2l_0)$, since otherwise Case A2 of Step A would have forced removal of * from e_0 during some stage v such that $w_0 < v \leqq u_0$. If the * in question was received by e_0

during *Step* A of Stage w_0 then we must have: $2l_0 \leqq m(w_0 - 1)$ & $\varphi^2_{e_0}(\lambda^{w_0-1}_{2l_0}, \sigma^{w_0-1}_{2l_0})$ is defined and is *not* a member of $(\varepsilon - \alpha) \cup (\varepsilon - \beta)$. But since e_0 *does not lose the * in question, which it has on account of* $2l_0$, *between the end of Step* A *of Stage* w_0 *and the end of Stage* u_0, we see from the construction that Λ_{2l_0} *and* Σ_{2l_0} *suffer no detachments either during Step* B *of Stage* w_0 *or* during stages v such that $w_0 < v \leqq u_0$; hence, $\varphi^2_{e_0}(\lambda^{w_0-1}_{2l_0}, \sigma^{w_0-1}_{2l_0}) \notin (\varepsilon - \alpha) \cup (\varepsilon - \beta)$, *although* either $\lambda^{w_0-1}_{2l_0} = \lambda_{2l_0} \in \varepsilon - \alpha$ or $\sigma^{w_0-1}_{2l_0} = \sigma_{2l_0} \in \varepsilon - \beta$ is true: contradiction. If, on the other hand, the * in question was received during Step B of Stage w_0, we must have: $2l_0 \leqq m_0(w_0)$ & $\varphi^2_{e_0}(\lambda^{A,w_0}_{2l_0}, \sigma^{A,w_0}_{2l_0})$ is defined and is *not* a member of $(\varepsilon - \alpha) \cup (\varepsilon - \beta)$. But then, as in the case of reception via Step A, we obtain a contradiction. We must conclude, therefore, that our "minimal counterexample", $\varphi^2_{e_0}$, is not in fact a counterexample; i.e., $(\varepsilon - \alpha) \cup (\varepsilon - \beta)$ is not semirecursive. It now remains only to verify the degree-inequalities (2). To see that $\boldsymbol{\alpha} \leqq \boldsymbol{d}$, assume that (using $\boldsymbol{d}$) we have computed the first $n + 1$ elements $\lambda_0, \lambda_1, \ldots, \lambda_n$ of $\varepsilon - \alpha$. To find the next element, λ_{n+1}, of $\varepsilon - \alpha$, we wait for a stage s_n such that $(\forall s)[(s \geqq s_n \;\&\; i \leqq n) \Longrightarrow (i \leqq m(s) \;\&\; \lambda^s_i = \lambda_i)] \;\&\; n + 1 \leqq m(s_n)$. If $n + 1$ is odd, say, $n + 1 = 2m + 1$, then $\lambda^{s_n}_{n+1} = \lambda_{n+1}$ *unless* $m = h(k)$ holds for some number $k > s_n$. If $n + 1 = 2m + 1$ & $(\exists k)[k > s_n \;\&\; h(k) = m]$, then $\lambda^{k_1}_{n+1} = \lambda_{n+1}$ where $k_1 = (\mu k)[k > s_n \;\&\; h(k) = m]$. If, on the other hand, $n + 1$ is even, then $\lambda^{s_n}_{n+1} = \lambda_{n+1}$ unless Λ_{n+1} gets moved at an odd stage $t > s_n$ via Case A1 of Step A with $k_0 = n + 1$ or via Subcase B1(ii) of Step B with $\frac{n+1}{2} = h(t)$. But if such a move has not occurred by the time all members of ρh which are $\leqq \min\{\lambda^{s_n}_{n+1}, \sigma^{s_n}_{n+1}\}$ have appeared in ρh, it can *never* occur. So, we wait for a stage $t' > s_n$ such that $(\forall t)[t > t' \Longrightarrow h(t) > \min\{\lambda^{s_n}_{n+1}, \sigma^{s_n}_{n+1}\}]$. If Λ_{n+1} *has* moved from $\lambda^{s_n}_{n+1}$ by Stage t', we find its next location after $\lambda^{s_n}_{n+1}$ and repeat the foregoing observations. Since $\lim_{s\to\infty} \lambda^s_{n+1}$ exists, we shall locate it in this fashion after finitely many tries. So, since the entire procedure is clearly recursive in $\boldsymbol{d}$ (uniformly in n), we get $\boldsymbol{\alpha} = \boldsymbol{\varepsilon} - \boldsymbol{\alpha} \leqq \boldsymbol{d}$. By a parallel argument, $\boldsymbol{\beta} = \boldsymbol{\varepsilon} - \boldsymbol{\beta} \leqq \boldsymbol{d}$. Finally, it is clear from the construction that, for every n, we have

$$n \in \rho h \Longleftrightarrow n \in \{h(u) \mid u \leqq (\mu s)[2n + 1 \leqq m(s) \;\&\; \lambda^s_{2n+1} = \lambda_{2n+1}]\} \Longleftrightarrow$$
$$n \in \{h(u) \mid u \leqq (\mu t)[2n + 1 \leqq m(t) \;\&\; \sigma^t_{2n+1} = \sigma_{2n+1}]\};$$

hence $\boldsymbol{d} \leqq \boldsymbol{\varepsilon} - \boldsymbol{\alpha} = \boldsymbol{\alpha}$ & $\boldsymbol{d} \leqq \boldsymbol{\varepsilon} - \boldsymbol{\beta} = \boldsymbol{\beta}$. The proof of Theorem 5.1 is complete.

To conclude this section, we offer a few remarks on relativization. Let an infinite subset β of ε be called α-*retraceable*, α = some fixed subset of ε, *just in case*

$$(\exists e)[\beta \subseteqq \delta\varphi^\alpha_e \;\&\; (\forall n)[\varphi^\alpha_e(p_\beta(n+1)) = p_\beta(n)] \;\&\; \varphi^\alpha_e(p_\beta(0)) = p_\beta(0)].$$

The notions of α-*semirecursiveness* and α-*regressiveness* are correspondingly obvious relativizations to α of the ordinary semirecursivity and regressiveness concepts. The proof of Theorem 5.1 relativizes routinely to a proof of

Theorem 5.1^R *Let* $\boldsymbol{c}$, $\boldsymbol{d}$ *be degrees of unsolvability such that* $\boldsymbol{c} \leqq \boldsymbol{d}$ & $\boldsymbol{c} \neq \boldsymbol{d}$ & $\boldsymbol{d}$ *is* r.e. *in* $\boldsymbol{c}$; *and let* $\alpha \in \boldsymbol{c}$. *Then there exist* α*-retraceable sets* γ_1, γ_2 *such that* $\gamma_1 | \gamma_2$, γ_1 *is* r.e. *in* α, γ_2 *is* r.e. *in* α, $\boldsymbol{\gamma}_1 = \boldsymbol{\gamma}_2 = \boldsymbol{d}$ & $\gamma_1 \cup \gamma_2$ *is not* α*-semirecursive.*

Since [11], Theorem 3.2 relativizes straightforwardly, the failure of α-semirecursiveness for $\gamma_1 \cup \gamma_2$ in Theorem 5.1^R implies the failure of $[\gamma_1 \cup \gamma_2]$ to be an α-regressive isol. This of course falls short of giving an actual extension of Theorem 5.1, since we have γ_i merely α-retraceable for $i = 1, 2$, rather than retraceable. On the other hand, Theorem 5.1^R does pertain to a *class of degrees* not covered by the combination of Theorems 3.1 and 5.1, namely, those degrees $\boldsymbol{d}$ which are incomplete, non-r.e., and r.e. *in* some smaller degree. The existence of such degrees $\boldsymbol{d}$ is doubtless known on the basis of direct construction; however, we shall point out a very simple method of obtaining such $\boldsymbol{d}$ from two theorems, due to Yates, which are most certainly well known. In the first place, if we relativize completely the proof of [22], Theorem 1 we obtain for any fixed degree $\boldsymbol{a}$ a pair α, β of sets such that

$$\boldsymbol{\alpha} \geqq \boldsymbol{a} \;\&\; \boldsymbol{\beta} \geqq \boldsymbol{a} \;\&\; \boldsymbol{\alpha} \neq \boldsymbol{a} \;\&\; \boldsymbol{\beta} \neq \boldsymbol{a} \;\&\; \alpha \text{ is r.e. in } \boldsymbol{a} \;\&\; \beta \text{ is r.e. in } \boldsymbol{a}$$
$$\& \; (\forall \boldsymbol{b})\,[(\boldsymbol{b} \leqq \boldsymbol{\alpha} \;\&\; \boldsymbol{b} \leqq \boldsymbol{\beta}) \Longrightarrow \boldsymbol{b} \leqq \boldsymbol{a}].$$

Now, by [23] the degree $\boldsymbol{a}$ with which we start can be assumed to satisfy the condition

$$\mathbf{0} \leqq \boldsymbol{a} \leqq \mathbf{0}' \;\&\; \mathbf{0} \neq \boldsymbol{a} \;\&\; \mathbf{0}' \neq \boldsymbol{a} \;\&\; (\forall \boldsymbol{b})\,[(\boldsymbol{b} \text{ an r.e. degree}$$
$$\& \; [\boldsymbol{b} \leqq \boldsymbol{a} \vee \boldsymbol{a} \leqq \boldsymbol{b}]) \Longrightarrow (\boldsymbol{b} = \mathbf{0} \vee \boldsymbol{b} = \mathbf{0}')].$$

If in fact we start with such an $\boldsymbol{a}$, it is easy to see that at least one of the degrees $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ arising from the relativization of [22], Theorem 1 to $\boldsymbol{a}$ must be both incomplete and non-r.e.

6 *Conclusion* It was shown by Hassett, in [9], that the class of well-behaved number-theoretic functions of two arguments which fail to map $\Lambda_R^\infty \times \Lambda_R^\infty$ into Λ_R is very extensive; similar results, set within a very general framework, have since been obtained by Ellentuck in his recent paper [7]. One could attempt to extend the results of the present article to the wider context of [7]; [9]; however, it seems to us more immediately interesting to give a full answer to the question only very fragmentarily dealt with in the preceding sections: is the failure of additive closure for Λ_R completely degree-independent? In particular, what about the class of *minimal* degrees? (The proofs given in sections 3 and 5, above, are applications of standard techniques; this is especially so in regard to section 5. *Minimal* $\boldsymbol{d}$, on the other hand, might be resistant to conventional procedures.)

Conjecture $\quad \Lambda_R + \Lambda_R \nsubseteq \Lambda_R$ *is totally independent of degree.*

REFERENCES

[1] Barback, J., "Two notes on regressive isols," *Pacific Journal of Mathematics*, vol. 16 (1966), pp. 407-420.

[2] Dekker, J. C. E., "The minimum of two regressive isols," *Mathematische Zeitschrift*, vol. 83 (1964), pp. 345-366.

[3] Dekker, J. C. E., "Infinite series of isols," *Proceedings of the Symposia on Pure Mathematics of the American Mathematical Society*, vol. 5 (1962), pp. 77-96.

[4] Dekker, J. C. E., and J. Myhill, "Recursive equivalence types," *University of California Publication in Mathematics* (New Series), vol. 3 (1960), pp. 67-214.

[5] Dekker, J. C. E., and J. Myhill, "Retraceable sets," *Canadian Journal of Mathematics*, vol. 10 (1958), pp. 357-373.

[6] Ellentuck, E., "Universal isols," *Mathematische Zeitschrift*, vol. 98 (1967), pp. 1-8.

[7] Ellentuck, E., "On the form of functions which preserve regressive isols," *Compositio Mathematica*, vol. 26 (1973), pp. 283-302.

[8] Friedberg, R., "A criterion for completeness of degrees of unsolvability," *The Journal of Symbolic Logic*, vol. 22 (1957), pp. 159-160.

[9] Hassett, M. J., "A closure property of regressive isols," *Rocky Mountain Journal of Mathematics*, vol. 2 (1972), pp. 1-24.

[10] Jockusch, C. G., Jr., *Ph.D. Thesis*, Massachusetts Institute of Technology, 1966.

[11] Jockusch, C. G., Jr., "Semirecursive sets and positive reducibility," *Transactions of the American Mathematical Society*, vol. 131 (1968), pp. 420-436.

[12] Jockusch, C. G., Jr., and T. G. McLaughlin, "Countable retracing functions and Π_2^0 predicates," *Pacific Journal of Mathematics*, vol. 30 (1969), pp. 67-93.

[13] Kleene, S. C., *Introduction to Metamathematics*, Van Nostrand, Princeton (1952).

[14] Martin, D. A., "Classes of recursively enumerable sets and degrees of unsolvability," *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, vol. 12 (1966), pp. 295-310.

[15] McLaughlin, T. G., "Trees and isols, Part I," *Rocky Mountain Journal of Mathematics*, vol. 5 (1975), pp. 401-418.

[16] McLaughlin, T. G., "Strong reducibility on hypersimple sets," *Notre Dame Journal of Formal Logic*, vol. VI (1965), pp. 229-234.

[17] Rogers, H., Jr., *Theory of Recursive Functions and Effective Computability*, McGraw-Hill Book Co., New York (1967).

[18] Sacks, G. E., *Degrees of Unsolvability*, Princeton University Press, Princeton (1963).

[19] Soare, R. I., "The Friedberg-Muchnik theorem re-examined," *Canadian Journal of Mathematics*, vol. 24 (1972), pp. 1070-1078.

[20] Yates, C. E. M., "Recursively enumerable sets and retracing functions," *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, vol. 8 (1962), pp. 331-345.

[21] Yates, C. E. M., "Three theorems on the degrees of recursively enumerable sets," *Duke Mathematical Journal*, vol. 32 (1965), pp. 461-468.

[22] Yates, C. E. M., "A minimal pair of recursively enumerable degrees," *The Journal of Symbolic Logic*, vol. 31 (1966), pp. 159-168.

[23] Yates, C. E. M., "Recursively enumerable degrees and the degrees less than o′," in *Sets, Models and Recursion Theory*, J. N. Crossley, ed., North Holland Co., Amsterdam (1967), pp. 264-271.

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