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Notre Dame Journal of Formal Logic
Volume XVIII, Number 3, July 1977
NDJFAM

SEMANTICS FOR S4.03

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Modal system S4.03, as shown in [2], is axiomatized by appending

I1 *ALCLpqCLMLqp*

to some base for S4 containing a primitive rule of necessitation. The purpose of this paper is to provide semantics for this system and its corresponding non-Lewis counterpart, K1.1.5, which is also introduced in [2]. The methods, lemmata, and terminology which we shall employ are taken from Hughes and Cresswell in [3], pp. 150-159.

In [3], p. 74, Hughes and Cresswell define an S4-model as an ordered triple $\langle W, R, V\rangle$, where W is a set of possible worlds, R is a reflexive and transitive accessibility relation holding among the members of W, and V is a value assignment satisfying the conditions stated in [3], p. 73. Now in order to construct a model for S4.03, we need only impose the additional stipulation that the accessibility relation in an S4-model be what we shall call "disjunctively symmetrical." We say that R is disjunctively symmetrical iff for every $w_i \in W$ there exists a w_j such that $w_i R w_j$ and for any $w_k, w_l \in W$ if $w_i R w_k$ and $w_j R w_l$, then either $w_k R w_i$ or $w_l R w_k$. Since modal system S4.03 is a proper extension of S4, we can demonstrate the soundness of our interpretation by simply showing that I1 is S4.03-logically true. This is accomplished in the following way.

Assume for the sake of reductio that $V(ALCpqCLMLqp, w_i) = 0$. Clearly it follows that

(1) $$V(LCLpq, w_i) = 0$$

and

(2) $$V(CLMLqp, w_i) = 0.$$

From (2) we obtain

(3) $$V(LMLq, w_i) = 1$$
(4) $$V(p, w_i) = 0.$$

Received April 4, 1975

Now, since R is disjunctively symmetrical, it follows that there exists a w_j such that $w_i R w_j$. Hence from (3) we obtain

(5) $$V(MLq, w_j) = 1.$$

But from (1) we obtain

(6) $$V(CLpq, w_k) = 0$$

and so

(7) $$V(Lp, w_k) = 1$$

and

(8) $$V(q, w_k) = 0.$$

Now it follows from (5) that

(9) $$V(Lq, w_l) = 1.$$

Again, in view of the consideration that R is disjunctively symmetrical, it must be the case that either $w_k R w_i$ or $w_l R w_k$. If $w_k R w_i$, then we obtain from (7)

(10) $$V(p, w_i) = 1$$

which is inconsistent with (4). If, on the other hand, $w_l R w_k$, then it follows from (9) that

(11) $$V(q, w_k) = 1.$$

But this contradicts (8). Consequently, either way we have a contradiction and so $V(ALCLpqCLMLqp, w_i) = 1$.

We now turn to the completeness theorem for S4.03. To deal with this system we must require that R be not only reflexive and transitive, but disjunctively symmetrical as well. This means that we have to add to the S4 proof that Theorem 2 holds for L (*cf.* [3], pp. 157-158), a proof that (1) for any $\Gamma_i \in \Gamma$ there exists a Γ_j subordinate to Γ_i; and (2) for any Γ_k, $\Gamma_l \in \Gamma$ if Γ_k is subordinate to Γ_i and Γ_l is subordinate to Γ_j, then either if $L\beta \in \Gamma_k$ then $\beta \in \Gamma_i$ or if $L\gamma \in \Gamma_l$ then $\gamma \in \Gamma_k$. We proceed with a proof of (1) and (2) in the following fashion.

(1) Let $L\beta \in \Gamma_i$, then since $CL\beta M\beta$ is obviously a thesis of S4.03, it follows that $CL\beta M\beta \in \Gamma_i$. Thus, we have (by Lemma 3) that $M\beta \in \Gamma_i$. Hence (by construction of Γ) there exists a Γ_j subordinate to Γ_i such that $\beta \in \Gamma_j$.
(2) Alternatively, what we need to prove here is that for any Γ_k, $\Gamma_l \in \Gamma$ if Γ_k is subordinate to Γ_i and Γ_l is subordinate to Γ_j, then if both $L\beta \in \Gamma_k$ and $L\gamma \in \Gamma_l$, then either $\gamma \in \Gamma_k$ or $\beta \in \Gamma_i$. Suppose that both $L\beta \in \Gamma_k$ and $L\gamma \in \Gamma_l$. Now since $ALCL\beta\gamma CLML\gamma\beta$ is a thesis of S4.03, we have $ALCL\beta\gamma CLML\gamma\beta \in \Gamma_i$ and so either $LCL\beta\gamma \in \Gamma_i$ or $CLML\gamma\beta \in \Gamma_i$.

If $LCL\beta\gamma \in \Gamma_i$, then (by construction of Γ_k) we have $CL\beta\gamma \in \Gamma_k$. But $L\beta \in \Gamma_k$ (by hypothesis); hence (by Lemma 3) $\gamma \in \Gamma_k$.

If $CLML\gamma\beta \epsilon \Gamma_i$, then since $L\gamma \epsilon \Gamma_l$ (by hypothesis) it must be the case that $ML\gamma \epsilon \Gamma_j$; and so $LML\gamma \epsilon \Gamma_i$. Thus we have (by Lemma 3) that $\beta \epsilon \Gamma_i$.

On either assumption then we have either $\gamma \epsilon \Gamma_k$ or $\beta \epsilon \Gamma_i$, and so the completeness theorem for S4.03 has been demonstrated.

In [2], it is shown that modal system K1.1.5 is axiomatized by appending

K1 $CLMpMLp$

to the basis of S4.03. Utilizing the same procedures sketched in [1], it is an easy matter to construct a semantic model for K1.1.5. We say that $\langle W, R, \vee \rangle$ is a K1.1.5-model if and only if (a) it is an S4.03-model; (b) there exists at least one *abnormal* $w_j \epsilon W$ such that for every *normal* $w_i \epsilon W$, $w_i R w_j$; and (c) $\vee$ is a value assignment not only satisfying the usual conditions, but also the additional conditions concerning the evaluation of wffs in abnormal words outlined in [1].

Quite obviously, the proofs for soundness and completeness will proceed in similar fashion as for the proofs of K1 given in [1].

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