Notre Dame Journal of Formal Logic
Volume XVIII, Number 3, July 1977
NDJFAM

THE CONTINUUM HYPOTHESIS IS INDEPENDENT OF SECOND-ORDER ZF

THOMAS S. WESTON

0 In the last few years, several authors have discussed formulations of set theory with underlying logic of order greater than one ([4], [5], [6], [8], [9]). Some of these writers have also considered the question of the utility of higher logic in deciding the continuum hypothesis ([4], [8], [9]). Kreisel, for example, has claimed that the **CH** is "decided" in second-order set theory ([8]). (Elsewhere, I have critically discussed the model-theoretic result to which Kreisel refers. See [12].)

This note presents a proof that in the usual (proof-theoretic) sense of "decided", second-order Zermelo-Frankel set theory ($\mathbf{ZF}^2$) does not decide **CH**. The proof is a straight-forward adaptation of L. Tharp's unpublished proof ([10]) that the **CH** is undecided in the set theory variously called **VBI** (for von Neuman-Bernays-Impredicative), Kelley-Morse, ([3], [7]) or **NQ** ([11]). A proof of independence of **CH** from **VBI** using an inaccessible cardinal has also been published by R. Chuaqui ([1]). The result presented in section **3** is that the first-order theory **VBI** is "almost the same theory" as $\mathbf{ZF}^2$, so that Tharp's results for **VBI** show the independence of **CH** from $\mathbf{ZF}^2$.

1 **VBI** is an impredicative extension of the more familiar von Neuman-Bernays set theory, **VB**. Recall that **VB** has axioms of Nullset, Pairing, Sumset, Infinity, and Powerset which concern only sets, as well as Foundation, Replacement, and Extensionality which quantify over classes (upper case variables) as well. The difference between **VB** and **VBI** is in the Comprehension Schema:

(1) $(X_1) \ldots (X_n)(\exists Y)(x \in Y \equiv \varphi(x, X_1, \ldots, X_n))$ *where* $\varphi(x, \ldots)$ *does not contain* Y *free.*

If we impose the additional limitation that $\varphi(x, \ldots)$ contain no bound class variables, we have **VB**.

2 $\mathbf{ZF}^2$ is a second-order theory with one-place predicate variables (upper case letters). Like **VBI**, its only non-logical constant is 'ϵ'. Equality for

Received January 5, 1976

individual variables is taken as a logical notion. In addition to the usual first-order logical axioms, $\mathbf{ZF}^2$ has two schemata for second-order quantification:

(2) $(P)(\varphi \supset \psi) \supset (\varphi \supset (P)\,\psi)$, *where the predicate variable* P *is not free in* φ,

and

(3) $(P)\,\varphi(P) \supset \check{\mathbf{S}}^{P(x)}_{\psi(x)}\,\varphi(P)$, *where the* $\check{\mathbf{S}}$ *means the result of substituting the formula* $\psi(x)$ *with individual free variable* x *(and perhaps others) for the predicate variable* P*. It is understood that if* $\psi(x)$ *is substituted for* $P(\alpha)$*, then free occurrences of* x *in* $\psi(x)$ *within* φ *are replaced by* α*. Substitution of* $\psi(x)$ *takes place subject to the following restrictions* (a) *no well-formed part of* φ *of the form* $(\alpha)\theta$*, where* α *is a free individual variable of* $\psi(x)$ *other than* x*, contains a free occurrence of* P*, and* (b) *for no free occurrence of* P *in* φ *of the form* $P(\alpha)$ *is it the case that* x *occurs free in a well-formed subformula of* ψ *of the form* $(\alpha)\theta$ (see [2], p. 192).

Ordinarily, we write $\check{\mathbf{S}}^{P(x)}_{\psi(x)}\,\varphi(x)$ as $\varphi(\psi(x))$. As will be shown below, the logical axiom schema (3) of $\mathbf{ZF}^2$ has essentially the same effect as the non-logical schema (1) of **VBI**.

For the remaining axioms of $\mathbf{ZF}^2$, we have Nullset, Pairing, Sumset, Infinity, and Powerset exactly as in **VBI**, while Extensionality is stated only for sets. Replacement and Foundation involve predicate quantification, and are the translations into primitive notation of the following:

(4) $(P)\,[(x)(y)(z)((P(\langle x, y\rangle)\ \&\ P(\langle y, z\rangle)) \supset y = z) \supset$
$(x)(\exists y)(z)(z \in y \equiv (\exists u)(u \in x\ \&\ P(\langle u, z\rangle)))]$

and

(5) $(P)\,[(\exists x)\,P(x) \supset (\exists x)(P(x)\ \&\ (y)(y \in x \supset \sim P(y)))]$

We want to establish a close relation between class quantification in **VBI** and predicate quantification in $\mathbf{ZF}^2$, but as it stands, $\mathbf{ZF}^2$ has nothing which corresponds to the Extensionality axiom of **VBI**:

(6) $(X)(Y)\,[(x)(x \in X \equiv x \in Y) \supset X = Y)]$

Indeed, equality of predicates is not even expressible in $\mathbf{ZF}^2$. This leads us to the following definition of equality for predicates of $\mathbf{ZF}^2$:

(7) $X = Y$ *for* $(\alpha)(X(\alpha) \equiv Y(\alpha))$, *where* α *is chosen so as to avoid clash of variables.*

With the aid of (7), it is very straightforward to prove not only Extensionality for predicates of $\mathbf{ZF}^2$, but the usual logical axioms of a theory with equality for predicates.

3 We define the following transformation: $\varphi \rightarrow \varphi^*$ from well-formed formulas of **VBI** to (abbreviated) formulas of $\mathbf{ZF}^2$: φ^* results from φ by replacing every atom of φ of the form $x \epsilon X$ by the predicate variable $X(x)$. We can now state the main

Theorem **VBI** $\vdash \varphi$ *if and only if* $\mathbf{ZF}^2 \vdash \varphi^*$.

Proof: All axioms of **VBI** except Comprehension and Extensionality are converted into axioms of $\mathbf{ZF}^2$ by the *-transformation. Conversely, all the non-logical axioms of $\mathbf{ZF}^2$ are converted into axioms of **VBI** by the inverse of the *-transformation. Extensionality is (an abbreviation of) a theorem of $\mathbf{ZF}^2$. Hence we need only prove the following two lemmas relating **VBI**'s Comprehension to the Substitution schema (3) of $\mathbf{ZF}^2$:

Lemma 1 *If* $\theta(x, x_1, \ldots, x_n, P_1, \ldots, P_k)$ *is a formula of* $\mathbf{ZF}^2$ *whose free set and predicate variables are among those shown then* $\mathbf{ZF}^2 \vdash (x_1) \ldots (x_n)(P_1) \ldots (P_k)(\exists P)(y)[P(y) \equiv \theta(y, x_1, \ldots, x_n, P_1, \ldots, P_k)]$ *where* y *is alphabetically the first variable not occurring in* θ.

Lemma 2 *If* $\varphi(X)$ *is a formula of* **VBI** *with* X *free, and the restrictions in* (3) *on* $\check{\mathsf{S}}$ *are satisfied for the substitution of the* **VBI** *formula* $\psi(x)$ *for* X *in* $\varphi(X)$, *then* **VBI** $\vdash [(X)\varphi(X) \supset \varphi(\psi(x))]$.

4 *Proof of Lemma 1:* We abbreviate $\theta(x, x_1, \ldots, x_n, P_1, \ldots, P_k)$ as $\theta(x)$, and we write $\theta(y)$ for the result of substituting y for free x's in $\theta(x)$. Let $\psi(x)$ be $\theta(x)$ and $\varphi(P)$ be $\sim(y)(\theta(y) \equiv P(y))$. Then:

(8) $(P) \sim (y)(\theta(y) \equiv P(y)) \supset \sim (y)(P(y) \equiv P(y))$

is an instance of the substitution schema (3). $\varphi(P)$ satisfies restrictions (a) and (b) of (3), and easy quantifier logic shows that (8) is equivalent to

(9) $(\exists P)(y)(P(y) \equiv \theta(y))$.

Hence by closure, we have:

(10) $\mathbf{ZF}^2 \vdash (x_1) \ldots (x_n)(P_1) \ldots (P_k)(\exists P)(y)(P(y) \equiv \theta(y, \ldots))$.

5 *Proof of Lemma 2:* Let $\varphi(X)$ and $\psi(x)$ satisfy the hypothesis of the lemma, and let us temporarily suppose that X is not free in $\psi(x)$. By the Comprehension Schema (1), we have

(11) $(\exists X)(x)(x \epsilon X \equiv \psi(x))$.

We want to show that (11) implies the desired conclusion

(12) $[(X)\varphi(X) \supset \varphi(\psi(x))]$,

for which we need the following

Sublemma: *Under the hypotheses of Lemma 2:*

(13) **VBI** $\vdash (X)[(x)(x \epsilon X \equiv \psi(x)) \supset (\varphi(X) \equiv \varphi(\psi(x)))]$.

Proof of Sublemma: By routine induction on the number of connectives and quantifiers in φ. The quantifier clause: $\varphi(X) \equiv (\alpha)\,\theta(X)$, where α is either a set or class variable. By the induction hypothesis,

(14) **VBI** $\vdash [(x)(x \epsilon X \equiv \psi(x)) \supset (\theta(X) \equiv \theta(\psi(x)))]$.

If α is not free in $(x)(x \in X = \psi(x))$, then (14) implies

(15) **VBI** $\vdash (x)(x \in X \equiv \psi(x)) \supset ((\alpha)\,\theta(X) \equiv (\alpha)\,\theta(\psi(x)))$.

Since by hypothesis $\varphi(\psi(x))$ satisfies (3)(a), α cannot be free in $\psi(x)$. α cannot be X because otherwise the substitution would not take place. So the sublemma follows.

From (11) and (13), the usual rules for distributing quantifiers over '$\supset$' yield

(16) $[(X)\,\varphi(X) \supset \varphi(\psi(x))]$.

Thus (16) is obtained on the assumption that X is not free in $\psi(x)$. In case X *is* free in $\psi(x)$, we first substitute for it some variable not used in any formula of the proof of (16), and then substitute X for that variable in (16). X does not thereby become bound in the consequent of (16) since the quantifier (X) binds only the antecedent. The lemma and theorem follow.

6 Corrollary *If* $\mathbf{ZF}^2$ *is consistent, then neither* $\mathbf{ZF}^2 \vdash \mathbf{CH}$ *nor* $\mathbf{ZF}^2 \vdash \sim\mathbf{CH}$.

Proof of Corrollary: If $\mathbf{ZF}^2$ is consistent, then by the Theorem, **VBI** is consistent. If **VBI** is consistent, then by Tharp's results, neither **VBI** $\vdash$ **CH** nor **VBI** $\vdash \sim$**CH**. Since **CH** contains no uppercase (class or predicate variables), we have **CH** = **CH***, so by the Theorem, neither $\mathbf{ZF}^2 \vdash \mathbf{CH}$ nor $\mathbf{ZF}^2 \vdash \sim\mathbf{CH}$.

REFERENCES

[1] Chuagui, R., "Forcing for the impredicative theory of classes," *The Journal of Symbolic Logic*, vol. 37 (1972), pp. 1-18.

[2] Church, A., *Introduction to Mathematical Logic*, Princeton University Press, Princeton (1956).

[3] Kelley, J., Appendix to *General Topology*, Van Nostrand Co., Princeton (1955).

[4] Kreisel, G., "Informal rigour and completeness proofs," in I. Lakatos, ed., *Philosophy of Mathematics*, North-Holland Publishing Co., Amsterdam (1967).

[5] Montague, R., "Set theory and higher-order logic," in J. N. Crossley and M. A. E. Dummett, *Formal Systems and Recursive Functions*, North-Holland Publishing Co., Amsterdam (1965), pp. 132-148.

[6] Montague, R., "Reductions of higher-order logic," in J. W. Addison, L. Henkin, and A. Tarski, *Theory of Models*, North-Holland Publishing Co., Amsterdam (1970).

[7] Morse, A., *A Theory of Sets*, Academic Press, New York (1965).

[8] Takeuti, G., "Formalization principle," in B. Van Rootselaar and J. F. Staal, *Logic, Methodology and Philosophy of Science, III*, North-Holland Publishing Co., Amsterdam (1968).

[9] Takeuti, G., "The universe of set theory," in *Foundations of Mathematics*, J. J. Bullof, *et al.*, eds., Springer-Verlag, New York (1969), pp. 74-129.

[10] Tharp, L., *Constructibility in Impredicative Set Theory*, unpublished Ph.D. dissertation, Massachusetts Institute of Technology, Cambridge, Massachusetts, 1965.

[11] Wang, H., "On Zermelo's and von Neumann's axioms for set theory," *Proceedings of the American National Academy of Sciences*, vol. 35 (1949), pp. 150-154.

[12] Weston, T., "Kreisel, the continuum hypothesis, and second order set theory," *Journal of Philosophical Logic*, vol. 5 (1976), pp. 281-298.

San Diego State University
San Diego, California