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SOME POST-COMPLETE EXTENSIONS OF S2 AND S3

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We shall take M, $\vee$, and $\neg$ as primitive connectives. Let $\mathcal{L}$ be the set of all wffs with these connectives. If α, $\beta \in \mathcal{L}$, we shall write $\alpha \dashv \beta$ for $\neg M \neg(\neg\alpha \vee \beta)$, and $\alpha \equiv \beta$ for $\neg[\neg(\alpha \dashv \beta) \vee \neg(\beta \dashv \alpha)]$. We let **f** and **t** denote the wffs $p \wedge \neg p$ and $\neg p \vee p$, respectively. If $\alpha \in \mathcal{L}$, we denote by $\mathcal{L}[\alpha]$ the smallest subset of $\mathcal{L}$ containing α and closed under the connectives M, $\vee$, and $\neg$. A *modal logic* **L** is a proper subset of $\mathcal{L}$ which is closed under the rules of uniform substitution and *modus ponens*, and contains all tautologies. If L_1 and L_2 are modal logics, then L_1 is an *extension* of L_2 iff $\mathsf{L}_2 \subseteq \mathsf{L}_1$. A modal logic is called *Post-complete* if it has no proper extensions. Let $p(\mathsf{L})$ be the number of Post-complete extensions of a modal logic **L**. Several papers have considered the problem of evaluating $p(\mathsf{L})$, for various modal logics **L** [1, 2, 3]. It has long been known that $p(S2) \geqslant \aleph_0$. Segerberg claims in [3] to prove that $p(S3) = 2^{\aleph_0}$: his proof is incorrect, but it may easily be modified to show that $p(S2) = 2^{\aleph_0}$ and that $p(S3) \geqslant \aleph_0$. Whether or not $p(S3) = \aleph_0$ remains an open question, to which this author believes the answer is probably affirmative. Most of the work on Post-complete systems uses the classical results of Lindenbaum and Tarski [4], and is therefore highly non-constructive. In fact, the only explicitly described Post-complete extensions of S3 in the literature known to the author are the systems S9 of [5] and F and Tr of [3]. This paper applies a variant of a theorem of Belnap and McCall [6] to construct some Post-complete extensions of the Lewis systems S2 and S3.

Let $\mathfrak{M} = \langle B, D, *\rangle$ be any matrix for a modal logic, where B is a Boolean algebra, D a set of distinguished elements, and $*$ interprets the possibility operator. Each element $\alpha \in \mathcal{L}[\mathbf{f}]$ determines an element $V_{\mathfrak{M}}(\alpha)$ of B, when interpreted in $\mathfrak{M}$ in the usual way.

Definition *The matrix $\mathfrak{M}$ is a functionally complete matrix* (**FCM**) *if:*

(i) for any $x \in B$, there is an $\alpha \in \mathcal{L}[\mathbf{f}]$ such that $V_{\mathfrak{M}}(\alpha) = x$.

and

(ii) for every $x \in B$, either $x \in D$ or $-x \in D$.

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Given any matrix $\mathfrak{M}$, we let $\mathsf{L}_{\mathfrak{M}} = \{\alpha \in \mathcal{L}: \mathfrak{M} \models \alpha\}$. The following lemma is essentially the theorem proved in [6], but is proved here for the sake of completeness.

Lemma *If $\mathfrak{M}$ is an* **FCM** *and* $\mathsf{L}_{\mathfrak{M}}$ *is a modal logic, then* $\mathsf{L}_{\mathfrak{M}}$ *is Post-complete.*

Proof: Let $\alpha \in \mathcal{L}$. If $\mathfrak{M} \dashv \alpha$, then there is a substitution instance α^* of α such that $\alpha^* \in \mathcal{L}[\mathbf{f}]$, and $\mathfrak{M} \dashv \alpha^*$. But then $V_{\mathfrak{M}}(\alpha^*) \notin D$, so $V_{\mathfrak{M}}(\neg\alpha^*) \in D$, and hence $\mathfrak{M} \models \neg\alpha^*$. Thus $\mathfrak{M} \dashv \alpha$ implies that α is inconsistent with $\mathsf{L}_{\mathfrak{M}}$, which proves the lemma.

Application 1: We construct a denumerably infinite collection of Post-complete extensions of S2, each closed under the rule of substitution of strict equivalents, and each having a finite characteristic matrix.

Let $n \geq 1$ be a fixed integer, and B_n be the Boolean algebra of subsets of $\{0, 1, \ldots, n\}$. Put $D_n = \{x \in B_n: 1 \in x\}$. Define $*_n\emptyset = \{0\}$; if $0 \leq j \leq n$, define $*_n\{j\}$ arbitrarily, subject to the conditions that

(1) $\{0, 1, j, j+1\} \subseteq *_n\{j\} \subseteq \{0, 1, 2, \ldots, j+1\}$ if $j < n$

and

(2) $\{0, 1, n\} \subseteq *_n\{n\}$.

If $x \in B_n$, define $*_n(x)$ by

(3) $*_n(x) = \bigcup_{j \in x} *_n\{j\}$.

We claim that $\mathfrak{M}_n = \langle B_n, D_n, *_n\rangle$ is an **FCM**. Indeed, define a sequence of wffs $\{\delta_i\}$ inductively, by

(4) $\delta_0 = M\mathbf{f}$; if $m \geq 1$, $\delta_m = M\delta_{m-1} \wedge \neg\delta_{m-1} \wedge \ldots \wedge \neg\delta_0$.

It is not hard to see that $V_{\mathfrak{M}_n}(\delta_k) = \{k\}$ whenever $0 \leq k \leq n$; it follows that $\mathfrak{M}_n$ is an **FCM**. By Theorem 3 of McKinsey [7], $\mathfrak{M}_n$ is a normal S2-algebra. Since B_n and B_l have different cardinalities for $n \neq l$, $\mathsf{L}_{\mathfrak{M}_n} \neq \mathsf{L}_{\mathfrak{M}_l}$. Thus $\{\mathsf{L}_{\mathfrak{M}_n} : n \geq 1\}$ is a collection of extensions of S2 having the desired properties.

Application 2: McCall and Vander Nat asked in [5] whether there are Post-complete modal systems with no finite characteristic matrix. Ulrich [8] has given an example of one; here we construct a nondenumerable family of such systems, each of which is an extension of S2. Let B be the Boolean algebra of all finite or cofinite subsets of $\{0, 1, \ldots\}$; let $D = \{x \in B: 1 \in x\}$. Define $*\emptyset = \{0\}$, and $*\{j\}$ arbitrarily for $0 \leq j$ subject to condition (1) above. Define $*(x)$ for $x \in B$ by (3). The wffs $\{\delta_i\}$ of (4) show that $\mathfrak{M} = \langle B, D, *\rangle$ is an **FCM**; since it is also a normal S2-algebra, the modal logic $\mathsf{L}_{\mathfrak{M}}$ is a Post-complete extension of S2 with no finite characteristic matrix. Let $\mathfrak{M}_1 = \langle B_1, D_1, *_1\rangle$ and $\mathfrak{M}_2 = \langle B_2, D_2, *_2\rangle$ be two distinct matrices obtained by the above construction; choose $j, k \leq 0$ such that $k \in *_1\{j\}$ but $k \notin *_2\{j\}$. Then $\mathfrak{M}_1 \models \delta_k \prec M\delta_j$, but $\mathfrak{M}_2$ rejects this wff. Hence $\mathsf{L}_{\mathfrak{M}_1} \neq \mathsf{L}_{\mathfrak{M}_2}$. A straightforward argument shows that the family of logics so constructed is nondenumerable.

Application 3: We determine a denumerably infinite collection of Post-complete extensions of S3, each finitely axiomatizable and each with a finite characteristic matrix. Fix an integer $N \geqslant 0$, and let B_N be the Boolean algebra of subsets of the set $S_N = \{0, 1, \ldots, N, \omega\}$. Let $D_N = \{x \in B_N: \omega \in x\}$. Define $*_N\emptyset = \{0\}$; if $0 \leqslant n \leqslant N$, put $*_N\{n\} = \{0, n, \omega\} \cup \{x \in S_N: n + 2 \leqslant x \leqslant N\}$; put $*_N\{\omega\} = \{0, \omega\}$. Define $*_N(x)$ by formula (3) for arbitrary $x \in B_N$. It is not difficult to verify that $\mathfrak{M}_N = \langle B_N, D_N, *_N\rangle$ is an S3 matrix. In the terminology of Kripke's model theory, $\mathfrak{M}_N$ corresponds to the frame with universe S_N, where 0 is the only non-normal world, ω sees every world, and if $0 < j \leqslant N$ then j sees $0, 1, \ldots, j - 2, j$. The theses of $\mathsf{L}_{\mathfrak{M}_N}$ are precisely the wffs which are verified in the world ω, in this frame. Define wffs X_n by

$$X_0 = M\mathbf{f};\ X_1 = \neg MM\mathbf{f};$$
$$\text{and if } n \geqslant 1,\ X_{n+1} = MX_0 \wedge MX_1 \wedge \ldots \wedge MX_{n-1} \wedge \neg MX_n.$$

It is not hard to show that whenever $0 \leqslant n \leqslant N$, $V_{\mathfrak{M}_N}(X_n) = \{n\}$. It follows that each $\mathfrak{M}_N$ is an **FCM**, and that the modal logics $\{\mathsf{L}_{\mathfrak{M}_N}: N \geqslant 0\}$ form a denumerably infinite family of distinct Post-complete extensions of S3. The reader will find that this construction is closely related to the paper [3] of Segerberg. The following theorem assures that each of these systems is finitely axiomatizable.

Theorem *Let $\mathfrak{M} = \langle B, D, *\rangle$ be a finite functionally complete* S3 *matrix. Then $\mathsf{L}_{\mathfrak{M}}$ is finitely axiomatizable.*

Proof: If $I = \{i_1, \ldots, i_n\}$ is any finite set, we shall write $\bigwedge\{\alpha_j: j \in I\}$ for $\alpha_{i_1} \wedge \ldots \wedge \alpha_{i_n}$, and $\bigvee\{\alpha_j: j \in I\}$ for $\alpha_{i_1} \vee \ldots \vee \alpha_{i_n}$. If I is empty, we take these expressions to represent $\mathbf{t}$ and $\mathbf{f}$, respectively. We may assume B is the Boolean algebra of subsets of $\{0, 1, \ldots, N\}$ for some $N \geqslant 0$, and $D = \{x \in B: 0 \in x\}$. For each n, $0 \leqslant n \leqslant N$, select a wff α_n such that $V_{\mathfrak{M}}(\alpha_n) = \{n\}$.

As axioms take Simons' axioms for S3 [9], together with

(5) $M\alpha_n \equiv \bigvee \{\alpha_j: j \in *\{n\}\}$ for each n, $0 \leqslant n \leqslant N$;
(6) $\alpha_n \wedge \alpha_m \equiv \mathbf{f}$ for each pair $n \neq m$, where $0 \leqslant n, m \leqslant N$;
(7) α_0;
(8) $\bigvee\{p \equiv \alpha_{i_1} \vee \ldots \vee \alpha_{i_k}: 0 \leqslant i_1 < \ldots < i_k \leqslant N\}$;
(9) $\alpha_0 \vee \ldots \vee \alpha_N \equiv \mathbf{t}$.

Let L be the extension of S3 defined by these axioms, with *modus ponens* and uniform substitution the only primitive rules of inference. This L will have the rule of substitution of strict equivalents as a derived rule, since this is true in any extension of Simons' axiomatization of S3. Clearly $\mathsf{L} \subseteq \mathsf{L}_{\mathfrak{M}}$; we must show that $\mathsf{L}_{\mathfrak{M}} \subseteq \mathsf{L}$. First we show that for all $\beta \in \mathcal{L}[\mathbf{f}]$,

(10) $\quad \vdash_{\mathsf{L}} \beta \equiv \bigvee \{\alpha_j: j \in V_{\mathfrak{M}}(\beta)\}$.

If β is the wff $\mathbf{f}$, the assertion is trivial; if γ and δ satisfy (10) and β is $\gamma \vee \delta$ or $M\gamma$, then clearly β satisfies (10). Now suppose β is $\neg\gamma$, where γ satisfies (10). Then:

(11) $\quad \vdash_{\mathsf{L}} \beta \equiv \neg\left(\bigvee\{\alpha_j: j \in V_{\mathfrak{M}}(\gamma)\}\right)$.

Now, from (6) one can show that whenever $i \neq j$, $\vdash_{\mathsf{L}} (\alpha_i \wedge \neg\alpha_j) \equiv \alpha_i$. Using (9) we get

$$\vdash_{\mathsf{L}} \neg\alpha_j \equiv (\alpha_0 \wedge \neg\alpha_j) \vee \ldots \vee (\alpha_N \wedge \neg\alpha_j).$$

Hence

$$\vdash_{\mathsf{L}} \neg\alpha_j \equiv \bigvee\{\alpha_i:\ i \neq j,\ 0 \leq i \leq N\}.$$

This, together with (11) and (6), shows that

$$\vdash_{\mathsf{L}} \beta \equiv \bigvee\{\alpha_i:\ i \notin V_{\mathfrak{M}}(\gamma)\}.$$

Hence (10) holds for all $\beta \in \mathcal{L}[\mathbf{f}]$.

By the above paragraph and (7) we have $\mathsf{L}_{\mathfrak{M}} \cap \mathcal{L}[\mathbf{f}] \subseteq \mathsf{L}$. Now let $\beta(p_1, \ldots, p_n)$ be any wff in the variables $p_1, \ldots, p_n$, and let $\Gamma = \{\alpha_{i_1} \vee \ldots \vee \alpha_{i_k}:\ 0 \leq i_1 < \ldots < i_k \leq N\}$. Using (8) and the substitution of strict equivalents, we get

$$\vdash_{\mathsf{L}} \bigvee\{\beta \equiv \beta(p_1/\gamma_1, \ldots, p_n/\gamma_n):\ \gamma_i \in \Gamma \text{ for } i = 1, \ldots, n\}.$$

If $\beta \in \mathsf{L}_{\mathfrak{M}}$, then each $\beta(p_1/\gamma_1, \ldots, p_n/\gamma_n)$ is in $\mathsf{L}_{\mathfrak{M}} \cap \mathcal{L}[\mathbf{f}] \subseteq \mathsf{L}$, and hence $\beta \in \mathsf{L}$. The theorem is now proved.

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