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THE LOGIC OF CLOSED CATEGORIES

MANFRED E. SZABO

0 *Introduction* In this paper,* we continue our study, initiated in [4, 5, 10] and reformulated in [11], of the connection between syntactic and semantic criteria for the "equivalence" and "normality" of formal proofs in intuitionist Gentzen systems. As in [11], we interpret proofs as morphisms in free closed categories. But by no longer requiring that these categories are "cartesian", we obtain a coarser equivalence relation than that in [11], still admitting a "reducibility relation" with the Church-Rosser property. As a by-product of this analysis, we are able to obtain necessary and sufficient conditions for the commutativity of diagrams in free closed categories.

1 *Closed categories* The theory of "closed categories" serves as a generalization for categories such as sets, R-modules over a commutative ring R, compactly generated Hausdorff spaces, small categories, etc., in which any two objects have a "tensor product" and in which the "hom-sets" themselves are again sets, R-modules, compactly generated Hausdorff spaces, small categories, etc. Formally, a *closed category* is a list $\langle \mathfrak{K}, \wedge, \supset, \mathrm{I}, \alpha, \lambda, \sigma, \Omega \rangle$ consisting of the following data:

(i) a category $\mathfrak{K}$;
(ii) a bifunctor $\wedge$: $\mathfrak{K} \times \mathfrak{K} \to \mathfrak{K}$ (called "tensor product");
(iii) a bifunctor $\supset$: $\mathfrak{K}^{\mathrm{op}} \times \mathfrak{K} \to \mathfrak{K}$ (called "internal hom");
(iv) a distinguished object I (called the "unit" of the tensor product);
(v) coherent natural isomorphisms α, λ, and σ with components

$$\alpha(A, B, C): A \wedge (B \wedge C) \to (A \wedge B) \wedge C,$$
$$\lambda(A): \mathrm{I} \wedge A \to A,$$

and

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$$\sigma(A, B): A \wedge B \rightarrow B \wedge A$$

in $\mathfrak{K}$ for all objects A, B, C of $\mathfrak{K}$;
(vi) a natural transformation Ω with bijections

$$\Omega(A, B, C): \mathrm{Hom}_{\mathfrak{K}}(A \wedge B, C) \rightarrow \mathrm{Hom}_{\mathfrak{K}}(B, A \supset C)$$

as components for all objects A, B, C of $\mathfrak{K}$.

The transformations α, λ, σ, and Ω are understood to be natural in each argument, and the "coherence" of α, λ, and σ means that any isomorphism μ in $\mathfrak{K}$, constructed from them by tensoring and composition, is unique, provided that no component of α, λ, and σ used in the construction of μ contains a repeated argument.

The following conditions, as given by MacLane [6] and simplified by Kelly [2], are necessary and sufficient for the coherence of α, λ, and σ:

(i) $\alpha(A, B, C) \wedge 1(D) \circ \alpha(A, B \wedge C, D) \circ 1(A) \wedge \alpha(B, C, D)$
$= \alpha(A \wedge B, C, D) \circ \alpha(A, B, C \wedge D)$;
(ii) $\sigma(B, A) \circ \sigma(A, B) = 1(A \wedge B)$;
(iii) $\sigma(A, C) \wedge 1(B) \circ \alpha(A, C, B) \circ 1(A) \wedge \sigma(B, C)$
$= \alpha(C, A, B) \circ \sigma(A \wedge B, C) \circ \alpha(A, B, C)$;
(iv) $\lambda(A) \wedge 1(C) \circ \alpha(\mathrm{I}, A, C) = \lambda(A \wedge C)$,

for all objects A, B, C of $\mathfrak{K}$. Here "$\circ$" denotes composition in $\mathfrak{K}$, and, for any object A of $\mathfrak{K}$, $1(A)$ stands for the identity on A. All undefined notions coincide with those in [7].

A closed category $\mathfrak{K}$ is *free* on a category $\mathfrak{A}$ if any functor F from $\mathfrak{A}$ to the "underlying" category $\mathsf{U}(\mathfrak{K}')$ of a closed category $\mathfrak{K}'$ extends uniquely to a functor F': $\mathfrak{K} \rightarrow \mathfrak{K}'$ which preserves the closed structure *exactly*, i.e., $F'(A \wedge B) = F'(A) \wedge F'(B)$, etc. The existence of free closed categories follows from Freyd's Adjoint Functor Theorem [1].

2 *The language* $\mathcal{L}$ *of* $\mathsf{F}(1)$ In this paper, we shall work exclusively with the free closed category $\mathsf{F}(1)$ on "the" discrete one-object category **1**. The object of **1** will be denoted by 0.

$\mathsf{Alph}(\mathcal{L})$, the *alphabet* of $\mathcal{L}$, is the set $\{0, \mathrm{I}, \wedge, \supset, (,)\}$.

$\mathsf{Term}(\mathcal{L})$, the set of *terms* of $\mathcal{L}$, is the smallest set of finite strings of symbols of $\mathsf{Alph}(\mathcal{L})$ such that

(i) $0 \in \mathsf{Term}(\mathcal{L})$;
(ii) $\mathrm{I} \in \mathsf{Term}(\mathcal{L})$;
(iii) if $A, B \in \mathsf{Term}(\mathcal{L})$, then $(A \wedge B)$ and $(A \supset B) \in \mathsf{Term}(\mathcal{L})$.

If $s_1 \ldots s_n \in \mathsf{Term}(\mathcal{L})$ by virtue of (ii) and (iii), then $s_1 \ldots s_n$ is called a "constant" term, and any term which enters into the construction of a term T at some stage is called a "subterm" of T. When writing down terms, we shall usually omit their outermost brackets.

$\mathsf{Form}(\mathcal{L})$, the set of *formulas* of $\mathcal{L}$, is the set of finite sequences $\langle A_1, \ldots A_n \rangle$ $(n \geq 1)$, where $A_i \in \mathsf{Term}(\mathcal{L})$.

Following Gentzen, we shall usually write $A_1, \ldots A_{n-1} \to A_n$ for $\langle A_1, \ldots, A_{n-1}, A_n\rangle$. "$A_1, \ldots, A_{n-1}$" is called the "antecedent" and "A_n" the "succedent" of $A_1, \ldots, A_{n-1} \to A_n$. The formula $\langle A\rangle$, in particular, will be written as "$\to A$". As in [12], capital Greek letters will often be used to abbreviate all or parts of a given antecedent.

The language $\mathcal{L}$ of $\mathsf{F}(1)$ is the list $\langle \mathsf{Alph}(\mathcal{L}), \mathsf{Term}(\mathcal{L}), \mathsf{Form}(\mathcal{L})\rangle$.

3 *The Gentzen system* **G** The deductive system **G** whose proofs are interpretable as morphisms of $\mathsf{F}(1)$ is the list $\langle \mathsf{Ax}(\mathbf{G}), \mathsf{Rules}(\mathbf{G}), \mathsf{Proofs}(\mathbf{G})\rangle$ consisting of the following data:

(I) The set of formulas $\mathsf{Ax}(\mathbf{G}) = \{\langle 0, 0\rangle, \langle \mathrm{I}\rangle\}$, whose members are called the "axioms" of **G**;
(II) The set of relations $\mathsf{Rules}(\mathbf{G}) = \{(\mathrm{C}), (\mathrm{W}), (\mathrm{H}), (\mathrm{R}), (\mathrm{P}), (\mathrm{K})\}$ on $\mathsf{Form}(\mathcal{L})$ whose members are called the "rules of inference" of **G**, and are specified as follows:

(a) $\langle \Gamma \to A; \Delta \to B; \Gamma, \Delta \to A \wedge B\rangle \in (\mathrm{C})$;
(b) $\langle \Gamma, A, B, \Delta \to C; \Gamma, A \wedge B, \Delta \to C\rangle \in (\mathrm{W})$;
(c) $\langle \Gamma \to A; \Delta, B, \Lambda \to C; \Delta, \Gamma, A \supset B, \Lambda \to \mathrm{C}\rangle \in (\mathrm{H})$;
(d) $\langle \Gamma, A, \Delta \to B; \Gamma, \Delta \to A \supset B\rangle \in (\mathrm{R})$;
(e) $\langle \Gamma \to A; \Gamma' \to A\rangle \in (\mathrm{P})$, where Γ' results from Γ by any non-trivial permutation of the terms of Γ;
(f) $\langle \Gamma \to A; \Gamma, \mathrm{I} \to A\rangle \in (\mathrm{K})$, for all finite sequences Γ and Δ of terms of $\mathcal{L}$ and all terms A, B, C.

The reader will appreciate the use of the semicolon for ordered pairs and triples. As in [11], we shall write

$$\frac{\Gamma \to A \quad \Delta \to B}{\Gamma, \Delta \to A \wedge B}\,(\mathrm{C}) \qquad \frac{\Gamma, A, B, \Delta \to C}{\Gamma, A \wedge B, \Delta \to C}\,(\mathrm{W})$$

etc., to indicate membership in (C), (W), etc.

Any member of a rule of inference will be called an *instance* or *application* of that rule of inference. The terms A and B in the above instances of (C), (W), (H), and (R), and the term I in the instance of (K) are called *active*, and all others *passive* occurrences of terms.

In order to define the proofs of **G**, we require the notion of a "tree". A tree will essentially be a finite inverted dyadic tree in the sense of Smullyan [9]. More precisely, we define a tree to be a list $\langle N, r, s, w\rangle$ consisting of the following data:

(i) a finite set N whose elements are called "nodes";
(ii) a distinguished node $r \in N$ called the "root" of the tree;
(iii) a function $s\colon N - \{r\} \to N$ called the "successor function";
(iv) a function $w\colon \{x \mid s^{-1}(x)$ has two elements$\} \to N \times N$ called the "well-ordering function".

These data satisfy the following axioms:

T1 *For all* $x \in N$, $s^{-1}(x)$ *contains at most two elements.*
T2 *For all* $x \in N$, *there exists an* $n \in \{0, 1, 2, \ldots\}$ *such that* $s^n(x) = r$.
T3 *For all* $x \in N$, $s^n(x) = x$ *iff* $n = 0$.
T4 *If* $w(x) = \langle y, z\rangle$, *then* $\{y, z\} = s^{-1}(x)$.

A node is called an "origin" if $s^{-1}(x)$ is empty, a "simple point" if $s^{-1}(x)$ is a singleton, and a "junction point" otherwise. The elements of $s^{-1}(x)$ are called the "predecessors" of x. If $w(x) = \langle y, z\rangle$, then y is called the "left predecessor" and z the "right predecessor" of x.

We define a *subtree* of $\langle N, r, s, w\rangle$ to be any tree obtained by restricting $\langle N, r, s, w\rangle$ to a suitable subset of N.

A *path* in a tree T is a sequence $S = \langle x, s(x), s^2(x), \ldots, s^n(x)\rangle$ of nodes in which x is an origin of T. The number of terms of S will be called the *length* of S. A *maximal path* in T is a path in which the last term is r. The maximum of the lengths of the maximal paths of T will be called the *height* of T.

(III) We now define the set $\mathsf{Proofs}(\mathbf{G})$ of proofs of $\mathbf{G}$ as follows: A *proof* is a pair $\langle P, T\rangle$, where T is a tree, N is the set of nodes of T, and $P: N \rightarrow \mathsf{Form}(\mathcal{L})$ is a function satisfying the following conditions:

(a) if x is an origin of T, then $P(x) \in \mathsf{Ax}(\mathbf{G})$;
(b) if x is a simple point of T, and $y = s(x)$, then $\langle P(x), P(y)\rangle$ is an instance of one of the rules (W), (R), (P), (K);
(c) if x is a junction point of T, and $w(x) = \langle y, z\rangle$, then $\langle P(y), P(z), P(x)\rangle$ is an instance of either (C) or (H).

A *subproof* of $\langle P, T\rangle$ is a proof obtained by restricting P to a suitable subtree of T. An instance of a rule of inference (Ri) in a proof P is *passive* with respect to an instance of a rule (Rj) if the active terms of (Ri) are passive with respect to (Rj) and conversely.

4 *Normal proofs* In this section, we interpret the proofs of $\mathbf{G}$ as morphisms of $\mathsf{F}(1)$, and define the "normal" proofs of $\mathbf{G}$ as certain "irreducible" proofs relative to the reducibility relation "$\geqslant$" still to be introduced. For this purpose, we find it convenient to make use of the following special notation: "$P: \Gamma \rightarrow A$" or "$\Gamma \xrightarrow{P} A$" stands for "P is a proof of $\Gamma \rightarrow A$", and "$\mathsf{Obj}(\mathsf{F}(1))$" and "$\mathsf{Morph}(\mathsf{F}(1))$" denote the classes of objects and morphisms of $\mathsf{F}(1)$, respectively. The interpretation of a proof $\langle P, T\rangle$ is defined by induction on the height of T. For this purpose it is clear from [4, 5] that we may assume that $\mathsf{Term}(\mathcal{L}) = \mathsf{Obj}(\mathsf{F}(1))$.

The Interpretation For the sake of simplicity, we shall think of $\Gamma, \Delta, \ldots$ below as representing individual objects $A, B, \ldots$ of $\mathsf{F}(1)$. In order to achieve a compatible bracketing of tensor products the interpretations of individual steps of a proof may therefore require compositions with isomorphisms not explicitly mentioned. By virtue of the coherence of α these isomorphisms are unique. A similar remark applies to the interpretation of the "cut" in Section **6**.

(i) $\langle 0, 0\rangle = 1(0)\colon 0 \to 0$;

(ii) $\langle \mathrm{I}\rangle = 1(\mathrm{I})\colon \mathrm{I} \to \mathrm{I}$;

(iii) $\dfrac{\to \mathrm{I}}{\mathrm{I} \to \mathrm{I}} = 1(\mathrm{I})\colon \mathrm{I} \to \mathrm{I}$;

(iv) if $A \xrightarrow{P} A = 1(A)\colon A \to A$, $B \xrightarrow{Q} B = 1(B)\colon B \to B$, and $C \xrightarrow{R} C = 1(C)\colon C \to C$, then

(a) $$\begin{array}{l} A \xrightarrow{P} A \qquad B \xrightarrow{Q} B \\ \hline A, B \to A \wedge B \\ \hline A \wedge B \to A \wedge B \end{array}$$ $= 1(A \wedge B)\colon A \wedge B \to A \wedge B$;

(b) $$\begin{array}{l} A \xrightarrow{P} A \qquad B \xrightarrow{Q} B \\ \hline A, A \supset B \to B \\ \hline A \supset B \to A \supset B \end{array}$$ $= 1(A \supset B)\colon A \supset B \to A \supset B$;

(c) $$\begin{array}{l} A \xrightarrow{P} A \qquad B \xrightarrow{Q} B \\ \hline A, B \to A \wedge B \qquad C \xrightarrow{R} C \\ \hline A, B, C \to (A \wedge B) \wedge C \\ \hline A, B \wedge C \to (A \wedge B) \wedge C \\ \hline A \wedge (B \wedge C) \to (A \wedge B) \wedge C \end{array}$$ $= \alpha(A, B, C)\colon A \wedge (B \wedge C) \to (A \wedge B) \wedge C$; (similarly for $\alpha^{-1}(A, B, C)$);

(d) $$\begin{array}{l} A \to A \\ \hline A, \mathrm{I} \to A \\ \hline \mathrm{I}, A \to A \\ \hline \mathrm{I} \wedge A \to A \end{array}$$ $= \lambda(A)\colon \mathrm{I} \wedge A \to A$; (similarly for $\lambda^{-1}(A)$);

(e) $$\begin{array}{l} B \xrightarrow{Q} B \qquad A \xrightarrow{P} A \\ \hline B, A \to B \wedge A \\ \hline A, B \to B \wedge A \\ \hline A \wedge B \to B \wedge A \end{array}$$ $= \sigma(A, B)\colon A \wedge B \to B \wedge A$;

(f) $$\begin{array}{l} A \xrightarrow{P} A \qquad B \xrightarrow{Q} B \\ \hline A, B \to A \wedge B \\ \hline B \to A \supset (A \wedge B) \end{array}$$ $= \eta(A, B)\colon B \to A \supset (A \wedge B)$, where "$\eta(A, B)$" stands for $\Omega(A, B, A \wedge B)(1(A \wedge B))$;

(g) $$\begin{array}{l} A \xrightarrow{P} A \qquad B \xrightarrow{Q} B \\ \hline A, A \supset B \to B \\ \hline A \quad (A \supset B) \to B \end{array} = \varepsilon(A, B)\colon A \quad (A \supset B) \to B,$$

where "$\varepsilon(A, B)$" stands for $\Omega^{-1}(A, B, A \supset B)(1(A \supset B))$;

(v) if $\Gamma \xrightarrow{P} A = f\colon C \to A$, and $\Delta \xrightarrow{Q} B = g\colon D \to B$,

then $\dfrac{\Gamma \xrightarrow{P} A \qquad \Delta \xrightarrow{Q} B}{\Gamma, \Delta \to A \wedge B} = f \wedge g\colon C \wedge D \to A \wedge B$;

(vi) (a) if $\Gamma, A, B, \Delta \xrightarrow{P} C = f\colon ((D \wedge A) \wedge B) \wedge E \to C$,

then $\dfrac{\Gamma, A, B, \Delta \xrightarrow{P} C}{\Gamma, A \wedge B, \Delta \to C} = f \circ \alpha(D, A, B) \wedge 1(E)\colon (D \wedge (A \wedge B)) \wedge E \to C$;

(b) if $\Gamma, A, B, \Delta \xrightarrow{P} C = g\colon D \wedge (A \wedge (B \wedge E)) \to C$,

then $\dfrac{\Gamma, A, B, \Delta \xrightarrow{P} C}{\Gamma, A \wedge B, \Delta \to C} = g \circ 1(D) \wedge \alpha^{-1}(A, B, E)\colon D \wedge ((A \wedge B) \wedge E) \to C$;

(c) if $\Gamma, A, B, \Delta \xrightarrow{P} C = h\colon (D \wedge (A \wedge B)) \wedge E \to C$,

then $\dfrac{\Gamma, A, B, \Delta \xrightarrow{P} C}{\Gamma, A \wedge B, \Delta \to C} = h\colon (D \wedge (A \wedge B)) \wedge E \to C$;

(d) if $\Gamma, A, B, \Delta \xrightarrow{P} C = k\colon D \wedge ((A \wedge B) \wedge E) \to C$,

then $\dfrac{\Gamma, A, B, \Delta \xrightarrow{P} C}{\Gamma, A \wedge B, \Delta \to C} = k\colon D \wedge ((A \wedge B) \wedge E) \to C$;

(vii) (a) if $\Gamma \xrightarrow{P} A = f\colon D \to A$, and $\Delta, B, \Lambda \xrightarrow{Q} C = g\colon E \wedge (B \wedge H) \to C$, then $\dfrac{\Gamma \xrightarrow{P} A \quad \Delta, B, \Lambda \xrightarrow{Q} C}{\Delta, \Gamma, A \supset B, \Lambda \to C} =$

$g \circ \varepsilon(A, B) \wedge 1(H) \circ \alpha(A, A \supset B, H) \circ f \wedge 1((A \supset B) \wedge H)$:
$E \wedge (D \wedge ((A \supset B) \wedge H)) \to C$;

(b) if $\Gamma \xrightarrow{P} A = f\colon D \to A$, and $\Delta, B, \Lambda \xrightarrow{Q} C = h\colon(E \wedge B) \wedge H \to C$, then $\dfrac{\Gamma \xrightarrow{P} A \quad \Delta, B, \Lambda \xrightarrow{Q} C}{\Delta, \Gamma, A \supset B, \Lambda \to C} =$

$h \circ (1(E) \wedge \varepsilon(A, B) \circ f \wedge 1(A \supset B)) \wedge 1(H) \circ \alpha^{-1}(E, D \wedge (A \supset B), H)$:
$E \wedge (D \wedge ((A \supset B \wedge H)) \to C$;

(viii) (a) if $\Gamma, A, \Delta \xrightarrow{P} B = f\colon C \wedge (A \wedge D) \to B$, then $\dfrac{\Gamma, A, \Delta \xrightarrow{P} B}{\Gamma, \Delta \to A \supset B} =$

$(1(A) \supset f \circ \alpha^{-1}(C, A, D) \circ \sigma(A, C) \wedge 1(D) \circ \alpha(A, C, D)) \circ \eta(A, C \wedge D)$:
$C \wedge D \to A \supset B$;

(b) if $\Gamma, A, \Delta \xrightarrow{P} B = g\colon (C \wedge A) \wedge D \to B$, then $\dfrac{\Gamma, A, \Delta \xrightarrow{P} B}{\Gamma, \Delta \to A \supset B} =$

$(1(A) \supset f \circ \sigma(A, C) \wedge 1(D) \circ \alpha(A, C, D)) \circ \eta(A, C \wedge D)\colon C \wedge D \to A \supset B$;

(ix) if $\Gamma \xrightarrow{P} A = f\colon B \to A$, then $\dfrac{\Gamma \xrightarrow{P} A}{\Gamma' \to A} = f \circ \kappa\colon C \to A$,

where $\kappa\colon C \to B$ is the unique isomorphism made up of components of α, 1, and σ which permutes the "factors" of B. The existence of κ is clear, and its uniqueness follows by coherence;

(x) if $\Gamma \xrightarrow{P} A = f\colon B \to A$, then $\dfrac{\Gamma \xrightarrow{P} A}{\Gamma, \mathrm{I} \to A} = f \circ \lambda(B) \circ \sigma(B, \mathrm{I})\colon B \wedge \mathrm{I} \to A$;

(xi) finally, the interpretation of any proof $A_1, \ldots, A_n \xrightarrow{P} B$ is the same as that of the proof Q obtained from P by appending $n - 1$ applications of (W) and introducing the symbols $\wedge$ from left to right; for example, the interpretation of $A_1, A_2, A_3 \xrightarrow{P} B$ is the same as that of the proof

$$\dfrac{\dfrac{A_1, A_2, A_3 \xrightarrow{P} B}{A_1 \wedge A_2, A_3 \to B}}{(A_1 \wedge A_2) \wedge A_3 \to B}.$$

The reader will appreciate the meaning of "=" in the above definition.

As a consequence of the coherence of α, λ, and σ, conditions (i)-(xi) clearly define a many-one mapping φ: $\mathsf{Proofs}(\mathbf{G}) \to \mathsf{Morph}(\mathsf{F}(1))$. Let E be the "equivalence kernel" of φ, i.e., $P \mathsf{E} Q$ iff $\varphi(P) = \varphi(Q)$. We shall show that E admits a reducibility relation $\geqslant$ (in the sense of [0]) with the Church-Rosser property, and define $P \in \mathsf{Proofs}(\mathbf{G})$ to be *normal* if $P \geqslant Q$ implies that $P = Q$.

Since it is clear from the coherence theorem in [3] that for any two constant objects K and L, there exists *precisely one* morphism $\kappa\colon K \to L$ in $\mathsf{F}(1)$, we first require the following facts in order to be able to define normal proofs involving constants:

Lemma *For any constant term K, the formula $\to K$ is provable without the use of rule* (P), *and the formula $\Gamma, K \to A$ is derivable from the formula $\Gamma \to A$, also without the use of rule* (P).

Proof: We carry out an induction on the number of occurrences of the symbols $\wedge$ and $\supset$ in K.

(i) (a) $\to$ I is an axiom, (b) $\dfrac{\Gamma \to A}{\Gamma, \mathrm{I} \to A}\,(\mathrm{K})$;

(ii) (a) $\dfrac{\to \mathrm{I} \quad \to \mathrm{I}}{\to \mathrm{I} \wedge \mathrm{I}}\,(\mathrm{C})$, (b) $\dfrac{\dfrac{\dfrac{\Gamma \to A}{\Gamma, \mathrm{I} \to A}\,(\mathrm{K})}{\Gamma, \mathrm{I}, \mathrm{I} \to A}\,(\mathrm{K})}{\Gamma, \mathrm{I} \wedge \mathrm{I} \to A}\,(\mathrm{W})$;

(iii) (a) $\dfrac{\dfrac{\to \mathrm{I}}{\mathrm{I} \to \mathrm{I}}\,(\mathrm{K})}{\to \mathrm{I} \supset \mathrm{I}}\,(\mathrm{R})$, (b) $\dfrac{\to \mathrm{I} \quad \dfrac{\Gamma \to A}{\Gamma, \mathrm{I} \to A}\,(\mathrm{K})}{\Gamma, \mathrm{I} \supset \mathrm{I} \to A}\,(\mathrm{H})$;

(iv) the induction step is similar.

We now define "$\geqslant$" to be the smallest binary relation on Proofs(**G**) such that

(i) $P \geqslant P$ for all P;
(ii) if $P \geqslant Q$ and $Q \geqslant R$, then $P \geqslant R$;
(iii) if $P \geqslant Q$ and B is a proof which results from A by the replacement of a subproof P of A by Q, then $A \geqslant B$;
(iv) $P \geqslant Q$, where Q is a proof in which the mutually passive instances of the rules of inference obey the following order of priorities:

(a) (C) precedes (R), (H), (P), (K), (W);
(b) (R) precedes (H), (P), (K), (W);
(c) (H) precedes (P), (K), (W);
(d) (P) precedes (K), (W);
(e) (K) precedes (W).

For example,

$$\dfrac{\dfrac{\Gamma \xrightarrow{P} A \quad \Delta, B, \Lambda \xrightarrow{Q} C}{\Delta, \Gamma, A \supset B, \Lambda \quad \to C} \quad \Phi \xrightarrow{R} D}{\Delta,\ \Gamma, A \supset B, \Lambda,\ \Phi \to C \wedge D} \geqslant \dfrac{\Gamma \xrightarrow{P} A \quad \dfrac{\Delta, B, \Lambda \xrightarrow{Q} C \quad \Phi \xrightarrow{R} D}{\Delta, B, \Lambda,\ \Phi \to C \wedge D}}{\Delta,\ \Gamma, A \supset B,\ \Lambda,\ \Phi \to C \wedge D};$$

$$\dfrac{\dfrac{\Gamma, A, \Delta, B, \Lambda, C, D, \Phi \xrightarrow{P} E}{\Gamma, A, \Delta, B, \Lambda, C \wedge D, \Phi \to E}}{\Gamma, B, \Delta, A, \Lambda, C \wedge D, \Phi \to E} \geqslant \dfrac{\dfrac{\Gamma, A, \Delta, B, \Lambda, C, D, \Phi \xrightarrow{P} E}{\Gamma, B, \Delta, A, \Lambda, C, D, \Phi \to E}}{\Gamma, B, \Delta, A, \Lambda, C \wedge D, \Phi \to E}, \text{ etc.}$$

(v) $P \geqslant Q$, where Q is a proof which does not contain two consecutive applications of (P), nor does it contain a (P) interchanging identical constant terms;

(vi) $P \geqslant Q$, where Q is a proof in which the orders of priority of (iv) are observed between (P) and (R), (H), (W), respectively, even if (P) is not passive relative to these rules;

(vii) $P \geqslant Q$, where Q is a proof in which antecedent $\supset$'s are introduced "from left to right".

For example,

$$\dfrac{\Phi \xrightarrow{R} D \quad \dfrac{\Psi \xrightarrow{P} E \quad \Gamma, A, \Delta, B, \Lambda \xrightarrow{Q} C}{\Gamma, A, \Delta, \Psi, E \supset B, \Lambda \to C}}{\Gamma, \Phi, D \supset A, \Delta, \Psi, E \supset B, \Lambda \to C} \geqslant \dfrac{\Psi \xrightarrow{P} E \quad \dfrac{\Phi \xrightarrow{R} D \quad \Gamma, A, \Delta, B, \Lambda \xrightarrow{Q} C}{\Gamma, \Phi, D \supset A, \Delta, B, \Lambda \to C}}{\Gamma, \Phi, D \supset A, \Delta, \Psi, E \supset B, \Lambda \to C};$$

(viii) $P \geqslant Q$, where Q is a proof in which antecedent $\wedge$'s are introduced "from left to right".

For example,

$$\dfrac{\dfrac{\Gamma, A, B, \Delta, C, D, \Lambda \to E}{\Gamma, A, B, \Delta, C \wedge D, \Lambda \to E}}{\Gamma, A \wedge B, \Delta, C \wedge D, \Lambda \to E} \geqslant \dfrac{\dfrac{\Gamma, A, B, \Delta, C, D, \Lambda \to E}{\Gamma, A \wedge B, \Delta, C, D, \Lambda \to E}}{\Gamma, A \wedge B, \Delta, C \wedge D, \Lambda \to E};$$

(ix) if $P\colon \Gamma \to A$ does not contain the axiom $\langle 0, 0\rangle$, then $P \geqslant Q$, where Q is the unique proof of $\Gamma \to A$ with the following properties:

(a) Q contains no instance of rule (P);
(b) the antecedents of the left premisses of any instances of rule (H) are empty;
(c) Q satisfies conditions (iv) and (vi)-(vii) above.

For example, if $\Gamma \to A$ is

$$(\mathrm{I} \wedge ((\mathrm{I} \supset \mathrm{I}) \supset \mathrm{I})) \wedge (\mathrm{I} \wedge \mathrm{I}) \to (\mathrm{I} \supset \mathrm{I}) \supset (\mathrm{I} \wedge (\mathrm{I} \supset \mathrm{I})),$$

then Q is the following tree:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\to \mathrm{I}}{\mathrm{I} \to \mathrm{I}}}{\to \mathrm{I} \supset \mathrm{I}} \quad \dfrac{\dfrac{\dfrac{\dfrac{\to \mathrm{I} \quad \dfrac{\dfrac{\to \mathrm{I} \quad \dfrac{\dfrac{\to \mathrm{I}}{\mathrm{I} \to \mathrm{I}}}{\to \mathrm{I} \supset \mathrm{I}}}{\to \mathrm{I} \wedge (\mathrm{I} \supset \mathrm{I})}}{\mathrm{I} \to \mathrm{I} \wedge (\mathrm{I} \supset \mathrm{I})}}{\mathrm{I} \supset \mathrm{I} \to \mathrm{I} \wedge (\mathrm{I} \supset \mathrm{I})}}{\to (\mathrm{I} \supset \mathrm{I}) \supset (\mathrm{I} \wedge (\mathrm{I} \supset \mathrm{I}))}}{\mathrm{I} \to (\mathrm{I} \supset \mathrm{I}) \supset (\mathrm{I} \wedge (\mathrm{I} \supset \mathrm{I}))}}{\mathrm{I}, \mathrm{I} \to (\mathrm{I} \supset \mathrm{I}) \supset (\mathrm{I} \wedge (\mathrm{I} \supset \mathrm{I}))}}{\mathrm{I}, (\mathrm{I} \supset \mathrm{I}) \supset \mathrm{I} \to (\mathrm{I} \supset \mathrm{I}) \supset (\mathrm{I} \wedge (\mathrm{I} \supset \mathrm{I}))}}{\mathrm{I}, (\mathrm{I} \supset \mathrm{I}) \supset \mathrm{I}, \mathrm{I} \to (\mathrm{I} \supset \mathrm{I}) \supset (\mathrm{I} \wedge (\mathrm{I} \supset \mathrm{I}))}}{\mathrm{I}, (\mathrm{I} \supset \mathrm{I}) \supset \mathrm{I}, \mathrm{I}, \mathrm{I} \to (\mathrm{I} \supset \mathrm{I}) \supset (\mathrm{I} \wedge (\mathrm{I} \supset \mathrm{I}))}}{\dfrac{\dfrac{\mathrm{I} \wedge ((\mathrm{I} \supset \mathrm{I}) \supset \mathrm{I}), \mathrm{I}, \mathrm{I} \to (\mathrm{I} \supset \mathrm{I}) \supset (\mathrm{I} \wedge (\mathrm{I} \supset \mathrm{I}))}{\mathrm{I} \wedge ((\mathrm{I} \supset \mathrm{I}) \supset \mathrm{I}), \mathrm{I} \wedge \mathrm{I} \to (\mathrm{I} \supset \mathrm{I}) \supset (\mathrm{I} \wedge (\mathrm{I} \supset \mathrm{I}))}}{(\mathrm{I} \wedge ((\mathrm{I} \supset \mathrm{I}) \supset \mathrm{I})) \wedge (\mathrm{I} \wedge \mathrm{I}) \to (\mathrm{I} \supset \mathrm{I}) \supset (\mathrm{I} \wedge (\mathrm{I} \supset \mathrm{I}))}}$$

(x) finally, let $P\colon \Gamma \to A$ be any proof satisfying conditions (iv) and (vi); let $P'\colon \Gamma' \to A$ result from P by the deletion of all instances of (W) at the end of P, and let $A_{i_1}, \ldots, A_{i_m}$ be the constant terms of Γ', listed in their occurrence from left to right in Γ'. Then it follows from the fact that (C), (R), (H), and (P) precede (K), that P' contains a subproof $P''\colon \Gamma'' \to A$, where Γ'' results from Γ' by the deletion of $A_{i_1}, \ldots, A_{i_m}$.

By condition (ix) and the lemma on constants, there exists a unique proof Q': Γ'', $A_{i_1}, \ldots, A_{i_m} \rightarrow A$ in which the A_{i_j} are introduced in the manner described in (ix). $P \geqslant Q$, where Q results from Q' by a single application of rule (P), if required, possibly followed by applications of (W), and has the further property that any of its subproofs whose last step is an application of rules (R) with an active constant term satisfy condition (ix). This completes the description of the reducibility relation $\geqslant$.

5 *The Church-Rosser property* In this section, we examine the connection between the relations $\geqslant$ and $\equiv$ on Proofs(**G**).

Theorem *If $P \geqslant Q$, then $P \equiv Q$.*

We omit the lengthy calculations required to establish this theorem. They are reasonably straightforward consequences of the axioms of a closed category. The following example illustrates the general method. By condition (iv) of $\geqslant$ rule (C) precedes rule (H), hence

$$\frac{\dfrac{\Gamma \xrightarrow{P} A \quad \Delta, B, \Lambda \xrightarrow{Q} C}{\Delta, \Gamma, A \supset B, \Lambda \rightarrow C} \quad \Phi \xrightarrow{R} D}{\Delta, \Gamma, A \supset B, \Lambda, \Phi \rightarrow C \wedge D} \geqslant \frac{\Gamma \xrightarrow{P} A \quad \dfrac{\Delta, B, \Lambda \xrightarrow{Q} C \quad \Phi \xrightarrow{R} D}{\Delta, B, \Lambda, \Phi \rightarrow C \wedge D}}{\Delta, \Gamma, A \supset B, \Lambda, \Phi \rightarrow C \wedge D}.$$

We must show that $\varphi(L.H.S.) = \varphi(R.H.S.)$. Suppose that

$$\varphi(\Gamma \xrightarrow{P} A) = E \xrightarrow{f} A;$$
$$\varphi(\Delta, B, \Lambda \xrightarrow{Q} C) = (G \wedge B) \wedge H \xrightarrow{g} C;$$
$$\varphi(\Phi \xrightarrow{R} D) = K \xrightarrow{h} D.$$

Then

$$\varphi(L.H.S.) = (g \circ (1(G) \wedge \varepsilon(A, B) \circ f \wedge 1(A \supset B)) \wedge 1(H) \circ \alpha^{-1}(G, E \wedge (A \supset B), H) \wedge h$$

and

$$\varphi(R.H.S.)$$
$$= g \wedge h \circ (1(G) \wedge \varepsilon(\Delta, B) \circ f \wedge 1(A \supset B)) \wedge 1(H) \circ \alpha^{-1}(G, E \wedge (A \supset B), H)) \wedge 1(K),$$

and therefore $\varphi(L.H.S.) = \varphi(R.H.S.)$ by the functoriality of $\wedge$. The other cases are proved similarly.

We now come to the main theorem of this section:

Church-Rosser Theorem *If $\varphi(P) = \varphi(Q)$ and P and Q prove the same formula, then there exists a unique normal proof R such that $P \geqslant R$ and $Q \geqslant R$.*

Proof: Since all rules of inference of **G** are cumulative, the theorem holds trivially for the proofs $0 \rightarrow 0$ and $\rightarrow$ I. Let us therefore assume that P and Q contain an application of a rule of inference. Then P and Q must in fact contain the same *number* of applications of rules (C), (R), (H), (K), and (W). In order to convince ourselves of this fact, we exhibit an algorithm for determining the rule by which a $\wedge$, $\supset$, or I was introduced into the formula $\Gamma \rightarrow A$ proved by P and Q. For this purpose, we label the $\wedge$'s and $\supset$'s as "$\underset{a}{\wedge}$", "$\underset{s}{\wedge}$", "$\underset{a}{\supset}$", or "$\underset{s}{\supset}$" in the following way:

(i) If $\Gamma \to A$ is $\Delta, B \wedge C, \Lambda \to A$, we write $\Delta, B \underset{w}{\wedge} C, \Lambda \to A$;
(ii) If $\Gamma \to A$ is $\Gamma \to B \wedge C$, we write $\Gamma \to B \underset{c}{\wedge} C$;
(iii) If $\Gamma \to A$ is $\Delta, B \supset C, \Lambda \to A$, we write $\Delta, B \underset{h}{\supset} C, \Lambda \to A$;
(iv) If $\Gamma \to A$ is $\Gamma \to B \supset C$, we write $\Gamma \to B \underset{r}{\supset} C$;
(v) If $\Gamma \to A$ is $\Delta, \mathrm{I} \to A$, we write $\Gamma, \underset{k}{\mathrm{I}} \to A$.

The following table, which is easily established by an induction on the heights of P and Q, provides the required algorithm for the labelling of all $\wedge$'s and $\supset$'s, and all I's introduced by rule (K):

(i)	If $B \underset{w}{\wedge} (C \wedge D)$, then $C \underset{w}{\wedge} D$;	(i′)	If $B \underset{c}{\wedge} (C \wedge D)$, then $C \underset{c}{\wedge} D$;
(ii)	If $(B \wedge C) \underset{w}{\wedge} D$, then $B \underset{w}{\wedge} C$;	(ii′)	If $(B \wedge C) \underset{c}{\wedge} D$, then $B \underset{c}{\wedge} C$;
(iii)	If $B \underset{w}{\wedge} (C \supset D)$, then $C \underset{h}{\supset} D$;	(iii′)	If $B \underset{c}{\wedge} (C \supset D)$, then $C \underset{r}{\supset} D$;
(iv)	If $(B \supset C) \underset{w}{\supset} D$, then $B \underset{h}{\supset} C$;	(iv′)	If $(B \supset C) \underset{c}{\wedge} D$, then $B \underset{r}{\supset} C$;
(v)	If $(B \wedge C) \underset{h}{\supset} D$, then $B \underset{c}{\wedge} C$;	(v′)	If $(B \wedge C) \underset{r}{\supset} D$, then $B \underset{w}{\wedge} C$;
(vi)	If $B \underset{h}{\supset} (C \wedge D)$, then $C \underset{w}{\wedge} D$;	(vi′)	If $B \underset{r}{\supset} (C \wedge D)$, then $C \underset{c}{\wedge} D$;
(vii)	If $(B \supset C) \underset{h}{\supset} D$, then $B \underset{r}{\supset} C$;	(vii′)	If $(B \supset C) \underset{r}{\supset} D$, then $B \underset{h}{\supset} C$;
(viii)	If $B \underset{h}{\supset} (C \supset D)$, then $C \underset{h}{\supset} D$;	(viii′)	If $B \underset{r}{\supset} (C \supset D)$, then $C \underset{r}{\supset} D$;
(ix)	If $A \underset{w}{\wedge} \mathrm{I}$, then $\underset{k}{\mathrm{I}}$;	(ix′)	If $\mathrm{I} \underset{w}{\wedge} A$, then $\underset{k}{\mathrm{I}}$;
(x)	If $A \underset{h}{\supset} \mathrm{I}$, then $\underset{k}{\mathrm{I}}$;	(x′)	If $\mathrm{I} \underset{r}{\supset} A$, then $\underset{k}{\mathrm{I}}$;

(xi) If I occurs as an antecedent term, then $\underset{k}{\mathrm{I}}$, and if I is not $\underset{k}{\mathrm{I}}$, then it stems from an instance of axiom $\langle \mathrm{I} \rangle$.

Since P and Q prove the same formula, they must thus also contain the same number of instances of axioms $\langle 0, 0\rangle$ and $\langle \mathrm{I}\rangle$, and can therefore differ at most in the following respects:

(i) the *order* of the axioms as they occur in these proofs from left to right;
(ii) the *order* of the application of rules (C), (R), (H), (K), and (W);
(iii) the *number* and *order* of applications of rule (P).

In order to prove the theorem, we must therefore show that there exist two finite sequences of proofs $\langle P_1, \ldots, P_n\rangle$ $(n \geqslant 1)$ and $\langle Q_1, \ldots, Q_m\rangle$ $(m \geqslant 1)$ with the following properties: (i) $P_1 = P$ and $Q_1 = Q$; (ii) $P_i \geqslant P_{i+1}$ $(1 \leqslant i \leqslant n - 1)$ and $Q_j \geqslant Q_{j+1}$ $(1 \leqslant j \leqslant m - 1)$; (iii) P_n and Q_m are normal; (iv) $P_n = Q_m$. Since the number of applications of rules (C), (R), (H), (K), and (W) in all P_i and Q_j is identical, and since the number of reduction pairs listed in conditions (iv)-(x) of $\geqslant$ is finite and $\geqslant$ is antisymmetric, the existence of the sequences $\langle P_1, \ldots, P_n\rangle$ and $\langle Q_1, \ldots, Q_m\rangle$ follows easily by an induction on the number of possible "contractions" of P and Q, where a *contraction* of a proof R to a proof R' is defined to be the replacement according to condition (iii) of $\geqslant$ of a subproof S of R by a proof T with the following properties: (i) $S \neq T$; (ii) $S \geqslant T$; (iii) T is "permutation-reduced", i.e., T does not contain two consecutive applications of rule (P) and contains no (P) interchanging identical constant terms. Thus the sequences $\langle P_1, \ldots, P_n\rangle$ and $\langle Q_1, \ldots, Q_m\rangle$ are obtained by replacing the proofs P_i and Q_j successively by two proofs P_{i+1} and Q_{i+1} which are permutation-reduced and which admit fewer contractions than P_i and Q_j. This process terminates after finitely many steps since there are only finitely many possible permutations of the applications of rules (C), (R), (H), (K), and (W). The following example serves as an illustration:

Suppose P, Q: $0, 0 \supset \mathrm{I}, 0, \mathrm{I} \to 0 \wedge \mathrm{I}$ and are given by trees (a) and (b) respectively:

(a)
$$\dfrac{0 \to 0 \quad \dfrac{\dfrac{\dfrac{0 \to 0 \quad \dfrac{\to \mathrm{I}}{\mathrm{I} \to \mathrm{I}}}{0, \mathrm{I} \to 0 \wedge \mathrm{I}}}{0, \mathrm{I}, \mathrm{I} \to 0 \wedge \mathrm{I}}}{\mathrm{I}, 0, \mathrm{I} \to 0 \wedge \mathrm{I}}}{0, 0 \supset \mathrm{I}, 0, \mathrm{I} \to 0 \wedge \mathrm{I}}$$

(b)
$$\dfrac{\dfrac{\dfrac{0 \to 0 \quad \dfrac{\dfrac{0 \to 0 \quad 0 \to \mathrm{I}}{0 \to 0 \wedge \mathrm{I}}}{0, \mathrm{I} \to 0 \wedge \mathrm{I}}}{0, 0, 0 \supset \mathrm{I} \to 0 \wedge \mathrm{I}}}{0, 0, 0 \supset \mathrm{I}, \mathrm{I} \to 0 \wedge \mathrm{I}}}{0, 0 \supset \mathrm{I}, 0, \mathrm{I} \to 0 \wedge \mathrm{I}.}$$

Then the sequences $\langle P_1, P_2, P_3, P_4, P_5\rangle$ and $\langle Q_1, Q_2\rangle$ establish the existence of the normal proofs P_5 and Q_2:

P_2:
$$\dfrac{0 \to 0 \quad \dfrac{\dfrac{\dfrac{\dfrac{0 \to 0 \quad \to \mathrm{I}}{0 \to 0 \wedge \mathrm{I}}}{0, \mathrm{I} \to 0 \wedge \mathrm{I}}}{0, \mathrm{I}, \mathrm{I} \to 0 \wedge \mathrm{I}}}{\mathrm{I}, 0, \mathrm{I} \to 0 \wedge \mathrm{I}}}{0, 0 \supset \mathrm{I}, 0, \mathrm{I} \to 0 \wedge \mathrm{I};}$$

P_3:
$$\dfrac{\dfrac{0 \to 0 \quad \dfrac{\dfrac{\dfrac{0 \to 0 \quad \to \mathrm{I}}{0 \to 0 \wedge \mathrm{I}}}{0, \mathrm{I} \to 0 \wedge \mathrm{I}}}{0, \mathrm{I}, \mathrm{I} \to 0 \wedge \mathrm{I}}}{0, 0, 0 \supset \mathrm{I}, \mathrm{I} \to 0 \wedge \mathrm{I}}}{0, 0 \supset \mathrm{I}, 0, \mathrm{I} \to 0 \wedge \mathrm{I};}$$

$P_4 = Q_1$; and $P_5 = Q_2$, where Q_2 is the normal proof obtained from Q_1 by interchanging the applications of (K) and (P) at the end of the proof.

Hence it remains to prove that P and Q contract to the same normal proof R. This will follow from the previous theorem and the fact that distinct normal proofs of the same formula interpret as distinct morphisms in $\mathsf{F}(1)$ under φ. For suppose that P and Q are two normal proofs of the formula $\Gamma \to A$. By virtue of conditions (iv), (ix), and (x) of $\geqslant$ we may ignore those applications of (R) in P and Q whose active antecedent terms are constant and may assume without loss of generality that Γ is empty or consists entirely of non-constant terms A_i of the form 0 or $B_i \supset C_i$. We may also neglect the case where Γ is empty, since this forces A to be constant if P (and hence also Q) contains no application of rule (R). Otherwise it is clearly sufficient to compare the respective subproofs of P and Q whose last lines are formulas with non-empty antecedents. Thus we need compare only two normal proofs P and Q of a formula $A_1, \ldots, A_p \to A$ such that

(i) all A_i are non-constant and of the form 0 or $B_i \supset C_i$;
(ii) the last application of a rule of inference is (C), (R), (H), or (P);
(iii) the active antecedent term of any application of (R) is non-constant;
(iv) the active terms of any application of (P) are non-constant.

We shall examine P and Q from three points of view:

(a) neither P nor Q ends with a (P);
(b) P ends with a (P), but Q does not;
(c) both P and Q end with a (P).

Suppose (a) holds. Then it follows from (iv) of $\geqslant$ and our previous remark that either both P and Q end with a (C), or both end with an (R), or both end with an (H). Then the following three possibilities arise:

(i) P is $\dfrac{\Gamma \xrightarrow{R} B \quad \Gamma \xrightarrow{S} B}{\Gamma,\, \Gamma \to B \wedge B}$, Q is $\dfrac{\Gamma \xrightarrow{S} B \quad \Gamma \xrightarrow{R} B}{\Gamma,\, \Gamma \to B \wedge B}$;

(ii) P is $\dfrac{\Gamma \xrightarrow{R} C}{\Delta \to B \supset C}$, Q is $\dfrac{\Lambda \xrightarrow{S} C}{\Delta \to B \supset C}$;

(iii) P is $\dfrac{\Gamma, B, \Delta \xrightarrow{R} B \quad \Gamma, B, \Delta \xrightarrow{S} B}{\Gamma, \Gamma, B, \Delta, B \supset B, \Delta \to B}$, Q is $\dfrac{\Gamma, B, \Delta \xrightarrow{S} B \quad \Gamma, B, \Delta \xrightarrow{R} B}{\Gamma, \Gamma, B, \Delta, B \supset B, \Delta \to B}$.

If the premisses in cases (i) and (iii) interpret as distinct morphisms in $\mathsf{F}(1)$, it is clear from the interpretation of (C) given earlier that P and Q interpret as distinct morphisms, hence $P \not\equiv Q$. If the premisses in case (ii) interpret as the same morphism and the active non-constant term B stands in different places in Γ and Λ, then it follows from the interpretation of (R) given earlier that the interpretations of P and Q contain different components of σ and hence represent distinct morphisms, and therefore once again $P \not\equiv Q$. If the premisses in case (ii) interpret as different morphisms, then it is clear from (iv) of $\geqslant$ and the fact that P and Q are "permutation-reduced", that P and Q again represent distinct morphisms, hence $P \not\equiv Q$.

Suppose (b) holds. Then the following possibilities arise:

(i) P is $\dfrac{\dfrac{\Gamma \xrightarrow{R} B \quad \Gamma \xrightarrow{S} B}{\Gamma,\, \Gamma \to B \wedge B}}{\Gamma,\, \Gamma \to B \wedge B}$, Q is $\dfrac{\Gamma \xrightarrow{R} B \quad \Gamma \xrightarrow{S} B}{\Gamma,\, \Gamma \to B \wedge B}$;

(ii) P is as in (i), Q is $\dfrac{\Gamma \xrightarrow{S} S \quad \Gamma \xrightarrow{R} B}{\Gamma,\, \Gamma \to S \wedge B}$;

(iii) P is $\dfrac{\dfrac{\Gamma \xrightarrow{R} C}{\Delta \to B \supset C}}{\Lambda \to B \supset C}\,(\mathrm{P})$, Q is $\dfrac{\Phi \xrightarrow{S} C}{\Lambda \to B \supset C}$;

(iv) P is $\dfrac{\dfrac{\Gamma, B, \Delta \xrightarrow{R} B \quad \Phi, B, \Psi \xrightarrow{S} B}{\Phi, \Gamma, B, \Delta, B \supset B, \Delta \to B}}{\Gamma, \Phi, B, \Psi, B \supset B, \Delta \to B}$, Q is $\dfrac{\Phi, B, \Psi \xrightarrow{S} B \quad \Gamma, B, \Delta \xrightarrow{R} B}{\Gamma, \Phi, B, \Psi, B \supset B, \Delta \to B}$.

The morphisms represented by P and Q in (i) are clearly distinct since in their interpretation P involves the composition of the morphism represented by Q with a non-constant component of σ. Similarly the morphisms represented by P and Q in (ii) are distinct since the component of σ involved is non-constant and since in the category of sets, for example, the values of the function represented by P are of the form $f(-) \times g(-)$ and those of the function represented by Q are of the form $g(-) \times f(-)$. In case (iii), P and Q represent different morphisms by the argument advanced in (a)(ii) and the fact that the morphism represented by P involves a composition with a non-constant component of σ. In case (iv), P and Q represent distinct morphisms by an argument analogous to (a)(iii) and (b)(iii). Hence in all cases $P \not\equiv Q$.

Suppose that (c) holds. Then the following possibilities arise:

(i) P is $\dfrac{\dfrac{\Gamma \xrightarrow{R} B \quad \Delta \xrightarrow{S} B}{\Gamma, \Delta \to B \wedge B}}{\Lambda \to B \wedge B}(\mathrm{P})$, Q is $\dfrac{\dfrac{\Delta \xrightarrow{S} B \quad \Gamma \xrightarrow{R} B}{\Gamma, \Delta \to B \wedge B}}{\Lambda \to B \wedge B}(\mathrm{P})$;

(ii) P is $\dfrac{\dfrac{\Gamma \xrightarrow{R} C}{\Delta \to B \supset C}}{\Lambda \to B \supset C}(\mathrm{P})$, Q is $\dfrac{\dfrac{\Phi \xrightarrow{S} C}{\Psi \to B \supset C}}{\Lambda \to B \supset C}(\mathrm{P})$;

(iii) P is $\dfrac{\dfrac{\Gamma, B, \Delta \xrightarrow{R} B \quad \Phi, B, \Psi \xrightarrow{S} B}{\Phi, \Gamma, B, \Delta, B \supset B, \Psi \to B}}{\Lambda \to B}(\mathrm{P})$;

Q is $\dfrac{\dfrac{\Phi, B, \Psi \xrightarrow{S} B \quad \Gamma, B, \Delta \xrightarrow{R} B}{\Gamma, \Phi, \Psi, B \supset B, \Delta \to B}}{\Lambda \to B}(\mathrm{P})$.

The arguments which show that P and Q in (i)-(iii) represent distinct morphisms if R and S interpret as different morphisms in $\mathsf{F}(1)$ are similar to the previous ones. Otherwise we may assume that P and Q differ only by ending with different instances of (P). Since all terms permuted by these (P)'s are non-constant, the morphisms represented by P and Q therefore involve compositions with different non-constant components of σ and therefore also represent distinct morphisms. Hence $P \not\equiv Q$. This concludes the proof of the theorem.

6 *The cut* If we let $\mathrm{Obj}(\mathfrak{C}) = \mathrm{Term}(\mathcal{L})$ and $\mathrm{Morph}(\mathfrak{C}) = \{P \mid P: A \to B$ is a normal proof of $\mathbf{G}$ for some $A, B \in \mathrm{Obj}(\mathfrak{C})\}$, then it is lengthy, but fairly routine, to prove that the system $\langle \mathrm{Obj}(\mathfrak{C}), \mathrm{Morph}(\mathfrak{C})\rangle$ is isomorphic to $\mathsf{F}(1)$, provided that $\mathrm{Morph}(\mathfrak{C})$ is "closed under composition", i.e., if $P: A \to B$, $Q: B \to C \in \mathrm{Morph}(\mathfrak{C})$, and $\varphi(P: A \to B) = f: A \to B$ and $\varphi(Q: B \to C) = g: B \to C$ in $\mathsf{F}(1)$, then there exists $R: A \to C \in \mathrm{Morph}(\mathfrak{C})$ such that $\varphi(R: A \to C) = g \circ f: A \to C$ in $\mathsf{F}(1)$.

For this purpose we introduce the "cut" (S) as an additional rule of inference on $\mathrm{Form}(\mathcal{L})$. By (S) we mean the ternary relation on $\mathrm{Form}(\mathcal{L})$ consisting of all lists of the form

$$\langle \Gamma \to A;\ \Delta, A, \Lambda \to B;\ \Delta, \Gamma, \Lambda \to B\rangle.$$

We append condition (c) in the definition of a "proof" relative to (S), and if P: $\Gamma \to A$ and Q: $\Delta, A, \Lambda \to B$ are two normal proofs of $\mathbf{G}$, we interpret the proof

$$\frac{\Gamma \xrightarrow{P} A \quad \Delta, A, \Lambda \xrightarrow{Q} B}{\Delta, \Gamma, \Lambda \to B}(\mathrm{S})$$

in $\mathsf{F}(1)$ as follows:

(a) if $\Gamma \xrightarrow{P} A = f: C \to A$ and $\Delta, A, \Lambda \xrightarrow{Q} B = g: (D \wedge C) \wedge E \to B$, then $\dfrac{\Gamma \xrightarrow{P} A \quad \Delta, A, \Lambda \xrightarrow{Q} B}{\Delta, \Gamma, \Lambda \to B} = g \circ ((1(D) \wedge f) \wedge 1(E)): (D \wedge C) \wedge E \to B$;

(b) if $\Gamma \xrightarrow{P} A = f: C \to A$ and $\Delta, A, \Lambda \xrightarrow{Q} B = h: D \wedge (C \wedge E) \to B$, then $\dfrac{\Gamma \xrightarrow{P} A \quad \Delta, A, \Lambda \xrightarrow{Q} B}{\Delta, \Gamma, \Lambda \to B} = h \circ (1(D) \wedge (f \wedge 1(E))): D \wedge (C \wedge E) \to B$.

It is understood that if $\Lambda = \emptyset$ and $\Delta \neq \emptyset$, $\Lambda \neq \emptyset$ and $\Delta = \emptyset$, or $\Lambda = \Delta = \emptyset$, then the cut interprets as $g \circ (1(D) \wedge f)$: $D \wedge C \to B$, $h \circ (f \wedge 1(E))$: $C \wedge E \to B$, and $h \circ f = g \circ f$: $C \to B$, respectively.

In order to have a collective name for these morphisms, we shall loosely call $g \circ ((1(D) \wedge f) \wedge 1(E))$, $g \circ (1(D) \wedge f)$, and $g \circ f$ "composites" of g and f. Similarly for h and f.

The following theorem then establishes the required closure under composition of Morph($\mathfrak{C}$):

Cut Definability Theorem *If P: $\Gamma \to A$ and Q: $\Delta, A, \Lambda \to B$ are two normal proofs of* **G**, *then there exists a normal proof R: $\Delta, \Gamma, \Lambda \to B$ such that R interprets as the composite of the interpretations of P and Q.*

Proof: We follow Gentzen [12] and carry out two inductions on the "degrees" and "ranks" of the cuts involved. Suppose that S: $\Delta, \Gamma, \Lambda \to B$ is the proof

$$\frac{\Gamma \xrightarrow{P} A \quad \Delta, A, \Lambda \xrightarrow{Q} B}{\Delta, \Gamma, \Lambda \to B}\ (\mathrm{S}).$$

The *degree* of S is the number of occurrences of the symbols $\wedge$ and $\supset$ in the term A eliminated by the cut. The *rank* of S is the sum of the "left and right ranks" of S. The *left rank* of S is the number of "consecutive" formulas in P whose succedent is A. The *right rank* of S is defined dually to be the number of "consecutive" formulas of Q containing the term A which is being eliminated by S, in the antecedent. Two formulas $\Phi \to C$ and $\Psi \to D$ are *consecutive* in a proof $\langle R, T\rangle$ if there exists a path $\langle R_1, \ldots, R_n\rangle$ in T such that $R(R_i) = \Phi \to C$ and $R(R_{i+1}) = \Psi \to D$ for some $1 \leqslant i \leqslant n - 1$.

Suppose that at least one of P and Q quotes an axiom of **G**. Then we must define four cases:

(i) $$\frac{0 \to 0 \quad 0 \to 0}{0 \to 0} = 0 \to 0;$$

(ii) $$\frac{\Gamma \xrightarrow{P} 0 \quad 0 \to 0}{\Gamma \to 0} = \Gamma \xrightarrow{P} 0;$$

(iii) $$\frac{0 \to 0 \quad \Delta, 0, \Lambda \xrightarrow{Q} B}{\Delta, 0, \Lambda \to B} = \Delta, 0, \Lambda \xrightarrow{Q} B;$$

(iv) $$\frac{\to \mathrm{I} \quad \dfrac{\Delta \xrightarrow{Q} B}{\Delta, \mathrm{I} \to B}}{\Delta \to B} = \Delta \xrightarrow{Q} B.$$

In all other cases we may assume that the last line of both P and Q is an application of a rule of inference (other than (S)). Suppose that the last line of P is an application of (H), (P), (K), or (W). We shall illustrate the case of (H). The remaining cases are similar.

$$\frac{\dfrac{\Theta \xrightarrow{M} C \quad \Phi, D, \Psi \xrightarrow{N} A}{\Phi, \Theta, C \supset D, \Psi \to A} \quad \Delta, A, \Lambda \xrightarrow{Q} B}{\Delta, \Phi, \Theta, C \supset D, \Psi, \Lambda \to B} = \frac{\Theta \xrightarrow{M} C \quad \dfrac{\Phi, D, \Psi \xrightarrow{N} A \quad \Delta, A, \Lambda \xrightarrow{Q} B}{\Delta, \Phi, D, \Psi, \Lambda \to B}}{\Delta, \Phi, \Theta, C \supset D, \Psi, \Lambda \to B}$$

The cases in which the right rank of $S > 1$ are dealt with similarly. For example,

$$\frac{\Gamma \xrightarrow{P} A \quad \dfrac{\Delta, C, \Phi, A, \Lambda \xrightarrow{N} D}{\Delta, \Phi, A, \Lambda \to C \supset D}}{\Delta, \Phi, \Gamma, \Lambda \to C \supset D} = \frac{\dfrac{\Gamma \xrightarrow{P} A \quad \Delta, C, \Phi, A, \Lambda \xrightarrow{N} D}{\Delta, C, \Phi, \Gamma, \Lambda \to D}}{\Delta, \Phi, \Gamma, \Lambda \to C \supset D}.$$

If P ends with an application of (K) and Q ends with an application of (P), an appropriate (P) may have to follow the simpler cut in order to derive the same formula. Hence we may assume that the cut S has rank 2. We must distinguish two cases:

(i)

$$\frac{\dfrac{\Phi \xrightarrow{L} C \quad \Psi \xrightarrow{M} D}{\Phi, \Psi \to C \wedge D} \quad \dfrac{\Delta, C, D, \Lambda \xrightarrow{N} B}{\Delta, C \wedge D, \Lambda \to B}}{\Delta, \Phi, \Psi, \Lambda \to B} = \frac{\Phi \xrightarrow{L} C \quad \dfrac{\Psi \xrightarrow{M} D \quad \Delta, C, D, \Lambda \xrightarrow{N} B}{\Delta, C, \Psi, \Lambda \to B}}{\Delta, \Phi, \Psi, \Lambda \to B};$$

(ii)

$$\frac{\dfrac{\Phi, C, \Psi \xrightarrow{L} D}{\Phi, \Psi \to C \supset D} \quad \dfrac{\Theta \xrightarrow{M} C \quad \Sigma, D, \Omega \xrightarrow{N} B}{\Sigma, \Theta, C \supset D, \Omega \to B}}{\Sigma, \Theta, \Phi, \Psi, \Omega \to B}$$

$$= \frac{\Theta \xrightarrow{M} C \quad \dfrac{\Phi, C, \Psi \xrightarrow{L} D \quad \Sigma, D, \Omega \xrightarrow{N} B}{\Sigma, \Phi, C, \Psi, \Omega \to B}}{\dfrac{\Sigma, \Phi, \Theta, \Psi, \Omega \to B}{\Sigma, \Theta, \Phi, \Psi, \Omega}\,\text{(P)}}$$

This completes the inductive definition of the normal proof R associated with the cut S. It is understood that the various proofs designated by L, M, and N above are normal. It is routine to verify by means of the interpretation of the rules of inference as operations in $\mathsf{F}(1)$ that in each case the definiens and the definiendum represent the same morphism. The proof of the theorem is therefore complete.

Corollary *The following equations hold between proofs with cut, provided L, M, N, P, Q, and R are normal:*

(i)

$$\frac{A \xrightarrow{1(A)} A \quad \Gamma, A, \Delta \xrightarrow{P} B}{\Gamma, A, \Delta \to B} = \Gamma, A, \Delta \xrightarrow{P} B;$$

(ii)

$$\frac{\Gamma \xrightarrow{Q} A \quad A \xrightarrow{1(A)} A}{\Gamma \to A} = \Gamma \xrightarrow{Q} A;$$

(iii)

$$\frac{\Psi \xrightarrow{L} A \quad \dfrac{\Phi \xrightarrow{M} B \quad \Gamma, A, \Delta, B, \Lambda \xrightarrow{N} C}{\Gamma, A, \Delta, \Phi, \Lambda \to C}}{\Gamma, \Psi, \Delta, \Phi, \Lambda \to C} = \frac{\Phi \xrightarrow{M} B \quad \dfrac{\Psi \xrightarrow{L} A \quad \Gamma, A, \Delta, B, \Lambda \xrightarrow{N} C}{\Gamma, \Psi, \Delta, B, \Lambda \to C}}{\Gamma, \Psi, \Delta, \Phi, \Lambda \to C};$$

(iv)

$$\frac{\dfrac{\Gamma \xrightarrow{L} A \quad \Delta, A, \Lambda \xrightarrow{M} B}{\Delta, \Gamma, \Lambda \to B} \qquad \Phi, B, \Psi \xrightarrow{N} C}{\Phi, \Delta, \Gamma, \Lambda, \Psi \to C} = \frac{\Gamma \xrightarrow{L} A \qquad \dfrac{\Delta, A, \Lambda \xrightarrow{M} B \quad \Phi, B, \Psi \xrightarrow{N} C}{\Phi, \Delta, A, \Lambda, \Psi \to C}}{\Phi, \Delta, \Gamma, \Lambda, \Psi \to C}.$$

We omit the lengthy calculations required to establish this corollary.

Since the equations in the previous corollary are precisely the axioms satisfied by the multimaps of a multicategory in the sense of [5] (*cf.* also [11]), we obtain the further

Corollary *The normal proofs of* **G** *constitute a Lambek multicategory.*

The results of this paper have an important application in the theory of categories, since the isomorphism of the system $\langle \mathsf{Ojb}(\mathfrak{C}), \mathsf{Morph}(\mathfrak{C})\rangle$ and $\mathsf{F}(1)$ yields necessary and sufficient conditions for the equality of morphisms in $\mathsf{F}(1)$:

Theorem *If* $f, g: A \to B$ *are two morphisms of* $\mathsf{F}(1)$, *then* $f = g$ *iff* $f = \varphi(P) = g$, *for some* $P \in \mathsf{Morph}(\mathfrak{C})$.

The advantage of this criterion for equality of morphisms in $\mathsf{F}(1)$ lies in the fact that the set of all normal proofs $P: A \to B$ in **G** is finite and computable. For any given objects A and B of $\mathsf{F}(1)$, we can therefore give a complete explicit description of $\mathsf{Hom}_{\mathsf{F}(1)}(A, B)$. The details of this algorithm are dealt with in [14].

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Sir George Williams Campus
Concordia University
Montreal, Quebec, Canada