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RECURSIVE AND RECURSIVELY ENUMERABLE MANIFOLDS. II

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CHAPTER IV—MORPHISMS, TYPES AND TYPE-DEGREES

The most unpleasant feature of the Theory of **REM**'s is that compositions of recursive maps are not necessarily recursive. I shall remedy this situation by considering some more restricted recursive maps, *morphisms*. Obviously, morphisms will reduce to classical recursive maps in case we consider enumerated sets only. My aim in this chapter* is to start a classification of **REM**'s using maps, more exactly: morphisms, between pairs of **REM**'s. Here, I have no analogy with the classical enumeration theory to follow: the content of Chapter III is sufficient for classification of enumerated sets; however, it is useless for comparison of atlases on disjoint sets, and for classification of **REM**'s.

By $\langle A, \mathfrak{A}\rangle$, $\langle B, \mathfrak{B}\rangle$, $\langle C, \mathfrak{C}\rangle$, I denote **REM**'s, with usual notation for atlases: $\mathfrak{A} = \{\alpha_p | p \in P\}$, $\mathfrak{B} = \{\beta_q | q \in Q\}$, $\mathfrak{C} = \{\gamma_r | r \in R\}$, ... Also I write A_p, B_q, C_r, . . ., for respective ranges of α_p, β_q, γ_r, ... Sometimes I shall use the **REM** $\langle M, \mathfrak{M}\rangle$, with $\mathfrak{M} = \{\mu_t | t \in T\}$ and M_t = range of μ_t. To shorten these notations, I shall write **a**, **b**, **c**, . . ., **m** for **REM**'s $\langle A, \mathfrak{A}\rangle$, $\langle B, \mathfrak{B}\rangle$, $\langle C, \mathfrak{C}\rangle$, . . ., $\langle M, \mathfrak{M}\rangle$ respectively.

Definition 4.1 (i) A map $f: A \to B$ is *compact* iff, for every $q \in Q$, $f^{-1}(B_q)$ can be covered by finite many A_p's.
(ii) ($\mathfrak{A}$-$\mathfrak{B}$-)recursive and compact maps are called *morphisms*; and *inmorphisms*, *surmorphisms*, and *bimorphisms* in case they are injective, surjective, and bijective respectively.
(iii) A morphism $f: A \to B$, such that each $f_{p,q}$ in (1.6) is injective, is called a *unimorphism*.

Lemma 4.1 *Composition of morphisms is a morphism.*

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Proof: Let $f: A \to B$ and $g: B \to C$ be morphisms, and let $h = g \circ f: A \to C$. We have to prove: for every pair $\langle p, r\rangle \in P \times R$ there is a p.r. function $f_{p,r}$, with domain $\mathbf{D}_{p,r} = \alpha_p^{-1}(f^{-1}(C_r))$ and such that

$$h(\alpha_p(n)) = \gamma_r(f_{p,r}(n)) \text{ for all } n \in \mathbf{D}_{p,r}. \tag{4.1}$$

(The fact that $h^{-1}(C_r)$ can be covered by finite many A_p's is trivial.) Suppose that $\{B_{q_1}, \ldots, B_{q_m}\}$ covers $g^{-1}(C_r)$, and let f_{p,q_i}, $i = 1, \ldots, m$, be partial recursive with

$$\alpha^{-1}(f^{-1}(B_{q_i} \cap g^{-1}(C_r)))$$

as domain, and such that

$$f(\alpha_p(n)) = \beta_{q_i}(f_{p,q_i}(n)) \text{ for } n \in \mathbf{D}_{f_{p,q_i}}.$$

Also, let $f_{q_i,r}$, $i = 1, \ldots, m$, be partial recursive, with $\beta_{q_i}^{-1}(g^{-1}(C_r))$ as domain, and such that

$$g(\beta_{q_i}(n)) = \gamma_r(f_{q_i,r}(n)) \text{ for } n \in \mathbf{D}_{f_{q_i,r}}.$$

Since (see Figure 4.1)

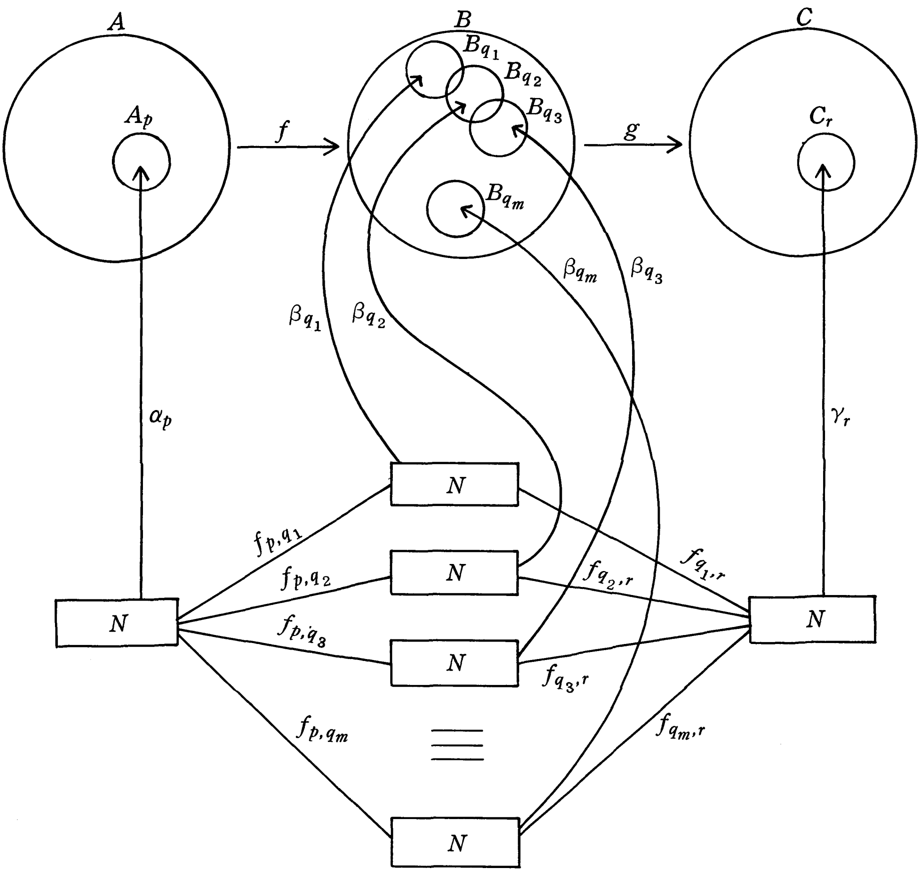

Figure 4.1

$$h(\alpha_p(n)) = g(f(\alpha_p(n))) = g(\beta_{q_i}(f_{p,q_i}(n))) \text{ for } n \in \mathsf{D}_{f_{p,q_i}},$$

i.e.,

$$h(\alpha_p(n)) = \gamma_p(f_{p,q_i}(f_{q_{i,r}}(n))) \text{ for } n \in \mathsf{D}_{q_{i,r}},$$

we can define a p.r. function $f_{p,r}$ such that $f_{p,r}(n)$ takes one of possible values $f_{p,q_i}(f_{q_i,r}(n))$ (for $i = 1, \ldots, m$); then, (4.1) will hold, and the domain of $f_{p,r}$ will be just as required.

I shall use morphisms for comparison of **REM**'s. From the definition of a recursively enumerable manifold it should be obvious that the cardinal of its carrier plays a definitive role in its behavior. I shall now make this role manifest.

Definition 4.2 (i) **a** is *weaker* (*1-weaker*) than **b**, in symbol $\mathbf{a} \underset{\mathsf{W}}{\leqslant} \mathbf{b}$ ($\mathbf{a} \underset{\mathsf{W}-1}{\leqslant} \mathbf{b}$) iff there is a morphism (unimorphism) $f\colon A \to B$.

(ii) $\mathbf{a} \underset{\mathsf{W}}{\equiv} \mathbf{b}$ ($\mathbf{a} \underset{\mathsf{W}-1}{\equiv} \mathbf{b}$) iff $\mathbf{a} \underset{\mathsf{W}}{\leqslant} \mathbf{b} \wedge \mathbf{b} \underset{\mathsf{W}}{\leqslant} \mathbf{a}$ ($\mathbf{a} \underset{\mathsf{W}-1}{\leqslant} \mathbf{b} \wedge \mathbf{b} \underset{\mathsf{W}-1}{\leqslant} \mathbf{a}$).

One could call equivalence classes under $\underset{\mathsf{W}}{\equiv}$ (respectively $\underset{\mathsf{W}-1}{\equiv}$) degrees; I prefer the name *types* (respectively 1-*types*). By $[a]_{\mathsf{W}}$ ($[a]_{\mathsf{W}-1}$) I shall denote the type (the 1-type) containing **a**.

Theorem 4.1 (i) *The* **IRM** $\mathbf{n} = \langle N, \{\mathsf{I}\}\rangle$, *where* I *is the identity on* N, *has the smallest type among all* **REM**'*s*.
(ii) *A genuine* **REM** $\mathbf{a} = \langle A, \mathfrak{A}\rangle$ *is in the type* $[\mathbf{n}]_{\mathsf{W}}$ *iff* $\mathfrak{A}$ *is finite.*
(iii) *The* **IRM** $\mathbf{a}' = \langle N, \mathfrak{A}'\rangle$, *where* $\mathfrak{A}' = \{\alpha_i' \mid i \in N\}$, $\alpha_i'(n) = \sigma^2(i, n)$, *has the smallest type among all genuine* **REM**'*s with at least denumerable atlases.*

Proof: (i) If $\mathbf{a} = \langle A, \mathfrak{A}\rangle$ is any **REM**, fix $p \in P$ and set $f = \alpha_p\colon N \to A$. Then f is a morphism of **n** into **a**, i.e., $\mathbf{n} \underset{\mathsf{W}}{\leqslant} \mathbf{a}$.
(ii) Suppose now that $\mathbf{a} \underset{\mathsf{W}}{\leqslant} \mathbf{n}$ and let $f\colon A \to N$ be a morphism. Then $A = f^{-1}(N)$ can be covered by finite many A_p's, i.e., $\mathfrak{A}$ must be finite.
(iii) If $\mathbf{b} = \langle B, \mathfrak{B}\rangle$, where $\mathfrak{B} = \{\beta_i \mid i \in N\}$, is genuine, then, for each $i \in N$, there is at least one $b_i \in B$ such that $b_i \in B_i - \bigcup_{\substack{j \neq i \\ j=0}}^{\infty} B_j$. ($B_i$ = range of β_i.) Define $f\colon N \to B$ by $f(\alpha_i'(n)) = b_i$ for all $n \in N$. f is, trivially, recursive. Also, $f^{-1}(B_i(= f^{-1}(\{b_i\}) = A_i'$; thus, f is a morphism. Similarly for larger cardinalities of $\mathfrak{B}$.

Theorem 4.2 *Let* $\aleph_0 \leqslant \bar{\bar{A}} < \bar{\bar{B}}$. *Then we can construct* **REM**'*s* (*even* **IRM**'*s*) $\mathbf{a} = \langle A, \mathfrak{A}\rangle$ *and* $\mathbf{b} = \langle B, \mathfrak{B}\rangle$ *such that* $[\mathbf{a}]_{\mathsf{W}} < [\mathbf{b}]_{\mathsf{W}}$.

Proof: We may suppose $A \subset B$. Let $B_0 \neq A$ be any denumerable subset of A and let β_0 be an indexing of B_0. Let $P = B - B_0$. To every $p \in P$ correspond the indexing $\beta_p\colon N \to B_0 \cup \{p\}$ defined by $\beta_p(n) = p$ for $n = 0$, and $\beta_p(n) = \beta_0(n - 1)$ for $n \geqslant 1$. (See Example 1.1.) Let $P_0 = A - B_0$. Set $\mathbf{a} = \langle A, \mathfrak{A}\rangle$ and $\mathbf{b} = \langle B, \mathfrak{B}\rangle$, where $\mathfrak{A} = \{\beta_p \mid p \in P_0\}$ and $\mathfrak{B} = \{\beta_p \mid p \in P\}$. Then I_A, the identity on A, is a morphism of A into B. However, there can be no morphism $f\colon B \to A$. To see this, remark that in case f is a morphism, $f^{-1}(A_p)$ can be covered by finite many B_p's. (A_p = range of β_p for $p \in P_0$.) Then, $B = f^{-1}(A)$ would be of the same cardinality as A.

If we look for some ways to classify **REM**'s, Theorems 4.1 and 4.2 suggest to start with **REM**'s of the same cardinality. For genuine **REM**'s, this implies (except in the most trivial case of finite atlases) that the atlases have to be of the same cardinality too; thus, we may assume that the enumerations in all atlases we consider are indexed by the same set of indices. Therefore, from now on, I suppose that **a**, **b**, **c**, . . ., are such that $\mathfrak{A} = \{\alpha_p \mid p \in P\}$, $\mathfrak{B} = \{\beta_p \mid p \in P\}$, $\mathfrak{C} = \{\gamma_p \mid p \in P\}$, ...

Definition 4.3 **a** is *reducible* (*1-reducible*) to **b**, in symbol **a** $\leqslant$ **b** (**a** $\leqslant_1$ **b**), iff there is a morphism (a unimorphims) $f: A \to B$ such that, for each $p \in P$, $f^{-1}(B_p) = A_p$.

Defining **a** $\equiv$ **b** $\longleftrightarrow$ **a** $\leqslant$ **b** $\wedge$ **b** $\leqslant$ **a** (respectively, **a** $\equiv_1$ **b** $\longleftrightarrow$ **a** $\leqslant_1$ **b** $\wedge$ **b** $\leqslant_1$ **a**), we call the equivalence classes under $\equiv$ (respectively $\equiv_1$) *type-degrees* (respectively *type-one-degrees*), in short **TD**'s (respectively **TOD**'s). [**a**] will denote the **TD** containing **a**, and $[\mathbf{a}]_1$ will denote the **TOD** containing **a**.

Since **a** $\leqslant$ **b** (**a** $\leqslant_1$ **b**) implies $f(A_p) \subset B_p$, and f is $\mathfrak{A}$-$\mathfrak{B}$-recursive, to every $p \in P$ corresponds a recursive (and injective) function $f_p: N \to N$, such that

$$f(\alpha_p(n)) = \beta_p(f_p(n)),\ \text{for all } n \in N. \tag{4.2}$$

Thus, if $\overline{A} = f(A)$, $\overline{\alpha}_p = f \circ \alpha_p$ and $\overline{\mathfrak{A}} = \{\overline{\alpha}_p \mid p \in P\}$, we have:

Lemma 4.2 **a** $\leqslant$ **b** *implies that* $(\overline{\mathbf{a}}) = \langle \overline{A}, \overline{\mathfrak{A}} \rangle$ *is an* **REM**, *which is effectively a submanifold of* **b**. *Thus, the atlas* $\overline{\mathfrak{A}}$ *is strongly reducible to the atlas* $\mathfrak{B}$.

Proof: $\overline{\alpha}_p(n) = \beta_p(f_p(n))$. Suppose that $B_p \cap B_{p_1} \neq \emptyset$ and let g_{p,p_1} be partial recursive and such that

$$\beta_p(n) = \beta_{p_1}(g_{p,p_1}(n)) \text{ for all } n \in \beta_p^{-1}(B_{p_1}).$$

Then

$$\overline{\alpha}_p(n) = \beta_{p_1}(g_{p,p_1}(f_p(n))) \text{ for all } n \in (\overline{\alpha}_p)^{-1}(B_{p_1}),$$

which shows that $(\overline{\mathbf{a}})$ is an **REM**. The remaining part of the proof is the matter of definitions (see remarks after Lemma 2.1, and the Definition 3.3).

Lemma 4.3 *Duplication of an* **REM** *does not change its* **TD**.

Proof: f and f^{-1} from Theorem 2.1 are morphisms, satisfying $f^{-1}(B_p) = A_p$ and $(f^{-1})^{-1}(A_p) = B_p$.

Theorem 4.3 *The class* $[U]$ *of all* **TD**'*s* (*of one fixed type*) *is an upper semi-lattice.*

Proof: Consider two **REM**'s **a** and **b** of two **TD**'s [**a**] and [**b**]. We may suppose that $A \cap B = \emptyset$ (Lemma 4.3). Define: $C = A \cup B$, $\gamma_p(2n) = \alpha_p(n)$ and $\gamma_p(2n + 1) = \beta_p(n)$; set $\mathfrak{C} = \{\gamma_p \mid p \in P\}$ and $C = \langle C, \mathfrak{C} \rangle$. Now, $f: A \to C$, defined by $f(x) = x$, satisfies $f(\alpha_p(n)) = \gamma_p(2n)$ and $f^{-1}(C_p) = A_p$, and $g: B \to C$, defined by $g(x) = x$, satisfies $g(\beta_p(n)) = \gamma_p(2n + 1)$ and $g^{-1}(C_p) = B_p$. Therefore, both are morphisms, and we obtain **a** $\leqslant$ **c** and **b** $\leqslant$ **c**. Suppose **d** $= \langle D, \mathfrak{D} \rangle$,

$\mathfrak{D} = \{\delta_p | p \in P\}$, satisfies $\mathbf{a} \leqslant \mathbf{d}$ and $\mathbf{b} \leqslant \mathbf{d}$. If $h_1\colon A \to D$ is a morphism, such that $h_1^{-1}(D_p) = A_p$, and if $h_2\colon B \to D$ is another morphism, such that $h_2^{-1}(D_p) = B_p$, let $h\colon C \to D$ be defined by

$$h(x) = \begin{cases} h_1(x) \text{ for } x \in A, \\ h_2(x) \text{ for } x \in B. \end{cases}$$

h is a morphism, as is easily checked. Since $h^{-1}(D_p) = C_p$, we obtain $\mathbf{c} \leqslant \mathbf{d}$, i.e., $[\mathbf{c}]$ is the least upper bound of $[\mathbf{a}]$ and $[\mathbf{b}]$.

Remark: $[\mathbf{c}]$ from the foregoing proof will be denoted by $[\mathbf{a}] \vee [\mathbf{b}]$.

To the **REM** $\mathbf{a} = \langle A, \mathfrak{A} \rangle$ we correspond its *cylindrification* $\mathbf{a}_{\mathsf{cyl}} = \langle A, \mathsf{Cyl}_{\mathfrak{A}} \rangle$, where $\mathsf{Cyl}_{\mathfrak{A}}$ is the cylindrification of $\mathfrak{A}$ (see Definition 3.4). In order to avoid confusion with notations for duplication of **REM**'s, we shall use the following notation for cylindrification:

$$\mathsf{Cyl}_{\mathfrak{A}} = \overline{\overline{\mathfrak{A}}} = \{\dot{\overline{\overline{\alpha}}}_p | p \in P\}, \text{ where } \overline{\overline{\alpha}}_p(\sigma^2(n, m)) = \alpha_p(m).$$

Consider the identity I_A on A as a map of $\mathbf{a}_{\mathsf{cyl}}$ into $\mathbf{a}$. Since

$$\mathsf{I}_A(\overline{\overline{\alpha}}_p(n)) = \mathsf{I}_A(\alpha_p(\sigma_2^2(n))) = \alpha_p(\sigma_2^2(n))$$

I_A is a morphism of $\mathbf{a}_{\mathsf{cyl}}$ onto $\mathbf{a}$, i.e., $\mathbf{a}_{\mathsf{cyl}} \leqslant \mathbf{a}$. From the other side, as a map of $\mathbf{a}$ into $\mathbf{a}_{\mathsf{cyl}}$, I_A is a unimorphism, since

$$\mathsf{I}_A(\alpha\ (n)) = \overline{\overline{\alpha}}(\sigma^2(0, n)).$$

Thus $\mathbf{a} \leqslant_1 \mathbf{a}_{\mathsf{cyl}}$.

Lemma 4.4 (i) $\mathbf{a} \leqslant_1 \mathbf{a}_{\mathsf{cyl}}$ and $\mathbf{a}_{\mathsf{cyl}} \leqslant \mathbf{a}$;
(ii) $\mathbf{b} \leqslant \mathbf{a}$ implies $\mathbf{b} \leqslant_1 \mathbf{a}_{\mathsf{cyl}}$;
(iii) $\mathbf{b} \leqslant \mathbf{a} \longleftrightarrow \mathbf{b}_{\mathsf{cyl}} \leqslant_1 \mathbf{a}_{\mathsf{cyl}}$.

Proof: (i) was proved above. (ii) If $f\colon B \to A$ is a morphism satisfying $f^{-1}(A_p) = B_p$, let each f_p be recursive and such that $f(\beta_p(n)) = \alpha_p(f_p(n))$. Then, $f(\beta_p(n)) = \overline{\overline{\alpha}}_p(\sigma^2(n, f_p(n)))$, which proves that f a unimorphism of $\mathbf{b}$ into $\mathbf{a}$, such that $f^{-1}(A_p) = B_p$. (iii) is now obvious (see the proof of Lemma 3.1).

Theorem 4.4 *Every* **TD** *contains a maximal* **TOD**.

Proof: Lemma 4.4 and a reasoning similar to the one of the proof of Theorem 3.6.

Example 4.1 Let us consider **TD**'s of all genuine denumerable **REM**'s, with denumerable atlases. We set $P = N$, and by A_i we denote the range of α_i.

Let $\mathbf{a}'$ be as in Theorem 4.1 (iii). If $\mathbf{a} \leqslant \mathbf{a}'$ and $f\colon A \to A' = \bigcup_{i=0}^{\infty} A_i' (= N)$ is a morphism, such that $f^{-1}(A_i') = A_i$ (A_i' = range of α_i'), then we must have $i \neq j \to A_i \cap A_j = \emptyset$. (Otherwise, if $x \in A_i \cap A_j$ and $x = \alpha_i(n)$ and $x = \alpha_j(m)$, f will have to send x into two disjoint sets A_i' and A_j'). Suppose now that $\mathbf{a}$ satisfies $i \neq j \to A_i \cap A_j = \emptyset$. Define $f\colon A' \to A$ by $f(\alpha_i'(n)) = \alpha_i(n)$. This gives $\mathbf{a}' \leqslant_1 \mathbf{a}$, i.e., we have

(i) **TD** $[\mathbf{a}']$ *consists exactly of all* **a** *such that* $i \neq j \rightarrow A_i \cap A_j \neq \emptyset$.

To measure the complexity of other **REM**'s in our family, to every $\mathbf{a} = \langle A, \mathfrak{A}\rangle$, $\mathfrak{A} = \langle \alpha_i \mid i \in N\rangle$ correspond its *measure* of *complexity* $F_{\mathbf{a}}: A \rightarrow 2^N$ by

$$F_{\mathbf{a}}(x) = \{i \in N \mid x \in A_i\} \text{ for } x \in A.$$

For example, if $\mathbf{a} \in [\mathbf{a}']$ then $F_{\mathbf{a}}(x) = \{i\}$ for $x \in A_i$. We have

(ii) *If* $f: A \rightarrow B$ *is a morphism satisfying* $f^{-1}(B_i) = A_i$ *for every* $i \in N$, *then* $F_{\mathbf{a}} = F_{\mathbf{b}} \circ f$.

To prove (ii) remark that $f^{-1}(B_i) = A_i$ implies

$$F_{\mathbf{a}}(x) = \{i \in N \mid x \in A_i\} = \{i \in N \mid f(x) \in B_i\} = F_{\mathbf{b}} \circ f(x),$$

and that $F_{\mathbf{a}} = F_{\mathbf{b}} \circ f$ implies, for every $x \in A$,

$$\{i \in N \mid x \in A_i\} = \{i \in N \mid f(x) \in B_i\}.$$

The same reasoning gives at once

(iii) *For a genuine* $\mathbf{a} = \langle A, \mathfrak{A}\rangle$, $\mathfrak{A} = \{\alpha_i \mid i \in N\}$, $\mathbf{b} \in [\mathbf{a}]$ *implies that there are morphisms* $f: B \rightarrow A$ *and* $g: A \rightarrow B$ *satisfying* $F_{\mathbf{b}} = F_{\mathbf{a}} \circ f$ *and* $F_{\mathbf{a}} = F_{\mathbf{b}} \circ g$.

Since there can be only at most denumerable many morphisms $f: A \rightarrow B$, satisfying $f^{-1}(B_i) = A_i$, i.e., $f(\alpha_i(n)) = \beta_i(f_i(n))$ where f_i is recursive, and since there is a continuum of possible $F_{\mathbf{a}}$'s, we obtain

(iv) *There is a continuum of* **TD**'*s of genuine denumerable* **REM**'*s with denumerable atlases; each such* **TD** *contains at most denumerable many members.*

One can relativize the foregoing notion of reducibility to submanifolds of a fixed **REM**, and obtain another analogy with the notion of reducibility for subsets of N. I shall discuss this relativization very briefly, in order to show an important difference with the classic theory.

Suppose we have fixed the **REM** $\mathbf{m} = \langle M, \mathfrak{M}\rangle$; we consider its effective submanifolds **a**, **b**, **c**, . . ., which are such that $\mathfrak{A} = \{\alpha_p \mid p \in P\}$, $\mathfrak{B} = \{\beta_p \mid p \in P\}$, $\mathfrak{C} = \{\gamma_p \mid p \in P\}$, ... Then we may say that **a** is $\mathfrak{M}$-*reducible* (respectively $\mathfrak{M}$-1-*reducible*) to **b**, in symbol $\mathbf{a} \leq_{\mathsf{M}} \mathbf{b}$ (respectively $\mathbf{a} \leq_{\mathsf{M}\text{-}1} \mathbf{b}$) iff there is an $\mathfrak{M}$-$\mathfrak{M}$-recursive (and injective) morphism $f: M \rightarrow M$, such that, for every $p \in P$ and $x \in A$

$$x \in A_p \leftrightarrow f(x) \in B_p.$$

One would expect for this reducibility the validity of Myhill's theorem:

if $\mathbf{a} \leq_{\mathsf{M}\text{-}1} \mathbf{b}$ *and* $\mathbf{b} \leq_{\mathsf{M}\text{-}1} \mathbf{a}$ *then there is an* $\mathfrak{M}$-$\mathfrak{M}$-*recursive permutation* $\tau: M \rightarrow M$ *of* M *onto* M, *such that* $\tau(A_p) = B_p$ *for all* $p \in P$.

The following example shows that such theorem is not true even in very elementary **REM**'s.

Example 4.2 Let M_0 be an infinite recursive subset of N, and let H be an infinite immune subset of N, disjoint from M_0; let $\mu_0: N \to M_0$ be recursive and increasing, with M_0 as range. Let $h: N \to H$ be increasing, with H as range. Define $\mu_1: N \to M_0 \cup H$ by $\mu_1(2n) = \mu_0(n)$ and $\mu_1(2n+1) = h(n)$. Let $M_1 = M_0 \cup H$ be the range of μ_1. Set $M = M_1$ and $\mathfrak{M} = \{\mu_0, \mu_1\}$. Since $M_0 \cap M_1 = M_0$ and $\mu_0^{-1}(M_0) = N$ and $\mu_1^{-1}(M_0) = \{2n \mid n \in N\}$, we can conclude easily that $\langle M, \mathfrak{M}\rangle$ is an **IRM**.

Now let $\alpha_0 = \mu_0$ and for $i \geq 1$

$$\alpha_i(n) = \begin{cases} h(i-1) \text{ for } n = 0, \\ \mu_0(n-1) \text{ for } n \geq 1. \end{cases}$$

Let A_i = range of α_i, $A = \bigcup_{i=0}^{\infty} A_i$ $(= M)$ and $\mathfrak{A} = \{\alpha_i \mid i \in N\}$. Then $\langle A, \mathfrak{A}\rangle$ is an **IRM**, which is effectively a submanifold of $\langle M, \mathfrak{M}\rangle$. Let β_i, $i \geq 0$, be defined by $\beta_0(n) = \mu_0(2n)$ and, for $i \geq 1$,

$$\beta_i(n) = \begin{cases} \mu_0(2i-1) \text{ for } n = 0, \\ \beta_0(n-1) \text{ for } n \geq 1. \end{cases}$$

Set B_i = range of β_i, $B = \bigcup_{i=0}^{\infty} B_i$ $(= M_0)$ and $\mathfrak{B} = \{\beta_i \mid i \in N\}$. Then $\langle B, \mathfrak{B}\rangle$ is an **IRM** which is effectively a submanifold of $\langle M, \mathfrak{M}\rangle$. Define $f: M \to M$ by $f(\mu_0(n)) = \mu_0(2n)$, $f(\mu_1(2n)) = \mu_0(2n)$ and $f(\mu_1(2n+1)) = \mu_0(2n+1)$; it is injective, recursive and, trivially, a morphism. Moreover,

$$x \in A_i \leftrightarrow f(x) \in B_i.$$

Similarly, $g: M \to M$ defined by

$$g(\mu_0(2n)) = \mu_0(n),\ g(\mu_0(2n+1)) = \mu_1(2n+1),\ g(\mu_1(4n)) = \mu_1(2n),$$
$$g(\mu_1(4n+2)) = \mu_1(2n+1),\ g(\mu_1(4n+1)) = \mu_1(4n+1),$$

and $g(\mu_1(4n+3)) = \mu_1(4n+3)$, is recursive, injective (and, trivially, a morphism) such that

$$x \in B_i \leftrightarrow g(x) \in A_i.$$

Now suppose: there is a bijective, recursive $\tau: M \to M$, such that $\tau(A_i) = B_i$ (and $\tau^{-1}(B_i) = A_i$). Since $A = M$, we obtain $\tau(A) = A \neq B$. Thus, such a permutation cannot exist.

CHAPTER V—SOME GENERAL POST-LIKE CONSIDERATIONS AND SOME SPECIAL MANIFOLDS

In this chapter I shall consider possibilities to extend notions of immunity, creativity and similar concepts from the classical recursive theory to subsets of a given **REM**. As it will become manifest, the most general case may be extremely empty: one can take as example an **REM** for which every local neighborhood consists of one point only. Thus, in order to be able to quote meaningful examples, I shall introduce first two special **REM**'s which have pleasant additional structures. Notations will be the same as at the beginning of Chapter 4.

Definition 5.1 An **REM** $\mathbf{a} = \langle A, \mathfrak{A}\rangle$ is *finitary* iff, for every $p_0 \in P$, the set $P_0 = \{p \in P | A_p \cap A_{p_0} \neq \emptyset\}$ is finite.

Theorem 5.1 (*Enumeration Theorem for Finitary* **REM**'s) *If* **a** *is injective and finitary then a set* $X \subset A$ *is* $\mathfrak{A}$-r.e. *iff there is* $\varphi: P \to N$ *such that* $X = \omega_\varphi$, *where*

$$\omega_\varphi = \bigcup_{p \in P} \alpha_p(\omega_{\varphi(p)}). \tag{5.1}$$

Proof: If X is $\mathfrak{A}$-r.e. then $X = \omega_\varphi$, for some $\varphi: P \to N$, defining $\omega_{\varphi(p)} = \alpha_p^{-1}(X)$. Conversely, for any $p_0 \in P$ let $P_0 = \{p_1, \ldots, p_s\}$, where P_0 is as in Definition 5.1. Then:

$$\alpha_{p_0}^{-1}(\omega_\varphi) = \bigcup_{i=0}^{s} \alpha_{p_0}^{-1} \circ \alpha_{p_i}(\omega_{\varphi(p_i)}),$$

and each member of this union is r.e. Thus, $\alpha_{p_0}^{-1}(\omega_\varphi)$ is r.e. for every $p_0 \in P$, i.e., ω_φ is $\mathfrak{A}$-r.e.

Similar is the situation with $\mathfrak{A}$-r.e. subsets of A^m. If

$$\omega_i^{(m)} = \{\langle n_1, \ldots, n_m\rangle \in N^m | \bigvee_y \underset{m}{T}(i, n_1, \ldots, n_m, y)\}$$

($\underset{m}{T}$ is the well-known primitive recursive predicate in the Kleene enumeration theorem), and

$$\omega_i^{\alpha_{p_1}, \ldots, \alpha_{p_m}} = \{\langle \alpha_{p_1}(n_1), \ldots, \alpha_{p_m}(n_m)\rangle | \langle n_1, \ldots, n_m\rangle \in \omega_i^{(m)}\} \tag{5.2}$$

then, in a finitary **REM** $\mathbf{a} = \langle A, \mathfrak{A}\rangle$, a set $X \subset A^m$ is $\mathfrak{A}$-r.e. iff there is $\varphi: P^m \to N$ such that $X = \omega_\varphi^{(m)}$, where

$$\omega_\varphi^{(m)} = \bigcup_{\langle p_1, \ldots, p_m\rangle \in P^m} \omega_{\varphi(p_1, \ldots, p_m)}^{\alpha_{p_1}, \ldots, \alpha_{p_w}}, \tag{5.3}$$

and **a** is injective.

It is obvious that, in case **a** is a finitary **REM**, both $\langle B, \mathfrak{B}\rangle$ from Theorem 2.1 and the graph of **a** (in case **a** is positive, respectively solvable) are finitary **REM**'s. Similarly, direct products and direct sums of finitary **REM**'s are finitary. At last, submanifolds of finitary **REM**'s are finitary.

Another well-behaved kind of **REM**'s are *amalgams*, i.e., **REM**'s **a** such that for all pairs $\langle p, p_1\rangle \in P^2$ for which $A_p \cap A_{p_1} \neq \emptyset$ we have $\alpha_p(n) = \alpha_{p_1}(n)$ for all $n \in \alpha_p^{-1}(A_{p_1}) = \alpha_{p_1}^{-1}(A_p)$. (In case of **IREM**'s, this reduces to: $\alpha_p^{-1} \circ \alpha_{p_1}$ are identities on their domains.) I have already given an illustration for lifting of addition and multiplication into amalgams. Let me now prove the general theorem about such lifting.

Theorem 5.2 (*Lifting of Functions in Injective Amalgams*) *Let the* **REM** $\mathbf{a} = \langle A, \mathfrak{A}\rangle$ *be an injective amalgam and* $\varphi: N^m \to N$ *a recursive function. For each* $p \in P$ *define a partial map* $\varphi_p: (A_p)^m \to A_p$ *by*

$$\varphi_p(\alpha_p(n_1), \ldots, \alpha_p(n_m)) = \alpha_p(\varphi(n_1, \ldots, n_m)). \tag{5.4}$$

Then, each φ_p *is an* $\mathfrak{A}$-$\mathfrak{A}$-*partial recursive map, and in case in which*

$\langle x_1, \ldots, x_m\rangle \in (A_p)^m \cap (A_{p_1})^m$ and $\varphi_p(x_1, \ldots, x_m) \in A_p \cap A_{p_1}$ or $\varphi_{p_1}(x_1, \ldots, x_m) \in A_p \cap A_{p_1}$ we have $\varphi_p(x_1, \ldots, x_m) = \varphi_{p_1}(x_1, \ldots, x_m)$.

Proof: I have to prove only the final part of the theorem. Let $\langle x_1, \ldots, x_m\rangle$ and φ_p satisfy $\langle x_1, \ldots, x_m\rangle \in (A_p)^m \cap (A_{p_1})^m$ and $\varphi_p(x_1, \ldots, x_m) \in A_p \cap A_{p_1}$. Then $x_i \in A_p \cap A_{p_1}$ for $i = 1, \ldots, m$, and if $x_i = \alpha_p(n_i)$ then $x_i = \alpha_{p_1}(n_i)$. Thus, taking any such n_i's, we have

$$\varphi_p(x_1, \ldots, x_m) = \varphi_p(\alpha_p(\alpha_p(n_1), \ldots, \alpha_p(n_m)) = \alpha_p(\varphi(n_1, \ldots, n_m)),$$

and, since $\alpha_p(\varphi(n_1, \ldots, n_m)) \in A_p \cap A_{p_1}$ it equals $\alpha_{p_1}(\varphi(n_1, \ldots, n_m))$; this gives

$$\begin{aligned}\varphi_p(x_1, \ldots, x_m) &= \alpha_{p_1}(n_1, \ldots, n_m)) \\ &= \varphi_{p_1}(\alpha_{p_1}(n_1), \ldots, \alpha_{p_1}(n_m)) = \varphi_{p_1}(x_1, \ldots, x_m).\end{aligned}$$

Thus, amalgams are **REM**'s suitable for computational purposes. This is not all. Let me call injective amalgams **I**-*amalgams*. Then we have

Theorem 5.3 (*Lifting of Sets in* **I**-*Amalgams*) *Let $\mathbf{a} = \langle A, \mathfrak{A}\rangle$ be an* **I**-*amalgam. If $E \subset N$ is an "$, \ldots,$"-subset of N, then $E_A = \bigcup_{p \in P} \alpha_p(E)$ is an "$\mathfrak{A}$-$, \ldots,$"-subset of A.*

Proof: Let $p_0 \in P$. Then:

$$\alpha_{p_0}^{-1}(E_A) = \bigcup_{\substack{p \in P \\ p \neq p_0}} \alpha_{p_0}^{-1}(\alpha_p(E)).$$

Since, for $n \in \alpha_{p_0}^{-1}(\alpha_p(E))$, $\alpha_{p_0}(n) = \alpha_p(n)$, we have $\alpha_{p_0}^{-1}(\alpha_p(E)) \subset E$, i.e., $\alpha_{p_0}^{-1}(E_A) = E$ for all $p_0 \in P$.

As I have already said, finitary **REM**'s and amalgams are well-suited for construction of examples in some analogies with Post's recursive theory. I shall illustrate this through several samples.

Definition 5.2 Let $\mathbf{a} = \langle A, \mathfrak{A}\rangle$ be any **REM**, and $X \subset A$. We say that X is "$\mathfrak{A}$-. . ." iff for every $p \in P$, the set $\alpha_p^{-1}(X)$ is an ". . ." subset of N.

As the first instance of Definition 5.2 let me consider the notion of finitude. $X \subset A$ is $\mathfrak{A}$-*finite* ($\mathfrak{A}$-*infinite*) iff every $\alpha_p^{-1}(X)$ is finite (infinite). This leaves aside a large family of subsets of A which are neither $\mathfrak{A}$-finite nor $\mathfrak{A}$-infinite. I shall call such sets $\mathfrak{A}$-*indefinite*.

Theorem 5.4 *Let $\mathbf{a} = \langle A, \mathfrak{A}\rangle$ be a finitary* **IRM**. *Then every $\mathfrak{A}$-infinite $\mathfrak{A}$-r.e. set contains an $\mathfrak{A}$-infinite $\mathfrak{A}$-recursive subset.*

Proof: Let $X \subset A$ be $\mathfrak{A}$-infinite and $\mathfrak{A}$-r.e. set. Then each $\alpha_p^{-1}(X)$ is an infinite r.e. set; thus, it contains an infinite recursive set, R_p say. Let

$$R = \bigcup_{p \in P} \alpha_p(R_p).$$

Let $p_0 \in P$ and let $P_0 = \{p_1, \ldots, p_s\}$ be as in Definition 5.1. Then

$$\alpha_{p_0}^{-1}(R) = R_{p_0} \cup \bigcup_{i=1}^{s} \alpha_{p_0}^{-1} \circ \alpha_{p_i}(R_{p_i}).$$

We have to prove only that each set $E_i = \alpha_{p_0}^{-1} \circ \alpha_{p_i}(R_{p_i})$ is recursive. Let

$D_i = \alpha_{p_0}^{-1}(A_{p_i})$ and $S_i = \alpha_{p_i}^{-1}(A_{p_0})$. Then both D_i and S_i are recursive sets, and both $\alpha_{p_i}^{-1} \circ \alpha_{p_0}: D_i \to S_i$ and $\alpha_{p_0}^{-1} \circ \alpha_{p_i}: S_i \to D_i$ are bijective p.r. functions. Remark that $E_i \subset D_i$. Let $y \in N - S_i$. (If $N - S_i = \emptyset$ we have to prove nothing.) Define $f_i: N \to S_i \cup \{y\}$ by

$$f_i(n) = \begin{cases} \alpha_{p_i}^{-1} \circ \alpha_{p_0}(n) & \text{for } n \in D_i, \\ y & \text{for } n \in N - D_i. \end{cases}$$

f_i is recursive, and $E_i = f_i^{-1}(R_{p_i})$, as the inverse image of a recursive set under a recursive function, is recursive.

Some $\mathfrak{A}$-notions have curious relation to classical notions. To give an example, $X \subset A$ is $\mathfrak{A}$-productive iff every $\alpha_p^{-1}(X)$ is productive, say under the recursive function f_p. (Thus, $\omega_i \subset \alpha_p^{-1}(X) \to f_p(i) \in \alpha_p^{-1}(X) - \omega_i$.) Suppose there exists an $\mathfrak{A}$-productive set X. Let E be any r.e. subset of A, say $E = \omega_\varphi = \bigcup_{p \in P} \alpha_p(\omega_{\varphi(p)})$, where $\omega_{\varphi(p)} = \alpha_p^{-1}(E)$, and suppose $\omega_\varphi \subset X$. This implies $\omega_{\varphi(p)} \subset \alpha_p^{-1}(X)$ and so $f_p(\varphi(p)) \in \alpha_p^{-1}(X) - \omega_{\varphi(p)}$. Lifting into A, we obtain

$$\{\alpha_p(f_p(\varphi(p))) \mid p \in P\} \subset X - \omega_\varphi.$$

We must say that $X \subset A$ is $\mathfrak{A}$-*creative* iff every $\alpha_p^{-1}(X)$ is creative. This implies that every $\alpha_p^{-1}(X)$ is r.e. with productive complement, i.e., $X \subset A$ *is* $\mathfrak{A}$-*creative iff it is* $\mathfrak{A}$-r.e. and $\complement X = A - X$ *is* $\mathfrak{A}$-*productive*. In case **a** is an **I**-amalgam, the set $K_A = \bigcup_{p \in P} \alpha_p(K)$, where K is any creative subset of N, is $\mathfrak{A}$-creative (Theorem 5.3). Already in a finitary **REM**, K_A is not necessarily $\mathfrak{A}$-creative.

By Definition 5.2 $X \subset A$ is $\mathfrak{A}$-*immune* ($\mathfrak{A}$-*simple*) iff every $\alpha_p^{-1}(X)$ is immune (simple). Here, I can prove

Theorem 5.5 *Let* **a** $= \langle A, \mathfrak{A} \rangle$ *be any* **REM**. *Then an* $\mathfrak{A}$-*infinite set* $X \subset A$ *is* $\mathfrak{A}$-*immune iff it does not contain any* $\mathfrak{A}$-*indefinite or any* $\mathfrak{A}$-*infinite* $\mathfrak{A}$-r.e. *subset of* A.

Proof: Let $X \subset A$ be $\mathfrak{A}$-immune; then, each $\alpha_p^{-1}(X)$ is immune. If $E \subset X$ is an $\mathfrak{A}$-infinite $\mathfrak{A}$-r.e. subset of X then each $\alpha_p^{-1}(E)$ is an infinite r.e. subset of the immune set $\alpha_p^{-1}(X)$. If E is $\mathfrak{A}$-indefinite then at least one $\alpha_p^{-1}(E)$ is an infinite r.e. subset of the immune set $\alpha_p^{-1}(X)$.

Conversely, suppose that X is $\mathfrak{A}$-infinite and does not contain any $\mathfrak{A}$-infinite or $\mathfrak{A}$-indefinite $\mathfrak{A}$-r.e. subset of A. Then, no $\alpha_p^{-1}(X)$ can contain an infinite r.e. set; moreover, each $\alpha_p^{-1}(X)$ is infinite, thus each one is immune, i.e., X is $\mathfrak{A}$-immune.

If **a** is an **I**-amalgam, then $S_A = \bigcup_{p \in P} \alpha_p(S)$, where $S \subset N$ is simple, is an $\mathfrak{A}$-simple subset of A. (Then $\complement S_A$ is the example of an $\mathfrak{A}$-immune set.) $\mathfrak{A}$-simple sets behave in many ways like simple subsets of N.

Theorem 5.6 *In any* **REM a**, *an* $\mathfrak{A}$-r.e. *set* $X \subset A$ *is* $\mathfrak{A}$-*simple iff, for every* $\mathfrak{A}$-*infinite* $\mathfrak{A}$-r.e. *set,* $X \cap E$ *is an* $\mathfrak{A}$-*infinite set, and, for every* $\mathfrak{A}$-*indefinite* $\mathfrak{A}$-r.e. *set* E, $X \cap E$ *is* $\mathfrak{A}$-*indefinite.*

Proof: First, if $X \subset A$ is $\mathfrak{A}$-simple it is $\mathfrak{A}$-r.e. and $\complement X$ is $\mathfrak{A}$-immune; by previous theorem, $\complement X$ does not contain any $\mathfrak{A}$-infinite or $\mathfrak{A}$-indefinite $\mathfrak{A}$-r.e. subset of A.

Let E be an $\mathfrak{A}$-infinite $\mathfrak{A}$-r.e. subset of A; then $X \cap E$ is not empty. Moreover, every $\alpha_p^{-1}(X \cap E) = \alpha_p^{-1}(X) \cap \alpha_p^{-1}(E)$ is the intersection of a simple set $\alpha_p^{-1}(X)$ and of an infinite r.e. set $\alpha_p^{-1}(E)$; thus, it is infinite, i.e., $X \cap E$ is $\mathfrak{A}$-infinite.

Let now E be an $\mathfrak{A}$-indefinite $\mathfrak{A}$-r.e. subset of A; then $X \cap E$ is not empty, and at least for one $p_0 \in P$, $\alpha_{p_0}^{-1}(X \cap E) = \alpha_{p_0}^{-1}(X) \cap \alpha_{p_0}^{-1}(E)$ is the intersection of a simple set $\alpha_{p_0}^{-1}(X)$ and an infinite r.e. set $\alpha_{p_0}^{-1}(E)$, i.e., it is infinite. Therefore, $X \cap E$ is $\mathfrak{A}$-indefinite (since at least one $\alpha_p^{-1}(E)$ is either empty or finite).

Conversely, let X be $\mathfrak{A}$-r.e. and such that $X \cap E$ is $\mathfrak{A}$-infinite for every $\mathfrak{A}$-infinite $\mathfrak{A}$-r.e. set E, and $\mathfrak{A}$-indefinite for every $\mathfrak{A}$-indefinite $\mathfrak{A}$-r.e. set E. Then, $\complement X$ cannot contain either one of those two kinds of sets, i.e., it is $\mathfrak{A}$-immune, by Theorem 5.5.

Corollary 5.6.1 (i) *The intersection of two $\mathfrak{A}$-simple sets is an $\mathfrak{A}$-simple set.*
(ii) *The union of two $\mathfrak{A}$-simple sets is either $\mathfrak{A}$-simple or has a complement which is not $\mathfrak{A}$-infinite.*

Proof: (i) Let X and Y be $\mathfrak{A}$-simple subsets of A. Then $\complement(X \cap Y) = \complement X \cup \complement Y$ is obviously $\mathfrak{A}$-infinite (both $\complement X$ and $\complement Y$ are $\mathfrak{A}$-infinite). Let now E be any $\mathfrak{A}$-infinite $\mathfrak{A}$-r.e. set. Then, by previous theorem, $E \cap X$ is $\mathfrak{A}$-infinite; it is, trivially, $\mathfrak{A}$-r.e. Then, anew by Theorem 5.6, $(E \cap X) \cap Y$ is $\mathfrak{A}$-infinite. Thus, $(X \cap Y) \cap E$ is $\mathfrak{A}$-infinite for every $\mathfrak{A}$-infinite $\mathfrak{A}$-r.e. set E. Let now E be $\mathfrak{A}$-r.e. and $\mathfrak{A}$-indefinite. Then, by previous theorem, $E \cap X$ is $\mathfrak{A}$-indefinite and $\mathfrak{A}$-r.e.; therefore, $(E \cap X) \cap Y$ is anew $\mathfrak{A}$-indefinite. By Theorem 5.6, $X \cap Y$ is $\mathfrak{A}$-simple (since it is, trivially, $\mathfrak{A}$-r.e.).

(ii) If X and Y are $\mathfrak{A}$-simple then $\complement(X \cup Y)$ is either $\mathfrak{A}$-infinite or it is not $\mathfrak{A}$-infinite. Suppose it is $\mathfrak{A}$-infinite. Then, since $\complement(X \cup Y) = \complement X \cap \complement Y$, it cannot contain any $\mathfrak{A}$-r.e. set E which is either $\mathfrak{A}$-infinite or $\mathfrak{A}$-indefinite; thus, by Theorem 5.5, it is $\mathfrak{A}$-immune.

Consider now notions of cohesiveness and maximality: $X \subset A$ is *$\mathfrak{A}$-cohesive* (*$\mathfrak{A}$-maximal*) iff each $\alpha_p^{-1}(X)$ is cohesive (maximal).

Theorem 5.7 *Let* $\mathfrak{a}$ *be any* **REM** *and $X \subset A$. Then:*

(i) *If X is $\mathfrak{A}$-cohesive then it is $\mathfrak{A}$-infinite and, for every* $\mathfrak{A}$-r.e. *set E, either $X \cap E$ or $X \cap \complement E$ is not $\mathfrak{A}$-infinite.*
(ii) *If X is $\mathfrak{A}$-infinite, and for every* $\mathfrak{A}$-r.e. *set E either $X \cap E$ or $X \cap \complement E$ is $\mathfrak{A}$-finite, then X is $\mathfrak{A}$-cohesive.*
(iii) *$Y \subset A$ is $\mathfrak{A}$-maximal iff it is* $\mathfrak{A}$-r.e. *and $\complement Y$ is $\mathfrak{A}$-cohesive.*

Proof: (i) If X is $\mathfrak{A}$-cohesive then each $\alpha_p^{-1}(X)$ is cohesive and so infinite. Thus, X is $\mathfrak{A}$-infinite. Further, if there is an $\mathfrak{A}$-r.e. set E, such that both

$X \cap E$ and $X \cap \complement E$ are $\mathfrak{A}$-infinite, then both $\alpha_p^{-1}(X) \cap \alpha_p^{-1}(E)$ and $\alpha_p^{-1}(X) \cap \complement\alpha_p^{-1}(E)$ would be infinite, for every $p \in P$; contradiction, since every $\alpha_p^{-1}(X)$ is cohesive.
(ii) If X satisfies the given conditions then, for every $p \in P$, either $\alpha_p^{-1}(X) \cap \alpha_p^{-1}(E)$ or $\alpha_p^{-1}(X) \cap \complement\alpha_p^{-1}(E)$ is finite for every r.e. set $\alpha_p^{-1}(E)$, i.e., $\alpha_p^{-1}(X)$ is cohesive (being already infinite).
(iii) Let Y be $\mathfrak{A}$-maximal. Then, each $\alpha_p^{-1}(Y)$ is maximal and each $\alpha_p^{-1}(\complement Y)$ is cohesive. Thus, $\complement Y$ is $\mathfrak{A}$-cohesive. Converse similar.

I believe to have exhibited enough samples for the local variant of Post's recursive theory. However, one can consider a variant which is global, i.e., independent of projections.

Definition 5.3 Let $\mathbf{a} = \langle A, \mathfrak{A}\rangle$ be any **REM**, and $X \subset A$. We say that X is *globaly infinite* (*globaly finite*) iff $\overline{\overline{X}} = \overline{\overline{A}}$ ($\overline{\overline{X}} < \overline{\overline{A}}$). (Obviously, $\overline{\overline{X}}$ and $\overline{\overline{A}}$ denote the cardinals of X and A respectively.)

I do not know yet how to define global productivity. However, I can handle such a variant of immunity.

Definition 5.4 Let $\mathbf{a}$ be an **REM** and $X \subset A$. Then:

(i) X is *globaly immune* iff it is globaly infinite and does not contain any globaly infinite $\mathfrak{A}$-r.e. set.
(ii) X is *globaly simple* iff it is $\mathfrak{A}$-r.e. and $\complement X$ is globaly immune.

Theorem 5.8 *Suppose the* **REM** $\mathbf{a} = \langle A, \mathfrak{A}\rangle$ *has the property that* $\overline{\overline{A}} = \overline{\overline{\mathcal{E}}}$, *where* $\mathcal{E}$ *is the family of all globaly infinite* $\mathfrak{A}$-r.e. *subsets of* A. *Then there are* $2^{\overline{\overline{A}}}$ *sets* $X \subset A$ *such that both* X *and* $\complement X$ *are globally immune.*

Proof: ($\overline{\sigma}$ will denote the cardinal of the ordinal σ.) Let σ be the smallest ordinal such that $\overline{\sigma} = \overline{\overline{A}}$. (Thus, for every $\eta < \sigma$, $\overline{\eta} < \overline{\overline{A}}$.) Well-order $\mathcal{E}$ into an ordinal sequence $\langle\omega_\xi\rangle_{\xi<\sigma}$. To each $\xi < \sigma$ correspond the ordered pair $\langle a_\xi, b_\xi\rangle$ of elements of A so that

(i) $a_\xi \neq b_\xi$, $a_\xi \in \omega_\xi$, $b_\xi \in \omega_\xi$

and

(ii) both a_ξ and b_ξ are not in $\bigcup_{\eta<\xi} \{a_\eta, b_\eta\}$. Let X consist of exactly one member of each pair $\langle a_\xi, b_\xi\rangle$. Then $\overline{\overline{X}} = \overline{\overline{\complement X}} = \overline{\overline{A}}$, and neither X nor $\complement X$ contains any ω_ξ. This choice may be done in $2^{\overline{\overline{A}}}$ different ways.

Relative to the existence of globaly simple sets I can prove

Theorem 5.9 *Let* $\mathbf{a} = \langle A, \mathfrak{A}\rangle$ *be an* **I**-*amalgam, such that, for every family* $\{E_p \mid p \in P\}$ *of non-empty sets* $E_p \subset A_p$, $\overline{\overline{\bigcup_{p\in P} E_p}} = \overline{\overline{A}}$ *iff at least one* E_p *is infinite. Then there is a globaly simple subset of* A (*which is also* $\mathfrak{A}$-*simple*).

Proof: Let $S \subset N$ be any simple set. Set $S_A = \bigcup_{p\in P} \alpha_p(S)$. (By Theorem 5.3, S_A is $\mathfrak{A}$-simple.) Now, S_A is globaly infinite, since each $\alpha_p(S)$ is infinite. Since $\alpha_{p_0}^{-1}(S_A) = S$ for every $p_0 \in P$, S_A is $\mathfrak{A}$-r.e. Now

$$\complement S_A = \bigcup_{p \in P} \{A_p - \alpha_p(S)\}$$

is globaly infinite, since each $A_p - \alpha_p(S)$ is infinite. If $\complement S_A$ contains a globaly infinite $\mathfrak{A}$-r.e. set E, then there is at least one $p_0 \in P$ such that $E \cap A_{p_0}$ is infinite. Then $\alpha_{p_0}^{-1}(E)$ will be an infinite subset of the immune set $\complement\, \alpha_{p_0}^{-1}(S)$.

If **a**$' = \langle N, \mathfrak{A}' \rangle$, $\mathfrak{A}' = \{\alpha_i' \mid i \in N\}$ where $\alpha_i'(n) = \sigma^2(i, n)$, is the **IRM** from the Theorem 4.1, then **a**$'$ does not satisfy the condition of Theorem 5.9. Let me show that in this case $S_A = \bigcup_{i=0}^{\infty} \alpha_i'(S)$ is *not* globaly infinite. Its complement $\complement S_A = \bigcup_{i=0}^{\infty} (A_i' - \alpha_i(S))$ is also globaly infinite (i.e., denumerable). However, by taking just one member $x_i \in A_i' - \alpha_i(S)$, we obtain the set $X = \{x_i \mid i \in N\}$ which is a globaly infinite $\mathfrak{A}$-r.e. subset of $\complement S_A$.

It is plausible that a slight change in the definition of global infinity in the case of the manifold **a**$' = \langle N, \mathfrak{A}' \rangle$ (say, adding: at least one $X \cap A_i'$ must be infinite) could give a more workable notion for the global immunity in **a**$'$. The generality of the notion of an **REM** suggests to consider global notions with respect to the cardinality of particular **REM**'s in question. I will restrain here from such relativization.

CHAPTER VI—THE CATEGORY OF REM's

In [5] Ershov applied the vocabulary of the Category Theory to the category of the enumerated sets. This application made possible a very general conception of precomplete and complete enumerations in terms of effective embeddings (or "**e**-partial objects" in terms of [5]).

In this chapter I engage into a similar venture with the category of **REM**'s; as a natural consequence of the notion of effective embedding I obtain an effective notion of finitely reducibility. Also, I consider a very strict generalization of precompleteness in order to illustrate a new notion—the *ordinalization* of an **REM**. I restrain myself from any detailed rendition of the content of [5], and I pursue only the directions which are really new in comparison with [5]. However, I like to point out the influence of Ershev's considerations upon the content of this chapter. I introduce very few categorical notions; thus, I give the corresponding definitions, in order to spare the students time and nerves. (For **REM**'s I use notations at the beginning of Chapter 4.)

A *category* $\aleph$ is a class of *objects* **a**, **b**, **c**, . . ., such that to each pair $\langle$**a**, **b**$\rangle$ corresponds a class $[\mathbf{a}, \mathbf{b}]_\aleph$ of *morphisms* f ("of **a** into **b**"), for which there is a partial operation $\circ$ (of "*composition*"), with following properties:

(K.1) If $h \circ g$ and $g \circ f$ are defined then $(h \circ g) \circ f = h \circ (g \circ f)$;
(K.2) To each object **a** corresponds an *identical* morphism $1_\mathbf{a} \in [\mathbf{a}, \mathbf{a}]_\aleph$, for which $1_\mathbf{a} \circ f = f$ and $g \circ 1_\mathbf{a} = g$, whenever the left sides are defined.

It is obvious that the class of all **REM**'s **a**, **b**, **c**, . . ., as objects, with the families of all morphisms f: **a** $\to$ **b**, . . ., (i.e., f: $A \to B$), with composition of morphisms, is a category. I shall denote this category by $\mathcal{E}$.

In the Category Theory, a morphism f: **a** $\to$ **b** is called an *isomorphism* iff there is a morphism g: **b** $\to$ **a** such that $g \circ f = \mathsf{I}_{\mathbf{a}}$ and $f \circ g = \mathsf{I}_{\mathbf{b}}$. Since $g \circ f = \mathsf{I}_{\mathbf{a}}$ and $f \circ g_1 = \mathsf{I}_{\mathbf{b}}$ imply easily $g = g_1$, g above is uniquely determined by f.

In the category $\mathcal{E}$ of all **REM**'s, the demand that $g \circ f = \mathsf{I}_{\mathbf{a}}$ (i.e., $g(f(x)) = x$ for all $x \in A$) and $f \circ g = \mathsf{I}_{\mathbf{b}}$ (i.e., $f(g(y)) = y$ for all $y \in B$) imply first that both f and g are bijective and, then, that $g = f^{-1}$. This gives

Theorem 6.1 *In the category $\mathcal{E}$, a morphism f:* **a** $\to$ **b** *is an isomorphism iff it is bijective and f^{-1}:* **b** $\to$ **a** *is a morphism.*

A category $\aleph_0$ is a *subcategory* of the category $\aleph$ iff $\aleph_0 \subset \aleph$ in obvious sense (for objects, morphisms, and composition); it is called a *full* subcategory of $\aleph$ iff, moreover, for every **a**, $\mathbf{b} \in \aleph_0$, $[\mathbf{a}, \mathbf{b}]_{\aleph_0} = [\mathbf{a}, \mathbf{b}]_{\aleph}$.

It should be obvious that the category $\mathcal{E}_I$, of all injective **REM**'s, is a full subcategory of $\mathcal{E}$. Also, the category $\mathcal{E}^0$, of all **RM**'s, is a full subcategory of $\mathcal{E}$. The category $\mathcal{E}_I^0$ of all **IRM**'s is a full subcategory both of $\mathcal{E}_0$ and of $\mathcal{E}_I$. At last, if $\mathcal{E}'$ denotes the class of all **REM**'s with inmorphisms (as morphisms), then $\mathcal{E}'$ is a subcategory of $\mathcal{E}$ which is not a full subcategory of $\mathcal{E}$.

In the Category Theory, a morphism f: **a** $\to$ **b** is a *monomorphism* (respectively an *epimorphism*) iff for any morphisms $g_0, g_1 \in [\mathbf{c}, \mathbf{a}]_{\aleph}$ (respectively $\in [\mathbf{b}, \mathbf{c}]_{\aleph}$), $f \circ g_0 = f \circ g_1$ (respectively $g_0 \circ f = g_1 \circ f$) implies $g_0 = g_1$. (See Figure 6.1.)

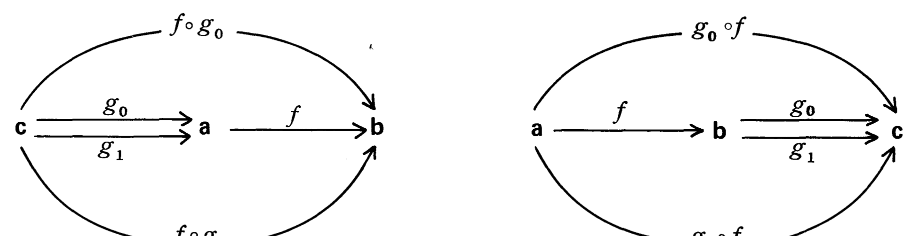

Figure 6.1

Theorem 6.2 *In the category $\mathcal{E}$ a morphism f:* **a** $\to$ **b** *is a monomorphism iff it is injective.*

Proof: Let f be a monomorphism. If it is not injective let $x_1 \neq x_0$ be such that $f(x_1) = f(x_0)$. Let $C = \{x_0, x_1\}$; define $\gamma: N \to C$ by $\gamma(2n) = x_0$, $\gamma(2n + 1) = x_1$ and set $\mathbf{c} = \langle C, \{\gamma\}\rangle$. Define g_0: **c** $\to$ **a** and g_1: **c** $\to$ **a** by $g_0(x_0) = x_0$, $g_0(x_1) = x_1$, $g_1(x_0) = x_1$ and $g_1(x_1) = x_0$. Then g_0 and g_1 are morphisms and $f \circ g_0 = f \circ g_1$ but $g_0 \neq g_1$. Thus, f must be injective. Converse obvious.

Let me remark that, in the category $\mathcal{E}$, every surjective morphism f: $\mathbf{a} \to \mathbf{b}$ is an epimorphism. However, I am unable to prove the converse of this proposition except in case $\mathbf{a}$ has a finite atlas.

In the Category Theory, the notion of embedding is usually given relative to functors. Ershov ([5]) introduces the notion "partial object of $\mathbf{m}$" as a pair $\langle \mathbf{a}, f \rangle$, where $\mathbf{a}$ and $\mathbf{m}$ are enumerated sets and f an injective $\{\alpha\}$-$\{\mu\}$-recursive map of A into M ($\mathbf{a} = \langle A, \{\alpha\}\rangle$, $\mathbf{m} = \langle M, \{\mu\}\rangle$); such a pair represents obviously an embedding of $\mathbf{a}$ into $\mathbf{m}$. This should explain my first definition.

Definition 6.1 (i) In the category $\mathcal{E}$, a pair $\langle \mathbf{a}, f \rangle$ is called an *embedding* of $\mathbf{a}$ into $\mathbf{m}$, iff f is a monomorphism of $\mathbf{a}$ into $\mathbf{m}$, such that each $f(A_p)$ can be covered by finite many M_t's.
(ii) Let $\langle \mathbf{a}, f \rangle$ and $\langle \mathbf{b}, g \rangle$ be embeddings into $\mathbf{m}$. We say that $\langle \mathbf{a}, f \rangle$ and $\langle \mathbf{b}, g \rangle$ are *equivalent* in m iff there is a bijective morphism h: $\mathbf{a} \to \mathbf{b}$, such that h^{-1} is also a morphism and $f = g \circ h$ (i.e., such that the diagram in Figure 6.2 commutes).

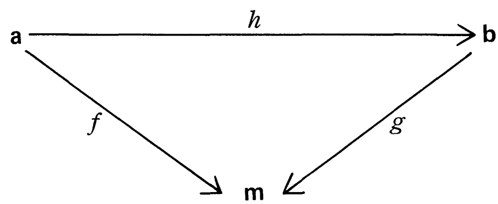

Figure 6.2

The lemma which follows will be needed later; however, I bring it now in order to illustrate the nature of embeddings, at least in a special case.

Lemma 6.1 *Let* $\langle \mathbf{a}, f \rangle$ *be an embedding into the positive* **REM** $\mathbf{m}$. *Then, to every pair* $\langle p, t \rangle \in P \times T$ *corresponds a* p.r. *arithmetic function* $g_{p,t}$, *with the set* $\mu_t^{-1}(f(A_p))$ *as domain, such that, for every* $k \in \mu_t^{-1}(f(A_p))$, $\mu_t(k) = f(\alpha_p(g_{p,t}(k)))$.

Proof: Since $f: A \to M$ is $\mathfrak{A}$-$\mathfrak{M}$-recursive, to every pair $\langle p, t \rangle \in P \times T$ corresponds a p.r. function $f_{p,t}$, with domain $\mathbf{D}_{p,t} = \alpha_p^{-1}(f^{-1}(M_t))$, such that

$$(6.1) \qquad f(\alpha_p(n)) = \mu_t(f_{p,t}(n)) \text{ for all } n \in \mathbf{D}_{p,t}.$$

Let $E_{p,t} = \mu_t^{-1}(f(A_p))$. Since

$$\begin{aligned} k \in E_{p,t} &\leftrightarrow \mu_t(k) \in f(A_p) \\ &\leftrightarrow \bigvee_u \mu_t(k) = f(\alpha_p(u)) \\ &\leftrightarrow \bigvee_u \mu_t(k) = \mu_t(f_{p,t}(u)) \wedge u \in \mathbf{D}_{p,t}, \end{aligned}$$

and since $\mathfrak{M}$ is a positive atlas, the set $E_{p,t}$ is r.e. for all p and t. Define now $g_{p,t}$ as follows: its domain is $E_{p,t}$ and for $k \in E_{p,t}$

$$g_{p,t}(k) = \begin{cases} \text{any } n \in \mathbf{D}_{p,t} \text{ such that either } k = f_{p,t}(n), \text{ or such that there is} \\ k_1 \text{ in the range of } f_{p,t}, \text{ such that } \mu_t(k) = \mu_t(k_1) \text{ and } k_1 = f_{p,t}(n). \end{cases}$$

$g_{p,t}(k)$ is obviously defined for all $k \in E_{p,t}$; since $\mathfrak{M}$ is positive $g_{p,t}$ is a partial recursive function. Now we have:

$$\alpha_p(g_{p,t}(k)) = \alpha_p \text{ (of some } n \in \mathbf{D}_{p,t} \text{ such that } \mu_t(k) = f(\alpha_p(n)))$$

i.e.,

$$f(\alpha_p(g_{p,t}(k))) = \mu_t(k) \text{ for all } k \in \mu_t^{-1}(f(A_p)).$$

In general case, let $\langle \mathbf{a}, f\rangle$ be an embedding into the **REM m**. Then f is an injective morphism, such that each $f(A_p)$ can be covered by finite many M_t's, say by $M_{t_0}^{(p)} \cup, \ldots, \cup M_{t_s}^{(p)}$. Since f is $\mathfrak{A}$-$\mathfrak{M}$-recursive, there are p.r. functions $\psi_i^{(p)}$, with domain $\mathbf{D}_{i,p} = \alpha_p^{-1}(f^{-1}(M_{t_i}^{(p)}))$, such that

$$f(\alpha_p(n)) = \mu_{t_i}(\psi_i^{(p)}(n)) \text{ for all } n \in \mathbf{D}_{i,p},$$

and $0 \leqslant i \leqslant s$.

Define $\mathfrak{A}' = \{\alpha_p' | p \in P\}$ by $\alpha_p' = f \circ \alpha_p$, and let A_p' be the range of α_p'. Then to every $p \in P$ corresponds a finite set $\{t_0, \ldots, t_s\} \subset T$, such that $\bigcup_{i=0}^{s} M_{t_i}$ covers A_p', and there are p.r. functions ψ_i, with domain $\mathbf{D}_{i,p}' = (\alpha_p')^{-1}(M_{t_i})$ such that

$$\alpha_p'(n) = \mu_{t_i}(\psi_i(n)) \text{ for all } n \in \mathbf{D}_{i,p}',$$

and $0 \leqslant i \leqslant s$. This shows that $\mathfrak{A}' \underset{\mathsf{F}}{\leqslant} \mathfrak{M}$ and that, moreover, each $(\alpha_p')^{-1}(M_{t_i})$ involved above is a r.e. set. (This was not demanded for finitely reducibility.) Therefore, in order to obtain an adequate characterization of embeddings and their equivalence, I shall introduce a slightly more restrictive definition of finitely reducibility.

Definition 6.2 Let $\mathfrak{A}$ and $\mathfrak{B}$ be atlases on a fixed set A. We say that $\mathfrak{A}$ is *effectively finitely reducible* to $\mathfrak{B}$, in symbol $\mathfrak{A} \underset{\mathsf{EF}}{\leqslant} \mathfrak{B}$, iff $\mathfrak{A} \underset{\mathsf{F}}{\leqslant} \mathfrak{B}$ and all sets $\alpha_p^{-1}(B_{q_i})$ in Definition 3.4 are recursively enumerable.

In an obvious way we define

$$\mathfrak{A} \underset{\mathsf{EF}}{\equiv} \mathfrak{B} \leftrightarrow \mathfrak{A} \underset{\mathsf{EF}}{\leqslant} \mathfrak{B} \wedge \mathfrak{B} \underset{\mathsf{EF}}{\leqslant} \mathfrak{A}$$

and we call the resulting atlas-degrees **EFAD**'s (*effectively finitely atlas-degrees*); in a similar way we can introduce $\underset{\mathsf{EF}_{-1}}{\leqslant}$, *effectively finitely one-reducibility*, and its degrees **EFAOD**'s. Obviously, **EFAD**'s form a subdivision of **FAD**'s. One should remark that $\mathfrak{A} \underset{\mathsf{EF}}{\equiv} \mathfrak{B}$ implies that $\mathfrak{A}$ and $\mathfrak{B}$ are compatible, i.e., that $\mathfrak{A} \cup \mathfrak{B}$ is an atlas on the set A in consideration. It is clear how one can extend $\underset{\mathsf{EF}}{\leqslant}$ to include subsets; thus, $\mathfrak{A} \underset{\mathsf{EF}}{\leqslant} \mathfrak{M}$ expresses the result of the discussion preceding Definition 6.2, and we have

Lemma 6.2 *Let* $\langle \mathbf{a}, f\rangle$ *be an embedding into* **m**, *let*

$$\alpha_p' = f \circ \alpha_p,\ \mathfrak{A}' = \{\alpha_p' | p \in P\} \text{ and } A' = f(A).$$

Then $\mathbf{a}' = f(\mathbf{a}) = \langle A', \mathfrak{A}'\rangle$ *is an* **REM**, *which is effectively a quasi-submanifold of* $\mathbf{m} = \langle M, \mathfrak{M}\rangle$, *such that* $\mathfrak{A}' \underset{\mathsf{EF}}{\leqslant} \mathfrak{M}$.

Following theorem establishes an important property of equivalence of embeddings.

Theorem 6.3 *Two embeddings* $\langle \mathbf{a}, f\rangle$ *and* $\langle \mathbf{b}, g\rangle$ *into* $\mathbf{m}$ *are equivalent in* $\mathbf{m}$ *iff* $f(A) = f(B)$ *and* $\mathfrak{A}' \underset{EF}{\equiv} \mathfrak{B}'$, *where* $\mathfrak{A}' = \{f \circ \alpha_p \mid p \in P\}$ *and* $\mathfrak{B}' = \{g \circ \beta_q \mid q \in Q\}$.

Proof: Suppose first that $\langle \mathbf{a}, f\rangle$ and $\langle \mathbf{b}, g\rangle$ are equivalent in $\mathbf{m}$, and let h be as in Definition 6.1 (ii). Since $h(A) = B$ and $f = g \circ h$, we obtain $f(A) = g(B)$. For given $p \in P$ let $\{B_{q_i}^{(p)} \mid 0 \leqslant i \leqslant s\}$ cover $h(A_p) = (h^{-1})^{-1}(A_p)$, and let $\varphi_i^{(p)}$ be p.r., with domain $\mathbf{D}_i^{(p)} = \alpha_p^{-1}(h^{-1}(B_{q_i}))$, satisfying

(6.2) $$h(\alpha_p(n)) = \beta_{q_i}(\varphi_i^{(p)}(n)),\ n \in \mathbf{D}_i^{(p)},\ 0 \leqslant i \leqslant s.$$

Define: $\alpha_p' = f \circ \alpha_p$, $\beta_q' = g \circ \beta_q$, $\mathfrak{A}' = \{\alpha_p' \mid p \in P\}$ and $\mathfrak{B}' = \{\beta_q' \mid q \in Q\}$. Then, (6.2) and $f = g \circ h$ imply

(6.3) $$\alpha_p'(n) = \beta_{q_i}'(\varphi_i^{(p)}(n)) \text{ for } n \in \mathbf{D}_i^{(p)},$$

and for all $i = 0, \ldots, s$. Now,

$$\begin{aligned} \mathbf{D}_i^{(p)} &= \alpha_p^{-1}(h^{-1}(B_{q_i})) = \alpha_p^{-1}(f^{-1}(g(B_{q_i}))) \\ &= \alpha_p^{-1}(f^{-1}(B_{q_i}')) = (\alpha_p')^{-1}(B_{q_i}'), \end{aligned}$$

and (6.3) becomes

(6.4) $$\alpha_p'(n) = \beta_{q_i}'(\varphi_i^{(p)}(n)) \text{ for } n \in (\alpha_p')^{-1}(B_{q_i}'),$$

where each $(\alpha_p')^{-1}(B_{q_i}')$ is a r.e. set. This gives $\mathfrak{A}' \underset{EF}{\leqslant} \mathfrak{B}'$. Symetric reasoning, with h^{-1} instead of h, gives $\mathfrak{B}' \underset{EF}{\leqslant} \mathfrak{A}'$, i.e., $\mathfrak{A}' \underset{EF}{\equiv} \mathfrak{B}'$.
Conversely, suppose that $f(A) = g(B)$ and that (6.4) holds, with $(\alpha_p')^{-1}(B_{q_i}')$ r.e. Since f and g are injective, we can define $h = g^{-1} \circ f$ and obtain, from (6.4),

$$h(\alpha_p(n)) = \beta_{q_i}(\varphi_i^{(p)}(n)) \text{ for } n \in \alpha_p^{-1}(h^{-1}(B_{q_i})),$$

which implies that h is a bijective isomorphism. Then, by a similar reasoning, one proves that h^{-1} is a morphism too. (Remark: $h = g^{-1} \circ f$ implies $h^{-1}(B_q) = f^{-1}(g(B_q))$.) Since $g(B_q)$ can be covered by finite many M_t's, and since each $f^{-1}(M_t)$ can be covered by finite many A_p's it follows that $h^{-1}(B_q)$ can be covered by finite many A_p's.

Theorem 6.3 induces a one-sided correspondence between embeddings into $\mathbf{m}$ and effectively quasi-submanifolds of $\mathbf{m}$, whose atlases are effectively finitely reducible to $\mathfrak{M}$. To explain this correspondence better, let us remark that, for an embedding $\langle \mathbf{a}, f\rangle$ into $\mathbf{m}$, the **REM** $\mathbf{a}' = f(\mathbf{a}) = \langle A', \mathfrak{A}'\rangle$, where $A' = f(A)$, $\mathfrak{A}' = \{\alpha_p' \mid p \in P\}$, $\alpha_p' = f \circ \alpha_p$, satisfies not only the condition $\mathfrak{A}' \underset{EF}{\leqslant} \mathfrak{M}$, but also the supplementary

Condition F: *For every* $t \in T$, *the set* $A' \cap M_t$ *can be covered by finite many local neighborhoods* A_p'.

To see this, remark that (since f is a morphism) $f^{-1}(M_t) = f^{-1}(A' \cap M_t)$ can be covered by finite many A_p's, say, by $A_{p_0} \cup \ldots \cup A_{p_s}$. Then the relation $f^{-1}(A' \cap M_t) \subset \bigcup_{i=0}^{s} A_{p_i}$ implies

$$A' \cap M_t \subset \bigcup_{i=0}^{s} f(A_{p_i}) = \bigcup_{i=0}^{s} A'_{p_i}.$$

Definition 6.3 The **REM** $\mathbf{a}' = \langle A', \mathfrak{A}' \rangle$ will be called **m**-*effective* iff:
(i) **a′** is an effectively quasi-submanifold of **m**;
(ii) $\mathfrak{A}' \underset{\mathsf{EF}}{\leqslant} \mathfrak{M}$,

and

(iii) $\mathfrak{A}'$ satisfies condition F above.

Corollary 6.3.1 *There is a bijective correspondence between embeddings into* **m** *and* **m**-*effective quasi-submanifolds of* **m**, *under which two embeddings* $\langle \mathbf{a}, f \rangle$ *and* $\langle \mathbf{b}, g \rangle$ *are equivalent iff* $\mathfrak{A}' = f(\mathfrak{A})$ *and* $\mathfrak{B}' = g(\mathfrak{B})$ *are in the same* **EFAD**.

Proof: One part of this corollary is an immediate consequence of Theorem 6.3. To prove the converse part, we have only to correspond to each **m**-effective **REM** $\mathbf{a}' = \langle A', \mathfrak{A}' \rangle$ a corresponding embedding $\langle \mathbf{a}, f \rangle$, in such a way that $\mathbf{a}' = f(\mathbf{a})$. First, define $f: \mathbf{a}' \to \mathbf{m}$ to be just the identity on A'. Since $\mathfrak{A}' \underset{\mathsf{EF}}{\leqslant} \mathfrak{M}$, then to every $p \in P$ (I suppose $\mathfrak{A}' = \{\alpha'_p \mid p \in P\}$ correspond finite many t's, say $t_0, t_1, \ldots, t_s$, such that $M_{t_0} \cup M_{t_1} \cup \ldots \cup M_{t_s}$ covers A'_p, where A'_p is the range of α'_p, and there are p.r. functions $f_0, f_1, \ldots, f_s$ such that

$$\alpha'_p(n) = \mu_{t_i}(f_i(n)) \text{ for } n \in (\alpha'_p)^{-1}(M_{t_i}),$$

$0 \leqslant i \leqslant s$, where each $(\alpha'_p)^{-1}(M_{t_i})$ is a r.e. set. Thus,

$$f(\alpha'_p(n)) = \mu_{t_i}(g_i(n)) \text{ for } n \in \mathsf{D}_{p,i},$$

where $\mathsf{D}_{p,i}$ = domain of $g_i = f_i \mid (\alpha'_p)^{-1}(M_{t_i})$. This proves that f is $\mathfrak{A}'$-$\mathfrak{M}$-recursive. It is also a morphism, since $f^{-1}(M_t) = A' \cap M_t$ can be covered by finite many A'_p's. Thus, $\langle \mathbf{a}', f \rangle$ is the embedding in question.

Following lemma introduces cylindrification into the study of **EFAD**'s and **EFAOAD**'s.

Lemma 6.3 *Let* **a** *be effectively a quasi-submanifold of* **m**, *such that* $\mathfrak{A} \underset{\mathsf{EF}}{\leqslant} \mathfrak{M}$ (*respectively let* **a** *be* **m**-*effective*). *Denote by* $\mathbf{a}_{\mathrm{cyl}} = \langle A, \mathrm{Cyl}_{\mathfrak{A}} \rangle$ *the cylindrification of* **a**. *Then,* $\mathbf{a}_{\mathrm{cyl}}$ *is effectively a quasi-submanifold of* **m**, *such that* $\mathrm{Cyl}_{\mathfrak{A}} \underset{\mathsf{EF}}{\leqslant} \mathfrak{M}$ (*respectively then* $\mathbf{a}_{\mathrm{cyl}}$ *is* **m**-*effective*).

Proof: If $\alpha_p^{-1}(M_t)$ is r.e. then

$$(\bar{\alpha}_p)^{-1}(M_t) = \{\sigma^2(n, k) \mid k \in \alpha_p^{-1}(M_t)\}$$

is also r.e.

Theorem 6.4 *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be atlases on* A *and* B *respectively, where* $A \subset B \subset M$. *Then:*

(i) $\mathfrak{A} \underset{\mathsf{EF}\text{-}1}{\leqslant} \mathsf{Cyl}_{\mathfrak{A}}$ *and* $\mathsf{Cyl}_{\mathfrak{A}} \underset{\mathsf{EF}}{\leqslant} \mathfrak{A}$;
(ii) $\mathfrak{B} \underset{\mathsf{EF}}{\leqslant} \mathfrak{A}$ *implies* $\mathfrak{B} \underset{\mathsf{EF}\text{-}1}{\leqslant} \mathsf{Cyl}_{\mathfrak{A}}$;
(iii) $\mathfrak{B} \underset{\mathsf{EF}}{\leqslant} \mathfrak{A} \leftrightarrow \mathsf{Cyl}_{\mathfrak{B}} \underset{\mathsf{EF}\text{-}1}{\leqslant} \mathsf{Cyl}_{\mathfrak{A}}$.

Moreover, if $\mathfrak{A} \underset{\mathsf{EF}}{\leqslant} \mathfrak{M}$ *and* $\mathfrak{B} \underset{\mathsf{EF}}{\leqslant} \mathfrak{M}$ *then* $\mathsf{Cyl}_{\mathfrak{A}} \underset{\mathsf{EF}}{\leqslant} \mathfrak{M}$ *and* $\mathsf{Cyl}_{\mathfrak{B}} \underset{\mathsf{EF}}{\leqslant} \mathfrak{M}$.

Proof: Previous lemma and the proof of Lemma 3.1.

Corollary 6.4.1 (i) *Every* **EFAD** *contains a maximal* **EFAOD**.
(ii) *The* **EFAD***'s on a fixed set form an upper semi-lattice.*

Let me point out that Example 3.1 demonstrates that on N there is no difference between **FAD**'s and **EFAD**'s. (See later Theorem 6.6 for a more general statement.)

The nature of embeddings will be illustrated in large measure by considerations of principal atlases.

Definition 6.4 (i) Let $\mathfrak{A}$ be an atlas on $A \subset M$. We say that $\mathfrak{A}$ is *effectively principal* (in **m**) iff $\mathfrak{A} \underset{\mathsf{EF}}{\leqslant} \mathfrak{M}$ and, for every other atlas $\mathfrak{B}$ on A, $\mathfrak{B} \underset{\mathsf{EF}}{\leqslant} \mathfrak{M}$ implies $\mathfrak{B} \underset{\mathsf{EF}}{\leqslant} \mathfrak{A}$.
(ii) An embedding $\langle \mathbf{a}, f\rangle$ into **m** is effectively principal (in **m**) iff $\mathfrak{A}' = f(\mathfrak{A})$ is an effectively principal atlas (in **m**).

Theorem 6.5 *If* $\mathbf{m} = \langle M, \mathfrak{M}\rangle$ *is positive and* $A \subset M$ *an* $\mathfrak{M}$-r.e. *set, then there is at least one atlas* $\mathfrak{A}$ *on* A *which is effectively principal.*

Proof: See the proof of Theorem 3.7.

Theorem 6.6 *Every embedding* $\langle \mathbf{a}, f\rangle$ *into a positive* **REM m** *is effectively principal* (*in* **m**).

Proof: Let $\alpha_p' = f\circ\alpha_p$, $A_p' = f(A_p)$ = range of α_p' and $\mathfrak{A}' = \{\alpha_p' | p \in P\}$. Suppose $\mathfrak{B}$ is an atlas on $A' = \bigcup_{p\in P} A_p'$ which satisfies $\mathfrak{B} \underset{\mathsf{EF}}{\leqslant} \mathfrak{M}$. Then, for given $q \in Q$, B_q can be covered by finite many M_t's say by $\bigcup_{i=0}^{s} M_{t_i}^{(q)}$, and there are p.r. functions $f_i^{(q)}$ such that

$$\beta_q(n) = \mu_{t_i}(f_i^{(q)}(n)) \text{ for all } n \in \beta_q^{-1}(M_{t_i}),$$

$0 \leqslant i \leqslant s$, where each $\beta_q^{-1}(M_{t_i})$ is a r.e. set. Since $f\colon A \to M$ is a morphism, each $f^{-1}(M_{t_i})$ can be covered by finite many A_p's, say by $\bigcup_{j=0}^{s_i} A_{p_{i,j}}$. Then $\bigcup_{j=0}^{s_i} A_{p_{i,j}}'$ covers $A' \cap M_{t_i}$; thus, the finite family $\{A_{p_{i,j}}' \mid 0 \leqslant i \leqslant s,\ 0 \leqslant j \leqslant s_i\}$ covers B_q. Moreover,

$$\beta_q^{-1}(M_{t_i}) = \bigcup_{j=0}^{s_i} \beta_q^{-1}(A_{p_{i,j}}'),$$

where each set $\beta_q^{-1}(A_{p_{i,j}}')$ is r.e. To verify this last statement remark that

$$\begin{aligned} n \in \beta_q^{-1}(A_{p_{i,j}}') &\leftrightarrow \beta_q(n) \in f(A_{p_{i,j}}) \cap M_{t_i} \\ &\leftrightarrow \mu_{t_i}(f_i^{(q)}(n)) \in f(A_{p_{i,j}}) \cap M_{t_i} \wedge n \in \beta_q^{-1}(M_{t_i}) \\ &\leftrightarrow \bigvee_u \mu_{t_i}(f_i^{(q)}(n)) = f(\alpha_{p_{i,j}}(u)) \wedge n \in \beta_q^{-1}(M_{t_i}). \end{aligned}$$

Since f is $\mathfrak{A}$-$\mathfrak{M}$-recursive, there are p.r. functions $h_{i,j}$, with domain $\mathbf{D}_{i,j} = \alpha_{p_{i,j}}^{-1}(f^{-1}(M_{t_i}))$, such that

$$f(\alpha_{p_{i,j}}(u)) = \mu_{t_i}(h_{i,j}(u)) \text{ for all } u \in \mathbf{D}_{i,j}.$$

Thus

$$n \in \beta_q^{-1}(A'_{p_{i,j}}) \leftrightarrow \bigvee_u (u \in \mathbf{D}_{i,j} \wedge \mu_{t_i}(f_i^{(q)}(n)) = \mu_{t_i}(h_{i,j}(u)) \wedge n \in \beta_q^{-1}(M_{t_i})).$$

Since $\mathbf{m}$ is positive and $\beta_q^{-1}(M_{t_i})$ r.e. it follows that $\beta_q^{-1}(A'_{p_{i,j}})$ is r.e. Now, if $g_{p,t}$'s are as in Lemma 6.1, we obtain that, for all $n \in \beta_q^{-1}(A'_{p_{i,j}})$,

$$\beta_q(n) = \mu_{t_i}(f_i^{(q)}(n)) = f(\alpha_p(g_{p_{i,j}}, t_i(f_i^{(q)}(n)))) = \alpha'_p(g_{p_{i,j}}, t_i(f_i^{(q)}(n)));$$

since each $\beta_q^{-1}(A'_{p_{i,j}})$ is r.e. and the family of all $A_{p_{i,j}}$ covers B_q, we obtain $B \underset{\mathrm{EF}}{\leqslant} A$.

Thus, embeddings into a positive **REM** $\mathbf{m}$ correspond to effectively principal atlases on fixed subsets of M, which, moreover, satisfy condition F (preceding Definition 6.3).

Definition 6.5 An embedding $\langle \mathbf{a}, f\rangle$ into $\mathbf{m}$ is *effective* iff it is effectively principal (in $\mathbf{m}$) and $A' = f(A)$ is an $\mathfrak{M}$-r.e. subset of M.

Theorem 6.7 *Let $\langle \mathbf{a}, f\rangle$ be an embedding into $\mathbf{m}$ for which $A' = f(A)$ is* $\mathfrak{M}$-r.e. *Then, $\langle \mathbf{a}, f\rangle$ is effective iff for every $t \in T$, for which $A' \cap M_t \neq \emptyset$, the set $A' \cap M_t$ can be covered by finite many neighborhoods $A'_p = f(A_p)$, say by $A'_{p_0} \cup \ldots \cup A'_{p_s}$, and there are* p.r. *functions $g_{p_i,t}$, with domain $\mu_t^{-1}(A'_{p_i})$, such that, for every $k \in \mu_t^{-1}(A'_{p_i})$,*

$$\mu_t(k) = f(\alpha_{p_i}(g_{p_i,t}(k))),$$

$0 \leqslant i \leqslant s$.

Proof: Suppose first that $\langle \mathbf{a}, f\rangle$ is effective. Let $A' = f(A)$, $A'_p = f(A_p)$, $\alpha'_p = f \circ \alpha_p$, $\mathfrak{A}' = \{\alpha'_p | p \in P\}$. Then $\mathfrak{A}'$ is effectively principal on the set A' (in $\mathbf{m}$). Let $T_0 \subset T$ be defined by

$$t \in T_0 \leftrightarrow \mu_t^{-1}(A') \neq \emptyset.$$

For every $t \in T_0$ let m_t be recursive with range $\mu_t^{-1}(A')$—which is a r.e. set (since A' is $\mathfrak{M}$-r.e.). Set $\beta_t(n) = \mu_t(m_t(n))$ for all $n \in N$ and all $t \in T_0$; let $\mathfrak{B} = \{\beta_t | t \in T_0\}$. Then $\mathfrak{B} \underset{\mathrm{EF}}{\leqslant} \mathfrak{M}$. Since $\mathfrak{B}$ is an atlas on A' and $\mathfrak{A}'$ is effectively principal there, we conclude: $\mathfrak{B} \underset{\mathrm{EF}}{\leqslant} \mathfrak{A}'$. This relation implies that each B_t can be covered by finite many sets A'_p, say by $\bigcup_{i=0}^{s} A'_{p_i}$, that each $\beta_t^{-1}(A'_{p_i})$ is r.e. and that there are p.r. functions f_i, with domain $\mathbf{D}_{t,i} = \beta_t^{-1}(A'_{p_i})$, satisfying

$$\beta_t(n) = \alpha'_{p_i}(f_i(n)) \text{ for } n \in \mathbf{D}_{t,i},$$

$0 \leqslant i \leqslant s$. Since $B_t = A' \cap M_t$ and $D_{t,i} = \beta_t^{-1}(A'_{p_i}) = m_t^{-1}(\mu_t^{-1}(A'_{p_i}))$, we obtain

$$m_t(\beta_t^{-1}(A'_{p_i})) = \mu_t^{-1}(A'_{p_i}) \cap m_t(N) = \mu_t^{-1}(A'_{p_i}) \cap \text{Range of } m_t$$
$$= \mu_t^{-1}(A'_{p_i}) \cap \mu_t^{-1}(A') = \mu_t^{-1}(A'_{p_i}),$$

which proves that every set $\mu_t^{-1}(A'_{p_i})$ is r.e. Remark now the following: for $k \in \mu_t^{-1}(A'_{p_i})$ there is at least one $u \in D_{t,i}$ such that $k = m_t(u)$. Therefore, the function u, defined for all $k \in \mu_t^{-1}(A'_{p_i})$ by $u(k) =$ some $y \in \mathbf{D}_{t,i}$ such that $k = m_t(y)$, is partial recursive, and

$$\beta_t(u(k)) = \mu_t(m_t(u(k))) = \mu_t(k)$$
$$= \alpha'_{p_i}(f_i(u(k))) = f \circ \alpha_{p_i}(f_i(u(k))) \text{ for } k \in \mu_t^{-1}(A'_{p_i}).$$

This, with $g_{p_i,t} = f_i \circ u$, completes the proof of the necessity of the condition of the theorem.

Suppose now that the condition of the theorem holds, and let $\mathfrak{B}$ be an atlas on A' such that $\mathfrak{B} \underset{\mathrm{EF}}{\leqslant} \mathfrak{M}$. Thus, for every $q \in Q$ there are $t_0, \ldots, t_s \in T$, such that $\bigcup_{i=0}^{s} M_{t_i}$ covers B_q, and there are p.r. functions h_i, with domain $\beta_q^{-1}(M_{t_i})$, satisfying

(6.5) $\qquad \beta_q(n) = \mu_{t_i}(h_i(n))$ for all $n \in \beta_q^{-1}(M_{t_i})$, and all $i = 0, \ldots, s$.

Suppose that $\bigcup_{j=0}^{e_i} A'_{p_{i,j}}$ covers $A' \cap M_{t_i}$; since $B_q \subset A'$, it follows that $\bigcup_{i=0}^{s} \bigcup_{j=0}^{e_i} A'_{p_{i,j}}$ covers B_q. By (6.5), $n \in \beta_q^{-1}(M_{t_i})$ implies $h_i(n) \in \mu_{t_i}^{-1}(B_q)$, i.e., $h_i(n) \in \mu_{t_i}^{-1}\left(\bigcup_{j=0}^{e_i} A'_{p_{i,j}}\right)$; thus, (6.5) and the condition of the theorem imply

(6.6) $\qquad \beta_q(n) = \alpha'_{p_{i,j}}(g_{p_{i,j},t_i}(h_i(n)))$ for all $n \in \beta_q^{-1}(A'_{p_{i,j}})$,

and $0 \leqslant i \leqslant s$, $0 \leqslant j \leqslant e_i$. This implies $\mathfrak{B} \underset{\mathrm{F}}{\leqslant} \mathfrak{A}'$. It remains to prove that each $\beta_q^{-1}(A'_{p_{i,j}})$ is a r.e. set. I shall prove: for every pair $\langle q, p\rangle \in Q \times P$, $\beta_q^{-1}(A'_p)$ is a r.e. set. By Definition 6.1 (i), A'_p can be covered by finite many M_t's, say by $M_{t_0} \cup \ldots \cup M_{t_s}$. Thus,

$$\beta_q^{-1}(A'_p) = \bigcup_{i=0}^{s} \beta_q^{-1}(A'_p \cap M_{t_i}).$$

Now, using (6.5)

$$n \in \beta_q^{-1}(A'_p) \leftrightarrow \beta_q(n) \in A'_p$$
$$\leftrightarrow \bigvee_{i=0}^{s} \mu_{t_i}(h_i(n)) \in A'_p \wedge n \in \beta_q^{-1}(M_{t_i})$$
$$\leftrightarrow \bigvee_{i=0}^{s} h_i(n) \in \mu_{t_i}^{-1}(A'_p) \wedge n \in \beta_q^{-1}(M_{t_i}).$$

By condition of the theorem, every set $\mu_{t_i}^{-1}(A'_p)$ is r.e., and by our supposition, every set $\beta_q^{-1}(M_{t_i})$ is also r.e. Thus, $\beta_q^{-1}(A'_p)$ is r.e., and we obtain $\mathfrak{B} \underset{\mathrm{EF}}{\leqslant} \mathfrak{A}'$.

Let me interpret Theorem 6.7 in case of enumerated sets. I shall suppose $\mathbf{m} = \langle M, \{\mu\}\rangle$ and $\mathbf{a} = \langle A, \{\alpha\}\rangle$. Then $\langle \mathbf{a}, f\rangle$ is an *embedding* into $\mathbf{m}$ iff $f\colon A \to M$ is an injective $\{\alpha\}$-$\{\mu\}$-recursive map. Let $A' = f(A)$ and

$\alpha' = f \circ \alpha$. Then $\{\alpha'\}$ is *effectively principal* iff $\mu^{-1}(A')$ is a r.e. set and for every enumeration β of A', for which $\beta = \mu \circ b$, where b is recursive, we have also $\beta = \alpha \circ b_1$, where b_1 is recursive. Thus, here, already the fact that $\{\alpha'\}$ is effectively principal implies that $\langle \mathbf{a}, f \rangle$ is effective. In this way, we obtain:

Corollary 6.7.1 *Let* $\mathbf{m} = \langle M, \{\mu\}\rangle$ *and* $\mathbf{a} = \langle A, \{\alpha\}\rangle$ *be enumerated sets and let* $f: A \to M$ *be an injective* $\{\alpha\}$-$\{\mu\}$*-recursive map of* A *into* M. *Let* $A' = f(A)$ *and* $\alpha' = f \circ \alpha$. *If* A' *is a* $\{\mu\}$-r.e. *set, then* $\{\alpha'\}$ *is principal on* A' *iff there is a* p.r. *function* g, *with domain* $\mu^{-1}(A')$, *such that, for all* $k \in \mu^{-1}(A')$, $\mu(k) = f(\alpha(g(k)))$.

Remark: In [5] (section **4**, Lemma 3) Ershov gives a proposition which differs from Corollary 6.7.1 in demanding that the domain of g *contains* $\mu^{-1}(A')$. Since $\mu^{-1}(A')$ is already a r.e. set, this demand reduces trivially to the demand of our Corollary 6.7.1.

Another interesting feature in the category $\mathcal{E}$ are retracts.

Definition 6.6 An embedding $\langle \mathbf{a}, f \rangle$ into $\mathbf{m}$ is a *retract* of $\mathbf{m}$ iff there is a morphism $h: \mathbf{m} \to \mathbf{a}$ such that $h \circ f = \mathsf{I}_{\mathbf{a}}$ ($\mathsf{I}_{\mathbf{a}}$ = identity on $\mathbf{a}$).

Remark that, for a retract $\langle \mathbf{a}, f \rangle$ of $\mathbf{m}$, $x \in A'$, where $A' = f(A)$, implies $f(h(x)) = x$. Namely, if $x = f(a)$, $a \in A$, then $h(x) = h(f(a)) = a$ and so $f(h(x)) = f(a) = x$.

Theorem 6.8 *Let* $\langle \mathbf{a}, f \rangle$ *be a retract of* $\mathbf{m}$. *Then, to every pair* $\langle p, t \rangle \in P \times T$ *corresponds a* p.r. *function* $g_{p,t}$, *whose domain contains the set* $\mu_t^{-1}(f(A_p))$, *such that*

$$\mu_t(n) = f(\alpha_p(g_{p,t}(n))) \text{ for all } n \in \mu_t^{-1}(f(A_p)).$$

Proof: Let $h: \mathbf{m} \to \mathbf{a}$ be as in Definition 6.6, let $g_{p,t}$ be p.r. with domain $\mathbf{D}_{p,t} = \mu_t^{-1}(h^{-1}(A_p))$, and such that

$$h(\mu_t(n)) = \alpha_p(g_{p,t}(n)) \text{ for all } n \in \mathbf{D}_{p,t}.$$

By the remark following Definition 6.6

$$f(h(\mu_t(n))) = \mu_t(n), \text{ i.e., } f(\alpha_p(g_{p,t}(n))) = \mu_t(n)$$

for all $\mu_t(n) \in f(A_p)$, i.e., for all $n \in \mu_t^{-1}(f(A_p))$. Now, remark that

$$\mathbf{D}_{p,t} = \mu_t^{-1}(h^{-1}(A_p)) = \mu_t^{-1}(h^{-1}(h(f(A_p)))) \supset \mu_t^{-1}(f(A_p)).$$

There are some difficulties in the adaptation of the notion of "pre-complete" to **REM**'s. For enumerated sets such difficulties do not exist, since the enumerated set $\mathbf{n} = \langle N, \{\mathsf{I}\}\rangle$, where I is the identity on N, is a universal reference-manifold for enumerated sets: every such set can be considered as embedded into $\mathbf{n}$. Such a universal manifold does not exist for **REM**'s. However, for **REM**'s of a fixed cardinality, I can define a half-substitute for this reference-manifold.

To every **REM** $\mathbf{b} = \langle B, \mathfrak{B}\rangle$, $\mathfrak{B} = \{\beta_q \mid q \in Q\}$, I shall correspond its *ordinalization* $\sigma(\mathbf{b}) = \langle \Omega_\sigma, H_\sigma\rangle$, where $\langle \Omega_\sigma, H_\sigma\rangle$ is as in Example 1.3, with

$\alpha_\xi(n) = \xi + n$, as follows: σ is the smallest ordinal whose cardinal is $\overline{\overline{Q}}$. In the same time I shall well-order Q in the order type of σ, and I shall set $Q = \{q_\xi | \xi < \sigma\}$. $\sigma(\mathbf{b})$ will serve as an etalon for **b** and for all **REM**'s with atlases of the same cardinality as Q, the index-set for the atlas $\mathfrak{B}$. (This is almost equivalent with: "with the same cardinality as B".) In $\mathfrak{B}$ considering morphisms h: $\sigma(\mathbf{b}) \to \mathbf{b}$, I shall say that such a morphism is *rigid* iff (see Example 1.3 for notations) $h(U_\xi) \subset B_{q_\xi}$. With all this, $\sigma(\mathbf{b})$ is not subtle enough to characterize precompleteness without fault.

Definition 6.7 The **REM b** is *precomplete* iff for every effective embedding $\langle \mathbf{a}, f\rangle$ into $\sigma(\mathbf{b})$ and every morhpism g: $\mathbf{a} \to \mathbf{b}$ there is a rigid morphism h: $\sigma(\mathbf{b}) \to \mathbf{b}$ such that $g = h \circ f$ (see Figure 6.3).

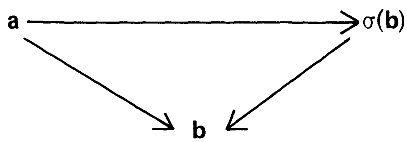

Figure 6.3

I can give only a necessary condition for precompleteness.

Theorem 6.9 *If* **b** *is precomplete, then for every family* $\{\varphi_q | q \in Q\}$ *of arithmetical* p.r. *functions there is a family* $\{f_q | q \in Q\}$ *of recursive functions, such that, for every* $q \in Q$.

(6.7) $$\beta_q(\varphi_q(n)) = \beta_q(f_q(n)) \text{ for all } n \in \mathbf{D}_q,$$

where $\mathbf{D}_q$ *is the domain of* φ_q.

The proof of Theorem 6.9 is straightforward. It should be obvious that either a change in condition on g and f in Definition 6.7, or on h, there could make possible a full characterization of precompleteness. I will not enter into the discussion of those changes at this place. I introduced Definition 6.7 only in order to outline the possibilities and needs for future constructions.

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