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TWO SYSTEMS OF PRESUPPOSITION LOGIC

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Introduction In *Introduction to Logical Theory* ([17], p. 175) Strawson says:

> It is self-contradictory to conjoin S with the denial of S' if S' is a necessary condition of the truth, simply, of S. It is a different kind of logical absurdity to conjoin S with the denial of S' if S' is a necessary condition of the *truth or falsity* of S. The relation between S and S' in the first case is that S entails S'. We need a different name for the relation between S and S' in the second case; let us say . . . that S *presupposes* S'.

This notion of presupposition has subsequently been employed in both philosophical and linguistic studies,[1] and it is generally held that a distinction between entailment and presupposition is important in connection with Russell's theory of descriptions. In the passage we have quoted, Strawson makes it clear that he considers presupposition to be a genuine logical relation. If it is, then it should be possible to give an account of its logic; but if in doing so classical two-valued propositional calculus is employed the following difficulty arises. Strawson's definition above gives that if S presupposes S' then

(1) $S \supset S'$,
(2) $\sim S \supset S'$.

But then since

(3) $S \vee \sim S$,

it follows that

(4) S'.

In effect, if presupposition is represented by any formula equivalent to or implying the conjunction of (1) and (2) then any proposition presupposed by another is true. Clearly, this is not what Strawson intended, since the main reason for introducing a notion of presupposition was to have available machinery for the description of cases of failure of presupposition. This suggests that in order to give an account of the logic of presupposition it is

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necessary to adopt a three-valued system and thereby take seriously the claim that a proposition may be neither ture nor false. Accordingly we provide a precise definition of the notion of presupposition in a three-valued propositional logic. Then by means of the construction of a formal system we derive a body of statements expressing consequences that follow from the definition.

When we had already obtained a number of results which we present here our attention was drawn to the paper by P. W. Woodruff, "Logic and Truth-Value Gaps", [18]. Woodruff also defines presupposition in a three-valued logic, and we were glad to find that our definitions are identical. There is then an overlap between some of his paper and Part One of ours; but we still felt it worthwhile to publish the material in Part One for two reasons. First, unlike Woodruff we have formulated the logic of presupposition in an axiomatic system and proved a large number of theorems. (His own treatment is natural-deductive.) Second, we have adopted what might be called a 'stringent' as contrasted with a 'lenient' definition of validity. That is, where 1, 2, and 3 are respectively the values true, void, and false, for us a formula of three-valued logic, A, is valid iff for all value assignments, V, $V(A) = 1$. This contrasts with the more common account in which a wff A is valid iff $V(A) = 1$ or 2, for all V.[2]

The axiomatic system of presupposition logic which we call **PRES** is presented in Part One, and a modal extension, $\mathbf{PRES_L}$, in Part Two. In the rest of this introduction we give an informal account of the basic ideas we use. The language of presupposition logic has an infinite stock of propositional variables, each of which ranges over the three truth-values 1, 2, and 3. There are eight logical constants, three of which are primitive and the rest introduced by definitions, as explained in Part One. The constants divide into two groups, the one-place logical operators in one group and the two-place logical connectives in the other. The constants are

Group 1: $\sim$ (negation), **T** (truth).

Group 2: $\vee$ (alternation), & (conjunction), $\supset$ (implication), $\leftrightarrow$ (Kleene-equivalence), $\rightarrow$ (presupposition), $\equiv$ (like-value equivalence).

The meanings of the constants are given in the following truth-tables.

p	$\sim p$	$\mathbf{T}p$
1	3	1
2	2	3
3	1	3

$\vee$	1	2	3
1	1	1	1
2	1	2	2
3	1	2	3

&	1	2	3
1	1	2	3
2	2	2	3
3	3	3	3

$\supset$	1	2	3
1	1	2	3
2	1	2	2
3	1	1	1

$\leftrightarrow$	1	2	3
1	1	2	3
2	2	2	2
3	3	2	1

$\rightarrow$	1	2	3
1	1	3	3
2	1	1	1
3	1	3	3

$\equiv$	1	2	3
1	1	3	3
2	3	1	3
3	3	3	1

The tables for negation, alternation, conjunction, implication, and Kleene-equivalence are those given by Kleene in 1938 in the system which Rescher ([13], p. 34) calls K_3; hence our expression 'Kleene-equivalence'. The table for presupposition is obtained by computing the table for $\mathsf{T}(A \vee \sim A) \supset \mathsf{T}B$. We shall later prove that $\mathsf{T}(A \vee \sim A)$ is equivalent to $(\mathsf{T}A \vee \mathsf{T} \sim A)$ which can be read "either A is true or A is false". Hence the formula $\mathsf{T}(A \vee \sim A) \supset \mathsf{T}B$ can be read "If A is either true or false then B is true", which (at least partially, see Part Two) captures Strawson's informal definition of presupposition.

The table for Kleene-equivalence gives $A \leftrightarrow B$ the value 1 when both A and B have the value 1 or 3 but not when both A and B have the value 2. So if the intuitive idea of equivalence between A and B, that for all values $\mathsf{V}(A) = \mathsf{V}(B)$, is to be captured in three-valued logic, then some notion of equivalence stronger than Kleene-equivalence is required in which (A equivalent B) has the value 1 iff $\mathsf{V}(A) = \mathsf{V}(B)$ and has the value 3 otherwise.[3] To meet this need we employ the idea of like-value equivalence. Before we had read Woodruff's paper we defined $(A \equiv B)$ as $((A \rightarrow B) \leftrightarrow (B \rightarrow A))$ and computed the table for '$\equiv$' accordingly. This gave the table shown here, but the definition proved clumsy to use in proofs. Woodruff's definition of like-value equivalence is easier to use, and we have adopted it here. The definition is given in Part One.

We have already noted how the constants divide into two groups syntactically. From the tables it can be seen that they also divide into two groups semantically. In one group are the constants 'T', '$\rightarrow$', and '$\equiv$', in the other the rest. For each of the constants in the first group, whatever the input values may be the output value is always either 1 or 3. In other words, these constants alone eliminate the indeterminate value 2, and for this reason we call them *decisive*.[4]

One other notion is important in our system, that of a T-*formula*. A T-formula is any wff of **PRES** in which no propositional variable is outside the scope of some occurrence of the operator 'T'. Evidently, T-formulae themselves must be either true or false, the value 2 having been eliminated by the occurrences of 'T' within the formula.[5] For example, '$\mathsf{T}p$', '$\mathsf{T}(p \vee q)$', '$(\mathsf{T}p \supset \mathsf{T}q)$', are all T-formulae; 'p', '$\mathsf{T}p \supset p$', '$q \vee \mathsf{T}q$', are not. It follows from the definitions to be given in Part One that any wff whose main connective is either '$\equiv$' or '$\rightarrow$' is a T-formula.

Part One: The System **PRES**

1. *Vocabulary* The propositional variables $p, q, r, \ldots$; the constants $\sim$, $\vee$, **T**; and parentheses.

2. *Formation Rules*
(1) Any propositional variable alone is a wff.
(2) If A is wff, so are $(\sim A)$, $(\mathbf{T}A)$.
(3) If both A and B are wff, then so is $(A \vee B)$.
(4) If A is wff, and B is obtained from A by rewriting any part of A according to the abbreviative definitions below, then B is wff.
(5) Only expressions satisfying (1)-(4) are wff.

3. *Abbreviative Definitions*
[Def $\supset$] $(A \supset B)$ for $(\sim A \vee B)$.
[Def &] $(A \,\&\, B)$ for $\sim(\sim A \vee \sim B)$.
[Def $\leftrightarrow$] $(A \leftrightarrow B)$ for $((A \supset B) \,\&\, (B \supset A))$.
[Def $\rightarrow$] $(A \rightarrow B)$ for $(\mathbf{T}(A \vee \sim A) \supset \mathbf{T}B)$.
[Def $\equiv$] $(A \equiv B)$ for $((\mathbf{T}A \leftrightarrow \mathbf{T}B) \,\&\, (\sim \mathbf{T} \sim A \leftrightarrow \sim \mathbf{T} \sim B))$.

4. *Axioms and Rules* The tables for $\sim$, $\vee$, and **T** have been axiomatized by Lennart Åqvist, see [1], and we adopt his system here. As above, we use $A, B, C, \ldots$ as schematic for wff of **PRES**, and also $P, Q, R, \ldots$ as schematic for wff of classical propositional calculus (**PC**). The axioms are:

A1. $\mathbf{T}(p \vee q) \leftrightarrow (\mathbf{T}p \vee \mathbf{T}q)$.
A2. $\mathbf{T}(p \,\&\, q) \leftrightarrow (\mathbf{T}p \,\&\, \mathbf{T}q)$.
A3. $\mathbf{T}p \supset p$.

The inference rules are:

R1. Modus ponens.
R2. Uniform substitution.
R3. From $\vdash A$ infer $\vdash \mathbf{T}A$.
R4. If $P \supset Q$ is one of the **PC** theorems:

PC1 $((p \supset q) \,\&\, (q \supset r)) \supset (p \supset r)$
PC2 $((p \supset q) \,\&\, (r \supset s)) \supset ((p \,\&\, r) \supset (q \,\&\, s))$
PC3 $((p \,\&\, q) \supset r) \supset (p \supset (q \supset r))$

and substitution in P yields an open formula (any formula of **PRES** which is not a **T**-formula is an open formula) which is a theorem of **PRES**, then the result of the same substitution in Q is a theorem of **PRES**.
R5. If P is a theorem of **PC**, and A is obtained from P by uniformly substituting not necessarily distinct **T**-formulae for each variable in P, then A is a theorem of **PRES**.
R6. If A is a theorem of **PRES** and either B or $\sim\sim B$ is a wff subformula of A, and C results from A by replacing B by $\sim\sim B$, or $\sim\sim B$ by B, then C is a theorem of **PRES**.

Proofs of the independence and consistency of the axioms are given in Åqvist, *cf.* [1], where it is also shown that the system is complete in the sense of having all and only valid formulae as theorems, though incomplete in the sense that it admits of a consistent proper extension.

5. *A Note on the Proofs* In giving proofs we write the number of a theorem or of a previous step in the proof followed by 'xRn =', where Rn is one of the rules R1-R6, and then write the formula deduced from the designated theorem or step by means of that rule. Where a rule has a double input we write '+' between the two formulae used as premisses. Thus an application of modus ponens is written '$A \supset B + A$ xR1 $= B$', or with reference numbers in place of '$A \supset B$' and 'A'. We write 'xDRn' when we use a derived rule DRn. In the case of R4 we write 'R4, PCn + m = K' where PCn is one of PC1-PC3, the open formula obtained by substitution in the antecedent is the theorem m, and the formula obtained from the consequent according to the rule is K. In the case of R5 we write 'R5, PCn = m', where PCn is either one of PC1-PC3 or else one of the following PC theorems:

PC4 $(p \supset (q \vee r)) \supset ((q \supset s) \supset (p \supset (s \vee r)))$
PC5 $(p \vee q) \supset (q \vee p)$
PC6 $(p \supset q) \supset ((q \supset r) \supset (p \supset r))$
PC7 $p \supset (p \vee q)$
PC8 $p \supset (q \vee p)$
PC9 $((p \supset q) \,\&\, (p \supset r)) \supset (p \supset (q \,\&\, r))$
PC10 $(p \,\&\, \sim q) \supset \sim (p \supset q)$
PC11 $((p \vee q) \,\&\, \sim q) \supset p$
PC12 $(p \supset (q \supset r)) \supset (q \supset (p \supset r))$
PC13 $(p \supset r) \supset ((p \vee q) \supset (r \vee q))$
PC14 $(q \supset r) \supset ((p \vee q) \supset (p \vee r))$
PC15 $(q \supset r) \supset ((p \supset q) \supset ((r \supset s) \supset (p \supset s)))$
PC16 $(p \supset (q \supset r)) \supset ((p \,\&\, q) \supset r)$
PC17 $(p \,\&\, q) \supset p$
PC18 $(p \,\&\, q) \supset q$

In connection with R2, R4, and R5 we do not list the substitutions in detail, since inspection of the proofs shows what this is. Definitional rewriting is shown by 'xDefC', where C is one of the nonprimitive constants. Where proofs of theorems are given in Åqvist, *cf.* [1], we omit them here, and give the number of the theorem in Åqvist's paper, preceded by 'AT'.

6. *Theorems*

[Th.1] $p \supset \sim \mathbf{T} \sim p$

(AT3)

[Th.2] $\mathbf{T}(p \vee q) \supset (\mathbf{T}p \vee \mathbf{T}q)$

(AT1)

[Th.2.1] $(\mathbf{T}p \vee \mathbf{T}q) \supset \mathbf{T}(p \vee q)$

(AT1.1)

[Th.3] $\mathbf{T}(p \,\&\, q) \supset (\mathbf{T}p \,\&\, \mathbf{T}q)$

(A2xDef$\leftrightarrow$) + (R5, PC17)xR1 = Th.3

[Th.3.1] $(\mathbf{T}p \,\&\, \mathbf{T}q) \supset \mathbf{T}(p \,\&\, q)$

(AT2)

[Th.4] $\mathbf{T}p \supset (\mathbf{T}q \supset \mathbf{T}(p \,\&\, q))$

(AT2.1)

[DR1] *From* $\vdash A$, $\vdash B$ *infer* $\vdash A \,\&\, B$

(1) A Hyp.
(2) B Hyp.
(3) $\mathbf{T}A$ (1)xR3
(4) $\mathbf{T}B$ (2)xR3
(5) $\mathbf{T}A \supset (\mathbf{T}B \supset \mathbf{T}(A \,\&\, B))$ (Th.4)xR2
(6) $\mathbf{T}(A \,\&\, B)$ ((5) + (3)xR1) + (4)xR1
(7) $(A \,\&\, B)$ ((A3)xR2) + (6)xR1

[DR2] *From* $\vdash A \supset B$, $\vdash B \supset C$ *infer* $\vdash A \supset C$, *provided* A, B, *and* C *are* **T**-*formulae.*

(1) $A \supset B$ Hyp.
(2) $B \supset C$ Hyp.
(3) $(A \supset B) \,\&\, (B \supset C)$ (1) + (2)xDR1
(4) $(A \supset C)$ (R5, PC1) + (3)xR1

[Th.5] $(\mathbf{T}p \supset p) \,\&\, (p \supset \sim\mathbf{T}\sim p)$

(AT4)

[Th.6] $\mathbf{T}p \supset \sim\mathbf{T}\sim p$

(R4, PC1) + (Th.5) = Th.6

[Th.7] $\mathbf{T}\sim p \supset \sim\mathbf{T}p$

(AT5)

[Th.8] $\mathbf{T}(p \supset q) \supset (\mathbf{T}p \supset \mathbf{T}q)$

R5, PC4, $\mathbf{T}(\sim p \vee q)/p$, $\mathbf{T}\sim p/q$, $\mathbf{T}q/r$, $\sim\mathbf{T}p/s$ =
(1) $(\mathbf{T}(\sim p \vee q) \supset (\mathbf{T}\sim p \vee \mathbf{T}q)) \supset ((\mathbf{T}\sim p \supset \sim\mathbf{T}p) \supset (\mathbf{T}(\sim p \vee q) \supset (\sim\mathbf{T}p \vee \mathbf{T}q)))$
((1) + (Th2.1xR2)xR1) + (Th.7)xR1 = (2) $\mathbf{T}(\sim p \vee q) \supset (\sim\mathbf{T}p \vee \mathbf{T}q)$
(2) xDef$\supset$ = Th.8

[Th.8.1] $\mathbf{T}(p \supset q) \supset (p \supset \mathbf{T}q)$

R5, PC5 = (1)$(\mathbf{T}p \vee \mathbf{T}q) \supset (\mathbf{T}q \vee \mathbf{T}p)$
(1) + (Th.2)xDR2 = (2) $\mathbf{T}(p \vee q) \supset (\mathbf{T}q \vee \mathbf{T}p)$
((2)xR6)xDef$\supset$ = (3) $\mathbf{T}(p \vee q) \supset (\sim\mathbf{T}q \supset \mathbf{T}p)$
(3) + (R5, PC16)xR1 = (4)$(\mathbf{T}(p \vee q) \,\&\, \sim\mathbf{T}q) \supset \mathbf{T}p$

(4) + (A3)xDR1 = (5)$((\mathbf{T}(p \vee q)\ \&\ {\sim}\mathbf{T}q) \supset \mathbf{T}p)\ \&\ (\mathbf{T}p \supset p)$
(R4, PC1) + (5) = (6) $(\mathbf{T}(p \vee q)\ \&\ {\sim}\mathbf{T}q) \supset p$
(R4, PC3) + (6) = (7) $\mathbf{T}(p \vee q) \supset ({\sim}\mathbf{T}q \supset p)$
((7)xDef⊃)xR6 = (8) $\mathbf{T}(p \vee q) \supset (\mathbf{T}q \vee p)$
((8)xR2)xDef⊃ = Th.8.1

[DR3] *From* $\vdash A \supset B$ *infer* $\vdash \mathbf{T}A \supset \mathbf{T}B$

(1) $A \supset B$ Hyp.
(2) $\mathbf{T}(A \supset B)$ (1)xR3
(3) $\mathbf{T}A \supset \mathbf{T}B$ (Th.8xR2) + (2)xR1

[Th.9] $\mathbf{T}p \supset \mathbf{TT}p$

(AT11)

[DR4] *From* $\vdash A \supset B$ *infer* $\vdash {\sim}B \supset {\sim}A$

(1) $A \supset B$ Hyp.
(2) $\mathbf{T}({\sim}A \vee B)$ ((1)xDef⊃)xR3
(3) $\mathbf{T} \sim A \vee \mathbf{T}B$ (Th.2xR2) + (2)xR1
(4) $\mathbf{T}B \vee \mathbf{T} \sim A$ (3)xR5, PC5
(5) $\mathbf{T}(B \vee {\sim}A)$ (Th.2.1xR2) + (4)xR1
(6) ${\sim}{\sim}B \vee {\sim}A$ ((A3xR2) + (5)xR1)xR6
(7) ${\sim}B \supset {\sim}A$ (6)xDef⊃

[DR5] *From* $\vdash A \supset B$ *infer* $\vdash {\sim}\mathbf{T} \sim A \supset {\sim}\mathbf{T} \sim B$

(1) $A \supset B$ Hyp.
(2) ${\sim}B \supset {\sim}A$ (1)xDR4
(3) $\mathbf{T} \sim B \supset \mathbf{T} \sim A$ (2)xDR3
(4) ${\sim}\mathbf{T} \sim A \supset {\sim}\mathbf{T} \sim B$ (3)xDR4

[DR6] *From* $\vdash A \supset B$, $\vdash B \supset A$ *infer* $\vdash A \equiv B$

(1) $A \supset B$ Hyp.
(2) $B \supset A$ Hyp.
(3) $\mathbf{T}A \leftrightarrow \mathbf{T}B$(((1)xDR3) + ((2)xDR3)xDR1)xDef ↔
(4) ${\sim}\mathbf{T} \sim A \leftrightarrow {\sim}\mathbf{T} \sim B$ Similarly by DR5
(5) $A \equiv B$ ((3) + (4)xDR1)xDef≡

[Th.9.1] $\mathbf{T}p \equiv \mathbf{TT}p$

(A3xR2) + Th.9xDR6 = Th.9.1

[Th.10] ${\sim}\mathbf{T}p \supset \mathbf{T} \sim \mathbf{T}p$

((A3)xR2)xDR4 = (1) ${\sim}\mathbf{T}p \supset {\sim}\mathbf{TT}p$
(2)$\mathbf{TT}p \vee \mathbf{T} \sim \mathbf{T}p$ (AT9)
((2)xR6)xDef⊃ = (3) ${\sim}\mathbf{TT}p \supset \mathbf{T} \sim \mathbf{T}p$
(1) + (3)xDR2 = Th.10

From this point on we give sample proofs only.

[Th.10.1] ${\sim}\mathbf{T}p \equiv \mathbf{T} \sim \mathbf{T}p$

[Th.11] $\mathbf{T}p \supset \mathbf{T}(p \vee q)$
[Th.12] $\mathbf{T}(p \vee q) \equiv (\mathbf{T}p \vee \mathbf{T}q)$
[Th.13] $\mathbf{T}(p \,\&\, q) \supset \mathbf{T}p$
[Th.13.1] $\mathbf{T}(p \,\&\, q) \supset \mathbf{T}q$
[Th.14] $\mathbf{T}(p \,\&\, q) \equiv (\mathbf{T}p \,\&\, \mathbf{T}q)$
[Th.15] $(\mathbf{T}p \,\&\, \mathbf{T}q) \supset \mathbf{T}(\mathbf{T}p \,\&\, \mathbf{T}q)$
[Th.15.1] $(\mathbf{T}p \,\&\, \mathbf{T}q) \equiv \mathbf{T}(\mathbf{T}p \,\&\, \mathbf{T}q)$
[Th.16] $(\mathbf{T}p \vee \mathbf{T}q) \supset \mathbf{T}(\mathbf{T}p \vee \mathbf{T}q)$
[Th.16.1] $(\mathbf{T}p \vee \mathbf{T}q) \equiv \mathbf{T}(\mathbf{T}p \vee \mathbf{T}q)$
[Th.17] $(\mathbf{T}p \supset \mathbf{T}q) \supset \mathbf{T}(\mathbf{T}p \supset \mathbf{T}q)$

R5, PC13, $\sim\mathbf{T}p/p$, $\mathbf{T}q/q$, $\mathbf{T}\sim\mathbf{T}p/r$ =
(1)$(\sim\mathbf{T}p \supset \mathbf{T}\sim\mathbf{T}p) \sim ((\sim\mathbf{T}p \vee \mathbf{T}q) \supset (\mathbf{T}\sim\mathbf{T}p \vee \mathbf{T}q))$
(1) + (Th.11)xR1 = (2)$(\sim\mathbf{T}p \vee \mathbf{T}q) \supset (\mathbf{T}\sim\mathbf{T}p \vee \mathbf{T}q)$
((2)xR6)xDef⊃ = (3)$(\sim\mathbf{T}p \vee \mathbf{T}q) \supset (\sim\mathbf{T}\sim\mathbf{T}p \supset \mathbf{T}q)$
(R5, PC16) + (3)xR1 = (4)$((\sim\mathbf{T}p \vee \mathbf{T}q) \,\&\, \sim\mathbf{T}\sim\mathbf{T}p) \supset \mathbf{T}q$
(4) + (Th.9xR2)xDR2 = (5)$((\sim\mathbf{T}p \vee \mathbf{T}q) \,\&\, \sim\mathbf{T}\sim\mathbf{T}p) \supset \mathbf{TT}q$
(R5, PC3) + (5)xR1 = (6)$(\sim\mathbf{T}p \vee \mathbf{T}q) \supset (\sim\mathbf{T}\sim\mathbf{T}p \supset \mathbf{TT}q)$
((6)xDef⊃)xR6 = (7)$(\sim\mathbf{T}p \vee \mathbf{T}q) \supset (\mathbf{T}\sim\mathbf{T}p \vee \mathbf{TT}q)$
(7) + (Th.2.1xR2)xDR2 = (8)$(\sim\mathbf{T}p \vee \mathbf{T}q) \supset \mathbf{T}(\sim\mathbf{T}p \vee \mathbf{T}q)$
(8)xDef⊃ = Th.17

[Th.17.1] $(\mathbf{T}p \supset \mathbf{T}q) \equiv \mathbf{T}(\mathbf{T}p \supset \mathbf{T}q)$
[Th.18] $(p \rightarrow p) \supset \sim\mathbf{T}\sim p$

R5, PC8 = (1)$\mathbf{T}\sim p \supset (\mathbf{T}p \vee \mathbf{T}\sim p)$
(Th.2.1xR2) + (1)xDR2 = (2) $\mathbf{T}\sim p \supset \mathbf{T}(p \vee \sim p)$
(R5, PC9) + ((2) + (Th.7)xDR1)xRI =
(3)$\mathbf{T}\sim p \supset (\mathbf{T}(p \vee \sim p) \,\&\, \sim\mathbf{T}p)$
R5, PC10 = (4)$(\mathbf{T}(p \vee \sim p) \,\&\, \sim\mathbf{T}p) \supset \sim(\mathbf{T}(p \vee \sim p) \supset \mathbf{T}p)$
(3) + (4)xDR2 = (5)$\mathbf{T}\sim p \supset \sim(\mathbf{T}(p \vee \sim p) \supset \mathbf{T}p)$
((5)xDR4)xR6 = (6)$(\mathbf{T}(p \vee \sim p) \supset \mathbf{T}p) \supset \sim\mathbf{T}\sim p$
(6)xDef→ = Th.18

[Th.18.1] $\sim\mathbf{T}\sim p \supset (p \rightarrow p)$

R5, PC11 = (1)$((\mathbf{T}p \vee \mathbf{T}\sim p) \,\&\, \sim\mathbf{T}\sim p) \supset \mathbf{T}p$
(R5, PC3) + (1)xR1 = (2)$(\mathbf{T}p \vee \mathbf{T}\sim p) \supset (\sim\mathbf{T}\sim p \supset \mathbf{T}p)$
(Th.2xR2) + (2)xDR2 = (3) $\mathbf{T}(p \vee \sim p) \supset (\sim\mathbf{T}\sim p \supset \mathbf{T}p)$
(R5, PC12) + (3)xR1 = (4) $\sim\mathbf{T}\sim p \supset (\mathbf{T}(p \vee \sim p) \supset \mathbf{T}p)$
(4)xDef→ = Th.18.1

[Th.18.2] $(p \rightarrow p) \equiv \sim\mathbf{T}\sim p$
[Th.19] $(p \rightarrow \sim p) \equiv \sim\mathbf{T}p$
[Th.20] $p \rightarrow (p \supset p)$
[Th.21] $(\mathbf{T}p \rightarrow q) \supset (p \rightarrow q)$
[Th.22] $(p \rightarrow \mathbf{T}q) \supset (p \rightarrow q)$

R5, PC6 =
(1)$(\mathbf{T}(p \vee \sim p) \supset \mathbf{TT}q) \supset ((\mathbf{TT}q \supset \mathbf{T}q) \supset (\mathbf{T}(p \vee \sim p) \supset \mathbf{T}q))$

(R5, PC12) + (1)xR1 =
(2)$(\mathbf{TT}q \supset \mathbf{T}q) \supset ((\mathbf{T}(p \vee \sim p) \supset \mathbf{TT}q) \supset (\mathbf{T}(p \vee \sim p) \supset \mathbf{T}q))$
(((A3)xR2) + (2)xR1)xDef→ = Th.22

[Th.22.1] $(p \rightarrow q) \supset (p \rightarrow \mathbf{T}q)$
[Th.22.2] $(p \rightarrow q) \equiv (p \rightarrow \mathbf{T}q)$
[Th.23] $(p \rightarrow q) \supset \mathbf{T}(p \rightarrow q)$
[Th.23.1] $(p \rightarrow q) \equiv \mathbf{T}(p \rightarrow q)$
[Th.24] $(p \rightarrow q) \supset (\sim p \rightarrow q)$

(Th.2xR2) + (R5, PC5)xDR2 =
(1)$\mathbf{T}(\sim p \vee p) \supset (\mathbf{T}p \vee \mathbf{T} \sim p)$
(Th.2.1xR2) + (1)xDR2 =
(2)$\mathbf{T}(\sim p \vee p) \supset \mathbf{T}(p \vee \sim p)$
(2)xR6 =
(3)$\mathbf{T}(\sim p \vee \sim\sim p) \supset \mathbf{T}(p \vee \sim p)$
(R5, PC6) + (3)xR1 =
(4)$(\mathbf{T}(p \vee \sim p) \supset \mathbf{T}q) \supset (\mathbf{T}(\sim p \vee \sim\sim p) \supset \mathbf{T}q)$
(4)xDef→ = Th.24

[Th.24.1] $(\sim p \rightarrow q) \supset (p \rightarrow q)$
[Th.24.2] $(p \rightarrow q) \equiv (\sim p \rightarrow q)$
[Th.25] $(p \rightarrow q) \supset (\mathbf{T}p \supset \mathbf{T}q)$
[Th.25.1] $(p \rightarrow q) \supset (\mathbf{T}p \supset q)$
[Th.25.2] $(p \rightarrow q) \supset (\mathbf{T} \sim p \supset \mathbf{T}q)$
[Th.25.3] $(p \rightarrow q) \supset (\mathbf{T} \sim p \supset q)$
[Th.26] $((p \rightarrow q) \,\&\, (p \rightarrow r)) \supset (p \rightarrow (q \,\&\, r))$
[Th.26.1] $(p \rightarrow (q \,\&\, r)) \supset ((p \rightarrow q) \,\&\, (p \rightarrow r))$
[Th.26.2] $((p \rightarrow q) \,\&\, (p \rightarrow r)) \equiv (p \rightarrow (q \,\&\, r))$
[Th.27] $((p \rightarrow q) \vee (p \rightarrow r)) \supset (p \rightarrow (q \vee r))$
[Th.27.1] $(p \rightarrow (q \vee r)) \supset ((p \rightarrow q) \vee (p \rightarrow r))$
[Th.27.2] $((p \rightarrow q) \vee (p \rightarrow r)) \equiv (p \rightarrow (q \vee r))$
[Th.28] $((p \rightarrow r) \,\&\, (q \rightarrow r)) \supset ((p \vee q) \rightarrow r)$
[Th.29] $((p \rightarrow r) \vee (q \rightarrow r)) \supset ((p \,\&\, q) \rightarrow r)$
[Th.30] $(p \rightarrow q) \supset ((\mathbf{T}q \supset \mathbf{T}r) \supset (p \rightarrow r))$
[Th.30.1] $(p \rightarrow q) \supset (\mathbf{T}(q \supset r) \supset (p \rightarrow r))$
[Th.31] $(p \rightarrow q) \supset ((q \rightarrow r) \supset (p \rightarrow r))$

R5, PC15, $\mathbf{T}(p \vee \sim p)/p$, $\mathbf{T}q/q$, $\mathbf{T}(q \vee \sim q)/r$, $\mathbf{T}r/s$ =
(1) $(\mathbf{T}q \supset \mathbf{T}(q \vee \sim q)) \supset ((\mathbf{T}(p \vee \sim p) \supset \mathbf{T}q) \supset ((\mathbf{T}(q \vee \sim q) \sim \mathbf{T}r)$
$\supset (\mathbf{T}(p \vee \sim p) \supset \mathbf{T}r)))$
(R5, PC7) + (Th.2.1xR2)xDR2 = (2)$\mathbf{T}q \supset \mathbf{T}(q \vee \sim q)$
((1) + (2)xR1)xDef→ = Th.31

[DR7] *From* $\vdash A$, $\vdash A \rightarrow B$ *infer* $\vdash B$

In concluding this section we think it worth remarking that the following formulae are not theorems:

* $(p \rightarrow q) \supset (\mathbf{T}p \rightarrow q)$ (the converse of Th.21)
* $((p \vee q) \rightarrow r) \supset ((p \rightarrow r) \,\&\, (q \rightarrow r))$ (the converse of Th.28)

* $((p \& q) \rightarrow r) \supset ((p \rightarrow r) \vee (q \rightarrow r))$ (the converse of Th.29)
* $(p \rightarrow q) \supset \mathbf{T}(p \supset q)$ (*cf.* Th.25)
* $(p \rightarrow q) \supset (p \supset q)$ (*cf.* Th.25)
* $(p \rightarrow q) \supset ((q \supset r) \supset (p \rightarrow r))$ (*cf.* Ths.30, 30.1)

Part Two: The Paradoxes of Material Presupposition and Their Solution

The system **PRES** captures and develops some important aspects of our ordinary (—if somewhat ill-defined—) concept of presupposition. Yet, in certain other respects, it constitutes a patently inadequate systematization of that notion. In particular, as a result of the definition of '$\rightarrow$', and the substitution of '$\mathbf{T}(p \vee \sim p)$' for '$p$', and '$\mathbf{T}q$' for '$q$' in the **PC**-thesis '$\sim p \supset (p \supset q)$', we obtain as a thesis of **PRES**:

[Th.32] $\sim \mathbf{T}(p \vee \sim p) \supset (p \rightarrow q)$

In words: a void proposition presupposes any proposition whatever. By making the same substitutions in the **PC**-thesis '$q \supset (p \supset q)$', we obtain as another thesis of **PRES**:

[Th.33] $\mathbf{T}q \supset (p \rightarrow q)$

That is, it is proved that a true proposition is presupposed by any proposition whatever. These results arise from the incorporation of the **PC**-theses known as the paradoxes of material implication, and consequently they might themselves be called the paradoxes of material presupposition. Now, just as some (notably C. I. Lewis) have found the paradoxes of material implication an insuperable obstacle to the construal of '$\supset$' as a formalisation of the pre-systematic notion of implication, so it might be objected that the paradoxes of material presupposition count heavily against any claim for '$\rightarrow$' as representing the pre-systematic idea of presupposition. As a solution to this problem, it might be further suggested that we follow in Lewis' footsteps and 'strengthen' our original connective by placing it in the scope of a necessity operator, now regarding this whole complex as the appropriate formalisation of the notion under investigation. After all, the inadequacy of material presupposition as a formal rendering of presupposition and the inadequacy of material implication as a formal rendering of implication (or entailment) are basically due to the same thing: namely, that the informal notions themselves involve more than just the comparison of truth-values, which is all that their formal representatives may make reference to. It seems reasonable to expect that a treatment of the 'problem of presupposition', then, might usefully parallel the treatment of the 'problem of implication', that is, be effecting a modalisation of the underlying truth-functional logic. For the remainder of this paper, we shall be occupied in pursuing this suggestion.

The system **PRES** may be regarded, in the light of what has just been said, as a formal theory of material presupposition, and we may regard the system $\mathbf{PRES_L}$, to be developed presently, as a theory of strict presupposition, where 'p strictly presupposes q' is the intended verbalisation of

'$\mathbf{L}(p \rightarrow q)$', with '$\rightarrow$' defined as in **PRES**. Indeed, the most convenient way of developing a logic of strict presupposition proves to be by providing a modal extension of **PRES**. To do this, we define $\mathbf{PRES_L}$ as the system formed by adding to the axioms of **PRES** the two axioms and one rule of inference distinctive of that system called by Sobociński '**T**':

[A4] $\mathbf{L}(p \supset q) \supset (\mathbf{L}p \supset \mathbf{L}q)$
[A5] $\mathbf{L}p \supset p$

R7. *From* $\vdash A$ *infer* $\vdash \mathbf{L}A$

together with one axiom relating the necessity operator to the truth-operator:

[A6] $\mathbf{L}p \supset \mathbf{LT}p$

As a result of R7 and [A4], it is obvious that $\mathbf{PRES_L}$ contains the following derived rule:

[D.R.8] *From* $\vdash A \supset B$ *infer* $\vdash \mathbf{L}A \supset \mathbf{L}B$

Applying this rule to [A3], we get the first theorem to be stated here that is in $\mathbf{PRES_L}$ but not in **PRES**:

[Th.34] $\mathbf{LT}p \supset \mathbf{L}p$

which, given [A6] via an application of [D.R.6], yields:

[Th.35] $\mathbf{L}p \equiv \mathbf{LT}p$

In the same vein we have:

[Th.36] $\mathbf{L}p \supset \mathbf{TL}p$

(Th.34xR3) + (Th.8.1xR2)xR1 = (1) $\mathbf{LT}p \supset \mathbf{TL}p$
R4, PC1 + ((1) + A6xDR1) = Th.36

[Th.37] $\mathbf{L}p \equiv \mathbf{TL}p$

These first four theorems of $\mathbf{PRES_L}$ connect with some of the features we are going to want the semantics of a modal system with a truth-operator to have. For on the normal reading of '**L**' as 'it is necessarily true that', and given that the formal rendering of 'it is true that p' is provably equivalent ([Th.9.1]) to the formal rendering of 'it is true that it is true that p', one would certainly expect to have 'it is necessarily true that p' and 'it is necessarily true that it is true that p' likewise equivalent. [Th.35] states this equivalence, and also means that '**L**-formulae' (i.e., formulae in which no propositional variable lies outside the scope of an occurrence of '**L**') may be regarded as derivatively **T**-formulae. Another aspect of the same matter is that '**L**' is a decisive constant, in the sense introduced above,[6] since if 'p' is void, so that $\vee(p) = 2$, '$\mathbf{T}p$' is false (remembering that we want $\vee(\mathbf{L}p) = \vee(\mathbf{LT}p)$, and assuming that we have '**L**' as a decidedness-preserving constant, in the sense of footnote 5). Now we can see [Th.37] as a modal analogue to [Th.15.1], [Th.16.1], and [Th.17.1]. For these three theses say, in effect, that any truth-functional compound, A, all of whose components are **T**-formulae, is provably equivalent to its own 'truthing up', **T**A. [Th.37] carries this result into the area of modality. This ties in neatly with the semantics of the situation, for we want **L**A to be true in a world if and only if A is true, in all worlds possible relative to that world. If A is not true 'everywhere', in this sense, then A is not necessarily true, and **L**A is false. We rule out, in other words, the possibility that **L**A should actually take the value 2 in a world, even if A is assigned 2 in all relatively possible worlds. (Under such circumstances, since A is not everywhere true, **L**A is just plain false, and is assigned 3 in the world in question.) Having decided, then, on the value assignment policy:

$$\mathsf{V}(\mathsf{L}A, w_i) = 1 \text{ iff } \mathsf{V}(A, w_j) = 1 \text{ for all } w_j \text{ possible relative to } w_i,$$
$$\mathsf{V}(\mathsf{L}A, w_i) = 3 \text{ otherwise,}$$

for necessity, a natural choice of policy for the correlative modal notion, possibility, represented here by '**M**', would be:

$$\mathsf{V}(\mathsf{M}A, w_i) = 1 \text{ iff } \mathsf{V}(A, w_j) = 1 \text{ for some } w_j \text{ possible relative to } w_i,$$
$$\mathsf{V}(\mathsf{M}A, w_j) = 3 \text{ otherwise.}$$

Thus, we construe necessity as truth in all possible worlds and possibility as truth in at least one possible world, just as in the semantics of modal extensions of two-valued propositional calculus. The similarity at this point might tempt one to think that we are in for the unfolding of all the usual modal relations, the existence of the third value perhaps merely complicating the matter in some routine way. To develop some theorems on possibility, it might then be thought, one has just to define **M**A as $\sim\mathsf{L}\sim A$, in the usual way, and proceed. This would be a serious mistake, however, as a moment's reflexion reveals. For what $\sim\mathsf{L}\sim A$ (being equivalent to $\sim\mathsf{LT}\sim A$) says is that A is not everywhere false, whereas what we decided **M**A was to mean is that A is somewhere true. In three-valued logic these are by no means the same thing, since the former would be true, and the latter false, if A were assigned the value 2 in all possible worlds. Similarly, $\sim\mathsf{M}\sim A$ is to be distinguished from **L**A ($\equiv$**LT**A), since the former is true provided that A is nowhere false, while the latter is true only if A is, not just non-false everywhere, but actually true everywhere.[7] To obtain these relations, **M**A must be defined, not as $\sim\mathsf{L}\sim A$, which would be equivalent to $\sim\mathsf{LT}\sim A$, but in the following way:

[Def. **M**] **M**A for $\sim\mathsf{L}\sim\mathsf{T}A$

The reader may verify for himself that the semantic relations specified by our assignment policies for **L** and **M** are guaranteed to hold by this definition. We may now proceed to develop some theorems on the notions of necessity, possibility and truth as they interact in $\mathbf{PRES_L}$.

[Th.38] $\mathsf{L}p \supset \mathsf{T}p$
[Th.39] $p \supset \sim\mathsf{L}\sim p$
[Th.40] $\mathsf{T}p \supset \sim\mathsf{L}\sim p$

[Th.41] $\mathsf{L}p \supset \sim \mathsf{L} \sim p$

[Th.42] $\sim \mathsf{L}p \supset \mathsf{T} \sim \mathsf{L}p$

Th.10xR2 = (1) $\sim \mathsf{TL}p \supset \mathsf{T} \sim \mathsf{TL}p$

Th.36xDR5 = (2) $\sim \mathsf{T} \sim \mathsf{L}p \supset \sim \mathsf{T} \sim \mathsf{TL}p$

((1)xDR4) + (2)xDR2 = (3) $\sim \mathsf{T} \sim \mathsf{L}p \supset \sim\sim \mathsf{TL}p$

((3)xR6) + (A3xR2)xDR2 = (4) $\sim \mathsf{T} \sim \mathsf{L}p \supset \mathsf{L}p$

(4)xDR4xR6 = Th.42

[Th.43] $\sim \mathsf{L}p \equiv \mathsf{T} \sim \mathsf{L}p$

[Th.44] $\sim \mathsf{T} \sim p \supset \sim \mathsf{L} \sim p$

[Th.45] $\mathsf{L}p \supset \mathsf{M}p$

Th.41xR2 = (1) $\mathsf{LT}p \supset \sim \mathsf{L} \sim \mathsf{T}p$

A6 + (1)xDR2 = (2) $\mathsf{L}p \supset \sim \mathsf{L} \sim \mathsf{T}p$

(2)xDef**M** = Th.45

[DR9] *From* $\vdash A \supset B$ *infer* $\vdash \sim \mathsf{L} \sim A \supset \sim \mathsf{L} \sim B$

[Th.46] $\mathsf{M}p \supset \sim \mathsf{L} \sim p$

A3xDR9xDef**M** = Th.46

[Th.47] $\mathsf{L}p \supset \sim \mathsf{M} \sim p$

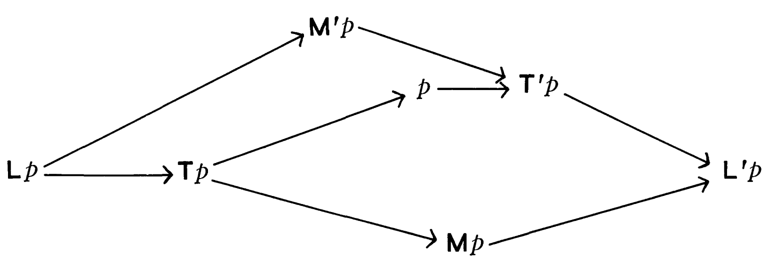
((Th.1xDR8)xR6)xDef**M** = Th.47

[Th.48] $\mathsf{T}p \supset \mathsf{M}p$

(Th.39xR2)xDef**M** = Th.48

[Th.49] $\sim \mathsf{M} \sim p \supset \sim \mathsf{T} \sim p$

These relations are pictured in the following diagram. An arrow between formulae signifies that the implication in that direction, but not its converse, is provable between those formulae. (Implications that are provable as a result of the transitivity of implication are not marked separately.) For convenience, we write a dash by an operator to symbolise the circumnegation of that operator; thus, $\mathsf{M}'p$ for $\sim \mathsf{M} \sim p$, etc.



Having thus demonstrated that the modal geography of a three-valued terrain introduces certain distinctive complications that might not have been anticipated, given a familiarity with two-valued modal logic, we may go on to deal with the more complex formulae of $\mathbf{PRES_L}$, including especially those pertaining to strict presupposition. One first point to clear away is the worry that the non-equivalence of L and $\sim\mathsf{M}\sim$ provides us with two distinct ways of modally strengthening material presupposition, and that we must, therefore, choose whether it is in fact $\mathsf{L}(p \rightarrow q)$ or rather $\sim\mathsf{M}\sim(p \rightarrow q)$ that should be regarded as representing strict presupposition. It is not difficult to see, however, that:

[Th.50] $\mathsf{L}(p \rightarrow q) \equiv \sim\mathsf{M}\sim(p \rightarrow q)$

We omit the proof (remarking only that it depends on [Th.23.1]). Fortunately, then, the worry about a modal *embarras de richesse* turns out to be unfounded in this context. One area in which such a worry legitimately arises, however, is that of entailment. We may suppose, so as not to multiply points of disputation, that, for two-valued logic, Lewis' strict implication adequately formalises the notion of entailment. There is, however, no single natural rendering in a three-valued logic with a truth-operator, of strict implication. Thus questions about 'whether or not all cases of presupposition are cases of entailment', etc., remain formally unsettleable, the relation of entailment having only been defined for two-valued logic, with the relation of presupposition defined only for three-valued logic.[8] We might say that the two notions, as they stand, are thus 'incommensurable'. Since we have good reason to believe that if presupposition is treated as a logical relation which may hold between individual propositions, it is not amenable to treatment in two-valued calculi (as argued in our introductory section), we may devote some attention to the question of how entailment is to be represented in three-valued logic (—we must have the two notions represented within the same logical system so as to compare their behaviour). Given that we want some sort of strict implication, we may examine three (among the several possible) candidates:

(A) $\mathsf{L}(p \supset q)\ [\equiv \mathsf{LT}(p \supset q)]$
(B) $\mathsf{L}(\mathsf{T}p \supset \mathsf{T}q)$
(C) $\sim\mathsf{M}(p \,\&\, \sim q)$

As to (C), we think we may dismiss its chances of representing entailment without much difficulty, if we accept the plausible condition on possible entailment relations that they be transitive. For (C) reduces by R6 and the definition of 'M' to '$\mathsf{L} \sim \mathsf{T}(p \,\&\, \sim q)$', which by Th.3, Th3.1, DR4, DR8, and DR6 is equivalent to '$\mathsf{L}\sim(\mathsf{T}p \,\&\, \mathsf{T} \sim q)$', whence by definitional rewriting and R6 we obtain as provably like-valued equivalent to (C) the formula '$\mathsf{L}(\mathsf{T}p \supset \sim\mathsf{T} \sim q)$'. It should be clear that '$\mathsf{L}(\mathsf{T}p \supset \sim\mathsf{T} \sim q)$' and '$\mathsf{L}(\mathsf{T}q \supset \sim\mathsf{T} \sim r)$' do not imply '$\mathsf{L}(\mathsf{T}p \supset \sim\mathsf{T} \sim r)$'. They would do so only if it were provable that '$\sim\mathsf{T}\sim q$' implied '$\mathsf{T}q$', that is, only if we eliminated the value 'void' from our logic.

Having excluded (C), we may compare (A) and (B). Both embody transitive relations (proofs omitted), and they are such that (A) implies but is not implied by (B). We state this as

[Th.51] $\mathsf{L}(p \supset q) \supset \mathsf{L}(\mathsf{T}p \supset \mathsf{T}q)$

(Th.8xDR8) + (A6xR2)xDR2 = Th.51

Semantically, this means that there are more circumstances under which (B) is true than under which (A) is true. (B) will be false only if there is some possible world in which 'p' is true and 'q' is either false or void, while (A) will be false then and also if there is any world in which 'p' is void and 'q' is void, or in which 'p' is void and 'q' is false. It is difficult to see what bearing such facts have on the question of which formula should be taken to represent entailment in **PRES$_L$** and perhaps it would be most sensible to regard both (A) and (B) as equally entailment-like, calling (A) 'external', and (B) 'internal' entailment (after the position of the '**T**' relative to the parentheses). There are some considerations, however, which weigh in favour of (B) as the more appropriate choice. One such consideration that might be brought to bear on the issue is the dictum that one proposition entails another when the truth of the latter follows from, or may be inferred from, the truth of the former. Such a view naturally inclines one to regard internal rather than external entailment as the 'genuine' entailment relation. If we accept this, then the ruling of the system **PRES$_L$** on the question of whether or not there are cases of presupposition that are not also cases of entailment[9] is in the negative. For one may easily prove

[Th.52] $\mathbf{L}(p \rightarrow q) \supset \mathbf{L}(\mathbf{T}p \supset \mathbf{T}q)$

Th.25xDR.8 = Th.52

This is in accord with the argument of Nerlich in [12] (in particular, with what he calls the *a fortiori* argument on this subject).[10] The argument is simply that since a proposition cannot be true or false unless what it presupposes is true, it cannot be true, *tout court*, unless what it presupposes is true, and hence from its truth one can infer the truth of what it presupposes, which is to say that it entails this latter. Should one wish to reject this conclusion, one might then opt for '$\mathbf{L}(p \supset q)$' as the more appropriate formalisation of entailment in three-valued logic, noting that the formula

$$* \ \mathbf{L}(p \rightarrow q) \supset \mathbf{L}(p \supset q)$$

is not provable in **PRES$_L$**, or alternatively one could choose to disregard the system **PRES$_L$** as a formalisation of presupposition. We ourselves are impressed by Nerlich's line of reasoning, and so naturally tend to regard the provability of [Th.52] as one of the system's assets.

Observe, most importantly, that while this means (on the 'internal' account of entailment) that the notions of presupposition and entailment are not disjoint, it does not mean that they are not distinct. That is, the converse of [Th.52] is certainly not provable. One could even give a special name to that class of entailments which are not cases of presupposition, for example, calling them 'affirmative entailments', so that:

$$p \text{ affirmatively entails } q =_{df} \mathbf{L}(\mathbf{T}p \supset \mathbf{T}q) \ \& \sim \mathbf{L}(p \rightarrow q)$$

or equivalently,

$$p \text{ affirmatively entails } q =_{df} \mathbf{L}(\mathbf{T}p \supset \mathbf{T}q) \ \& \sim \mathbf{L}(\mathbf{T}p \vee \sim p \supset \mathbf{T}q)$$

Then the distinction drawn by Strawson in [17] between one proposition's entailing another, and its presupposing that other might be redrawn, by those interested, as a distinction between the first proposition's affirmatively entailing the second, and its presupposing it. This would at least be a division into mutually exclusive classes.

Given that $\mathbf{PRES_L}$ is a modal extension of **PRES**, it is not surprising that many of the results established for material presupposition go over, *mutatis mutandis*, into our account of strict presupposition. We have already seen this in the case of [Th.52], the modal analogue of [Th.25], and offer the following further examples:

[Th.53] $\mathsf{L}(\mathsf{T}p \rightarrow q) \supset \mathsf{L}(p \rightarrow q)$ (= Th.21xDR8)

[Th.53.1] $\mathsf{L}(p \rightarrow \mathsf{T}q) \supset \mathsf{L}(p \rightarrow q)$ (= Th.22xDR8)

[Th.53.2] $\mathsf{L}(p \rightarrow q) \supset \mathsf{L}(p \rightarrow \mathsf{T}q)$ (= Th.22.1xDR8)

[DR10] *From* $\vdash A \supset B$, $\vdash B \supset A$ *infer* $\vdash \mathsf{L}A \equiv \mathsf{L}B$

(Proof by DR8 and DR6)

[Th.53.3] $\mathsf{L}(p \rightarrow q) \equiv \mathsf{L}(p \rightarrow \mathsf{T}q)$ (= Th.22 + Th.22.1xDR10)

[Th.54] $\mathsf{L}(p \rightarrow q) \supset \mathsf{L}(\sim p \rightarrow q)$ (= Th.24xDR8)

[Th.54.1] $\mathsf{L}(\sim p \rightarrow q) \supset \mathsf{L}(p \rightarrow q)$ (= Th.24.1xDR8)

[Th.54.2] $\mathsf{L}(p \rightarrow q) \equiv \mathsf{L}(\sim p \rightarrow q)$ (= Th.24 + Th.24.1xDR10)

[Th.55] $\mathsf{L}(p \,\&\, q) \supset \mathsf{L}p$

(Th.13xDR8) + (A6xR2)xDR2 = (1)$\mathsf{L}(p \,\&\, q) \supset \mathsf{LT}p$
(1) + Th.34xDR2 = Th.55

[Th.55.1] $\mathsf{L}(p \,\&\, q) \supset (\mathsf{L}p \,\&\, \mathsf{L}q)$

[Th.55.2] $(\mathsf{L}p \,\&\, \mathsf{L}q) \supset \mathsf{L}(p \,\&\, q)$

[Th.55.3] $\mathsf{L}(p \,\&\, q) \equiv (\mathsf{L}p \,\&\, \mathsf{L}q)$

[Th.56] $(\mathsf{L}(p \rightarrow q) \,\&\, \mathsf{L}(p \rightarrow r)) \supset \mathsf{L}(p \rightarrow (q \,\&\, r))$

Th.26xDR8 = (1) $\mathsf{L}((p \rightarrow q) \,\&\, (p \rightarrow r)) \supset \mathsf{L}(p \rightarrow (q \,\&\, r))$
(Th.55.2xR2) + (1)xDR2 = Th.56

[Th.56.1] $\mathsf{L}(p \rightarrow (q \,\&\, r)) \supset (\mathsf{L}(p \rightarrow q) \,\&\, \mathsf{L}(p \rightarrow r))$

[Th.56.2] $(\mathsf{L}(p \rightarrow q) \,\&\, \mathsf{L}(p \rightarrow r)) \equiv \mathsf{L}(p \rightarrow (q \,\&\, r))$

[Th.57] $\mathsf{L}p \supset \mathsf{L}(p \vee q)$

((Th.11xDR8) + A6xDR2) + (Th.34xR2)xDR2 = Th.57

[Th.57.1] $(\mathsf{L}p \vee \mathsf{L}q) \supset \mathsf{L}(p \vee q)$

[Th.58] $(\mathsf{L}(p \rightarrow q) \vee \mathsf{L}(p \rightarrow r)) \supset \mathsf{L}(p \rightarrow (q \vee r))$

(Th.27xDR8) + (Th.57.1xR2)xDR2 = Th.58

Note that because the converse of [Th.57.1] is not provable, we do not have the analogues for strict presupposition of [Th.27.1] and [Th.27.2].

[Th.59] $(\mathsf{L}(p \rightarrow r) \,\&\, \mathsf{L}(q \rightarrow r)) \supset \mathsf{L}((p \vee q) \rightarrow r)$

[Th.60] $(\mathsf{L}(p \rightarrow r) \vee \mathsf{L}(q \rightarrow r)) \supset \mathsf{L}((p \,\&\, q) \rightarrow r)$

[Th.61] $\mathsf{L}(p \rightarrow q) \supset (\mathsf{L}(\mathsf{T}q \supset \mathsf{T}r) \supset \mathsf{L}(p \rightarrow r))$

[Th.62] $\mathsf{L}(p \rightarrow q) \supset (\mathsf{L}(q \supset r) \supset \mathsf{L}(p \rightarrow r))$

[Th.63] $\mathsf{L}(p \rightarrow q) \supset (\mathsf{L}(q \rightarrow r) \supset \mathsf{L}(p \rightarrow r))$

And finally the analogues to [Th.32] and [Th.33]:

[Th.64] $\mathbf{L} \sim \mathbf{T}(p \vee \sim p) \supset \mathbf{L}(p \rightarrow q)$
[Th.65] $\mathbf{LT}q \supset \mathbf{L}(p \rightarrow q)$

which constitute the inevitable 'paradoxes of strict presupposition' and bring us back to where we began. In words: a necessarily void proposition presupposes any proposition whatever, and a necessarily true proposition is presupposed by any proposition whatever. At this point, just as at the corresponding point in the dispute about strict implication, there will be some who feel that the strict presupposition paradoxes are to be welcomed as showing us something important and easy to overlook in our concept of presupposition, and there will be others who will find in them grounds for rejecting strict presupposition as a formalisation of that presystematic concept. These latter might care to amuse themselves in the construction of formal systems differing from $\mathbf{PRES_L}$ in replacing the material implication of **PRES**, not by strict implication, but instead by Parry's 'analytic implication', by Anderson and Belnap's 'relevant implication' or their 'entailment', by McCall's 'connexive implication' or by any of the many other implicational connectives on the market today. We wish them the best of luck.

NOTES

1. Particularly relevant to our discussion is Austin ([2], p. 51). We have taken from here the term 'void' and adapted it to apply to propositions with unfulfilled presuppositions. (For Austin it is the utterance which is void in such cases.) Much recent philosophical interest in presupposition centres round what has become known as free logic. See van Fraassen's and Lambert's contributions to Lambert [10], and the references cited there. For the use of the notion of presupposition by linguists, see especially Lakoff [8] and [9]; Kiparsky and Kiparsky [7]; and Horn [6].

2. This means that the system of presupposition logic we present does not include classical propositional calculus. However, the language of classical propositional logic, considered purely syntactically, is a part of the language of presupposition logic.

3. The system K_3 has the advantage that the non-primitive constants are defined in terms of the primitive constants in the way familiar from two-valued propositional calculus.

4. The term 'decisive' is due in this connection to Rescher. Our usage differs slightly from his in that for him it is the truth-table of a constant rather than the constant itself that is decisive. See Rescher [13], p. 61.

5. This is true only because all the logical constants we have introduced are *decidedness-preserving*, by which we mean that whenever the input values are only 1 and/or 3 the output value also is only 1 or 3. This is a partial generalisation of Rescher's notion 'normal'. See Rescher [13], p. 55.

6. Notice that if we had not adapted Rescher's use of 'decisive' we would not have been able to apply this term to the non-truth-functional constant '**L**'.

7. The fact that the normal relations between '**L**' and '**M**' are skewed by the existence of a third value has been noticed before. A particularly lucid discussion is to be found in Segerberg (1967). Segerberg is concerned to provide a modal extension of a three-valued logic where the third value is the intended assignment for meaningless sentences rather than void propositions. This difference is reflected in Segerberg's choice of Bochvar's tables (see Rescher [13], p. 29) for the constants rather than Kleene's. It is important to see that in Segerberg's treatment and in ours modality exists alongside many-valuedness, in contrast, for example, with some of the work of Łukasiewicz, where modality is construed in terms of many-valuedness.

8. Our talk of the *relation* of presupposition here should be understood as follows: we say that the relation of presupposition holds between A and B when it is true that A presupposes B. We treat '$\rightarrow$' as a binary connective, reading '$p \rightarrow q$' as 'that p presupposes that q', and not as the name of a binary relation holding between sentences or between propositions. Similarly for entailment.

9. Implicitly Strawson, in [16], is arguing that there are cases of presupposition which are not cases of entailment, since he argues that Russell was wrong to say that 'The King of France is wise' entails 'The King of France exists'.

10. This argument appears in an almost identical form in Roberts [14].

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