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SECOND ORDER AND HIGHER ORDER UNIVERSAL DECISION ELEMENTS IN m-VALUED LOGIC

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Introduction An N'th order universal decision element of the r'th degree in m-valued logic may be defined as follows:

A functor $\Phi(\ ,\ .\ .\ .,\ )$ of n argument places corresponds to an N'th order universal decision element of the r'th degree if we may define all non-trivial functors of m-valued logic with N or less argument places, solely by substitution of the formulae $F_1(P_1), \ldots, F_r(P_1), \ldots, F_1(P_n), \ldots, F_r(P_n)$, $(F_1(P) =_T P)$, and the logical constants $1, \ldots, m$ in its argument places, the functor $\Phi(\ ,\ .\ .\ .,\ )$ being used only once in the definiens.

For 2-valued logic the case $N = 2$, $r = 1$ has been considered in detail by Sobociński [1] and Foxley [2]. This has been extended to higher values of N by a number of authors (see, for example, [3], [4]). For m-valued logic, Rose [5] and Loader [6], [7], [8] have considered the case $N = 1$ with various values of r. In this paper we will consider second and higher order universal decision elements of the first degree and show that in 3-valued logic there exists an 8 input functor that is suitable. Throughout we will denote the m values of m-valued logic by the integers $1, \ldots, m$. The entry points in the truth table of the formula $\Phi(P_1, \ldots, P_n)$ will be numbered according to the rule:

$$\text{entry point number, } j = \sum_{k=1}^{n} (x_k - 1) \cdot m^{n-k} + 1 \quad ,$$

where x_k is the value taken by the variable P_k, $k = 1, \ldots, m$.

1 *Lower Bounds* If $\Phi(P_1, \ldots, P_n)$ corresponds to an $(N - 1)$'th order universal decision element for m-valued logic we may immediately deduce the formula $\Phi_1(P_1, \ldots, P_{mn+1})$ of $mn + 1$ argument places which corresponds to an N'th order universal decision element as follows:

$$\Phi_1(P_1, \ldots, P_{mn+1}) =_T [P_1, \Phi(P_2, \ldots, P_{n+1}), \ldots, \Phi(P_{n(m-1)+2}, \ldots, P_{mn+1}), P_1],$$

where $[\ ,\ .\ .\ .,\ ]$, the generalized conditioned disjunction functor, is such that $[P, Q_1, \ldots, Q_m, P]$ takes the truth value of Q_i when P takes the truth

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value i, $i = 1, \ldots, m$. Thus for the cases $m = 3$ and $m = 4$ where we know that first order universal decision elements of m argument places exist [7], we may deduce second order universal decision elements of 10 and 17 argument places respectively and hence third order ones of 31 and 69 argument places. For the second order case we may deduce a lower bound for the number of argument places required, using an argument similar to that used by Sobociński [1] for the 2-valued case. First we make the following definition:

A binary functor will be said to be trivial if it satisfies one of the following conditions:

(i) all rows or all columns of its corresponding truth table are identical;
(ii) if $\Lambda_1(P, Q) =_T \Lambda_2(Q, P)$ where $\Lambda_1(\ ,\ )$ and $\Lambda_2(\ ,\ )$ are distinct functors, then one of $\Lambda_1(\ ,\ )$, $\Lambda_2(\ ,\ )$ is said to be trivial.

From this definition it is easily shown that the number of non-trivial binary functors in m-valued logic is given by

$$\tfrac{1}{2} m^m (M^2 + M - 2) \quad , \quad \text{where } M = m^{m(m-1)/2} \quad .$$

For $m = 2$ and $m = 3$ the above expression gives the values 8 and 10179 respectively.

Theorem 1 *If* $\Phi(\ , \ldots,\ )$ *is a functor of* n *argument places of* m*-valued logic and if* n *is such that the inequality*

$$(m + 2)^n - 2(m + 1)^n + m^n - 3^n + 2^{n+1} - 1 < (m^{m^2} + m^{m(m+1)/2} - 2m^m)(m - 1)/m$$

holds then $\Phi(\ , \ldots,\ )$ *cannot correspond to a second order universal decision element.*

Proof: It is easily shown that the number of different ways in which the argument places of the functor $\Phi(\ , \ldots,\ )$ may be filled from the set $\{P, Q, 1, \ldots, m\}$ such that at least one P and one Q occur is given by $(m + 2)^n - 2(m + 1)^n + m^n$. Of these possible substitutions $3^n - 2^{n+1} + 1$ will use entry point 1 since these substitutions must be such that the argument places are filled from the set $\{P, Q, 1\}$. Thus the number of substitutions which do not use entry point 1 is given by

$$(m + 2)^n - 2(m + 1)^n + m^n - 3^n + 2^{n+1} - 1 \quad .$$

Now suppose that $\Phi(P_1, \ldots, P_n)$ takes the truth value k, $k \in \{1, \ldots, m\}$, when $P_1, \ldots, P_n$ all take the truth value 1. Then in order to define those binary functors which have the first entry in their corresponding truth table different from k, we must use only those substitutions which do not utilize entry point 1. Now the number of symmetrical truth tables where the first entry is not k is given by $(m^{m(m+1)/2} - m)(m - 1)/m$ and the number of remaining non-trivial binary functors where the first entry in the corresponding truth table is not k is given by

$$\tfrac{1}{2}(m^{m^2} - m^{m(m+1)/2} - 2m^m + 2m)(m - 1)/m \quad .$$

Thus to define each of these we must use a substitution where entry point 1 is not utilized. Further, there must exist two such substitutions since if the truth table is symmetric we must be able to interchange P and Q without upsetting the definition, and if the truth table is non-symmetric there will be a corresponding trivial functor which must be definable by interchanging P and Q. Thus the number of such substitutions required is given by

$$2(m^{m(m+1)/2} - m)(m - 1)/m + (m^{m^2} - m^{m(m+1)/2} - 2m^m + 2m)(m - 1)/m$$

i.e.,

$$(m^{m^2} + m^{m(m+1)/2} - 2m^m)(m - 1)/m \quad .$$

But from the above the number of substitutions available is $(m + 2)^n - 2(m + 1)^n + m^n - 3^n + 2^{n+1} - 1$ and hence if the inequality stated in the theorem holds, $\Phi(\ ,\ldots,\ )$ cannot correspond to a second order universal decision element.

The table below shows the minimum value of n where the inequality of Theorem 1 breaks down, for $m = 2, \ldots, 10$.

m	2	3	4	5	6	7	8	9	10
minimum n	4	7	13	21	31	44	58	75	93

A less complicated method of finding a lower bound for n, for any N, may be achieved by simply considering the total number of substitutions available and the total number of functors of N argument places to be defined (including the trivial ones). We first prove the following theorem.

Theorem 2 *If* $\Phi(\ ,\ldots,\ )$ *is a functor of* n *argument places then the number of ways of substituting in the argument places of* $\Phi(\ ,\ldots,\ )$ *from the set* $\{P_1, \ldots, P_N, 1, \ldots, m\}$ $(N < m)$ *such that each* P_i $(i = 1, \ldots, N)$ *occurs at least once is given by*

$$\rho(N) = \sum_{j=0}^{N} (-1)^j \binom{N}{j} (m + N - j)^n \quad .$$

Proof: The proof is by strong induction on N. If $N = 1$ then the number of substitutions is $(m + 1)^n - m^n$ and this may be expressed in the form

$$\sum_{j=0}^{1} (-1)^j \binom{1}{j} (m + 1 - j)^n = \rho(1) \quad .$$

Now suppose that the result holds for $N = 1, \ldots, r$ and consider the case $N = r + 1$. The substitution set is $\{P_1, \ldots, P_{r+1}, 1, \ldots, m\}$ and the total number of ways of substituting in the argument places of the functor $\Phi(\ ,\ldots,\ )$ without restriction is $(m + r + 1)^n$. Now the number of substitutions such that each P_i $(i = 1, \ldots, r + 1)$ occurs at least once may be expressed as the total number of substitutions without restriction less the number of substitutions such that exactly k of the P_i occur at least once for all $k = 0, \ldots, r$. On the induction hypothesis the number of substitutions with any k of the P_i occurring at least once is given by $\binom{r+1}{k}\rho(k)$. Thus the required number of substitutions is given by

$$(m+r+1)^n - \sum_{k=0}^{r}\binom{r+1}{k}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(m+k-j)^n \quad .$$

Rearranging the terms in the double summation this may be written

$$(m+r+1)^n + \sum_{j=1}^{r+1}(-1)^j\binom{r+1}{j}(m+r+1-j)^n$$

$$= \sum_{j=0}^{r+1}(-1)^j\binom{r+1}{j}(m+r+1-j)^n = \rho(r+1) \quad .$$

Hence the result is proved.

From Theorem 2 we may deduce that $\Phi(\ ,\ldots,\ )$ cannot correspond to an N'th order universal decision element for m-valued logic if

$$\rho(N) < m^{m^N} \quad .$$

The table below shows the minimum value of n where this inequality breaks down for $m = 2, \ldots, 10$; $N = 1, \ldots, 5$.

m \ N	1	2	3	4	5
2	2	3	5	8	12
3	3	7	17	46	129
4	4	13	46	171	647
5	5	21	97	458	2185
6	6	32	177	1009	5811
7	7	44	290	1949	13162
8	8	58	445	3428	26566
9	9	75	645	5621	49163
10	10	93	898	8726	85028

2 *The* 3-*Valued Case* From above we know that if $\Phi(P_1, \ldots, P_n)$ corresponds to a second order universal decision element for 3-valued logic then $n \geqslant 7$. The number of different ways in which the argument places of the formula $\Phi(P_1, \ldots, P_n)$ may be filled from the set $\{P, Q, 1, 2, 3\}$ such that at least one P and one Q occur is given by $5^n - 2.4^n + 3^n$, which gives the value 47544 when $n = 7$.

Loader [7] describes a general method for finding universal decision elements and this method was adopted starting with an arbitrary formula $\Phi(P_1, \ldots, P_7)$. Initially the number of binary functors undefined was found to be 2070. After considering 1281 entry points the formula $\Phi_1(P_1, \ldots, P_7)$ was found where the number of undefined binary functors was 1055. However, this was achieved at the expense of over 400 hours machine time using an IBM 1130 configuration. At this stage an attempt was made to find a formula of 8 argument places corresponding to a universal decision element. Initially the formula

$$\Lambda(P_1, \ldots, P_8) =_T [P_1, \Phi_1(P_2, \ldots, P_8), \sim\Phi_1(P_2, \ldots, P_8), \sim\sim\Phi_1(P_2, \ldots, P_8), P_1]$$

was considered where $\sim$ corresponds to the cyclic negation functor of Post [9]. The number of undefined binary functors was found to be 12 and proceeding with the method the formula

$$\Lambda_1(P_1, \ldots, P_8) =_T [P_1, \Phi_2(P_2, \ldots, P_8), \sim\Phi_2(P_2, \ldots, P_8), \sim\sim\Phi_2(P_2, \ldots, P_8), P_1]$$

corresponding to a second order universal decision element was found.

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