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INFINITE SERIES OF REGRESSIVE ISOLS UNDER ADDITION

JUDITH L. GERSTING

1 *Introduction* Let E denote the collection of all non-negative integers (numbers), Λ the collection of all isols, Λ_R the collection of all regressive isols, and Λ_{TR} the collection of T-regressive isols. (T-regressive isols were introduced in [4].) We recall the definition of an infinite series of isols, $\sum_T a_n$, where $T \in \Lambda_R - E$ and a_n: $E \to E$:

$$\sum\nolimits_T a_n = \text{Req} \sum_0^\infty j(t_n, \nu(a_n))$$

where $j(x, y)$: $E^2 \to E$ is a one-to-one recursive function, t_n is any regressive function ranging over a set in T, and for any number n, $\nu(n) = \{x \mid x < n\}$. Infinite series of isols were introduced by J. C. E. Dekker in [2], where it was shown that $\sum_T a_n \in \Lambda$. In [1] J. Barback studied infinite series of the form $\sum_T a_n$ where $T \leqslant^* a_{n-1}$. The relation $T \leqslant^* a_{n-1}$ means that for any regressive function t_n ranging over a set in T, the mapping $t_n \to a_{n-1}$ has a partial recursive extension. Professor Barback proved that for $T \leqslant^* a_{n-1}$, $\sum_T a_n \in \Lambda_R$. Because

$$a_n \text{ recursive} \Rightarrow T \leqslant^* a_n \Rightarrow T \leqslant^* a_{n-1}$$

but not conversely, there are several conditions of varying strength on the function a_n such that $\sum_T a_n \in \Lambda_R$. It is also known [5] that $T \leqslant^* a_{n-1}$ is not a necessary condition for $\sum_T a_n$ to be a regressive isol.

The following questions were posed by Professor Barback. Let $T \in \Lambda_R - E$ and let a_n, b_n: $E \to E$ be functions such that $\sum_T a_n$, $\sum_T b_n \in \Lambda_R$:

(1) Does $\sum_T a_n + \sum_T b_n \in \Lambda_R$?
(2) Does $\sum_T a_n + \sum_T b_n = \sum_T (a_n + b_n)$?

The present paper provides some partial answers to these questions.

2 *Some results* We will assume throughout that $T \in \Lambda_R - E$ and that a_n, b_n: $E \to E$ with $\sum_T a_n$, $\sum_T b_n \in \Lambda_R$.

Theorem 1 *Let α and β be disjoint recursive sets with $\alpha \cup \beta = E$ such that*

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$\mathrm{T} \leqslant^* a_n$ on α and $\mathrm{T} \leqslant^* b_n$ on β, that is, for any regressive function t_n ranging over a set in T, there exist partial recursive functions f_α and f_β such that

$$(\forall n)\,[n \in \alpha \Rightarrow t_n \in \delta f_\alpha \text{ and } f_\alpha(t_n) = a_n \text{ and } n \in \beta \Rightarrow t_n \in \delta f_\beta \text{ and } f_\beta(t_n) = b_n].$$

Then

$$\textstyle\sum_{\mathrm{T}} a_n + \sum_{\mathrm{T}} b_n = \sum_{\mathrm{T}} (a_n + b_n).$$

Proof: Suppose first that α and β are both infinite sets. Let r_n be the strictly increasing function ranging over α and let s_n be the strictly increasing function ranging over β. Define sets $\bar{\alpha}$ and $\bar{\beta}$ by

$$\begin{aligned}\bar{\alpha} &= \Big[\bigcup_i (j(t_{s(i)}, b_{s(i)}), \ldots, j(t_{s(i)}, b_{s(i)} + a_{s(i)} - 1))\Big] \\ &\quad \cup \Big[\bigcup_i (j(t_{r(i)}, 0), \ldots, j(t_{r(i)}, a_{r(i)} - 1))\Big] \\ \bar{\beta} &= \Big[\bigcup_i (j(t_{s(i)}, 0), \ldots, j(t_{s(i)}, b_{s(i)} - 1))\Big] \\ &\quad \cup \Big[\bigcup_i (j(t_{r(i)}, a_{r(i)}), \ldots, j(t_{r(i)}, a_{r(i)} + b_{r(i)} - 1))\Big].\end{aligned}$$

Then $\sum_{\mathrm{T}} a_n = \mathrm{Req}\ \bar{\alpha}$ and $\sum_{\mathrm{T}} b_n = \mathrm{Req}\ \bar{\beta}$. Also, $\bar{\alpha} | \bar{\beta}$ and $\bar{\alpha} \cup \bar{\beta} = \sum_0^\infty j(t_n, \nu(a_n + b_n)) \in \sum_{\mathrm{T}} (a_n + b_n)$. The argument is easily modified to take care of the case where one of sets α or β is finite (or even empty).

Corollary 1 *If $\mathrm{T} \leqslant^* a_n$, then $\sum_{\mathrm{T}} a_n + \sum_{\mathrm{T}} b_n = \sum_{\mathrm{T}} (a_n + b_n)$.*

Thus under the condition $\mathrm{T} \leqslant^* a_n$, the answer to Question (2) is affirmative. Keeping $\mathrm{T} \leqslant^* a_n$, we investigate several conditions on the function b_n which result in affirmative answers to Question (1) as well.

Lemma 1 *If $\mathrm{T} \leqslant^* a_n$ and $\mathrm{T} \leqslant^* b_{n-1}$, then $\sum_{\mathrm{T}} (a_n + b_n) \in \Lambda_{\mathrm{R}}$.*

Proof: $\mathrm{T} \leqslant^* a_n$ and $\mathrm{T} \leqslant^* b_{n-1}$ implies $\mathrm{T} \leqslant^* (a_{n-1} + b_{n-1})$ which means (by Proposition 5 of [1]) that $\sum_{\mathrm{T}} (a_n + b_n)$ is regressive.

Theorem 2 *For $\mathrm{T} \leqslant^* a_n$ and $\mathrm{T} \leqslant^* b_{n-1}$ (or $\mathrm{T} \leqslant^* b_n$ or b_n recursive), $\sum_{\mathrm{T}} a_n + \sum_{\mathrm{T}} b_n \in \Lambda_{\mathrm{R}}$.*

Lemma 2 *If $\mathrm{T} \leqslant^* a_n$ and for all n, $a_n, b_n \geqslant 1$, then $\sum_{\mathrm{T}} a_n + \sum_{\mathrm{T}} b_n \in \Lambda_{\mathrm{R}}$.*

Proof: Let t_n be a regressive function ranging over a set in T. Let f be a partial recursive function such that $\rho t_n \subset \delta f$ and $(\forall n)[f(t_n) = a_n]$. Let

$$\alpha = \sum_0^\infty j_3(t_n, \nu(a_n), 0)$$

$$\beta = \sum_0^\infty j_3(t_n, \nu(b_n), 1).$$

Then $\alpha \in \sum_{\mathrm{T}} a_n$ and $\beta \in \sum_{\mathrm{T}} b_n$ while $\alpha | \beta$. Hence $\mathrm{Req}(\alpha \cup \beta) = \sum_{\mathrm{T}} a_n + \sum_{\mathrm{T}} b_n$. By the assumption that $\sum_{\mathrm{T}} b_n \in \Lambda_{\mathrm{R}}$, β is a regressive set. Let $\beta = \rho s_n$, where s_n is a regressive function; let $p(x)$ be a regressing function for s_n. We define by induction a function r_n such that r_n is a regressive function and r_n ranges over $\alpha \cup \beta$.

Let $r_0 = s_0$. Let $n \geqslant 1$, and assume that $r_0, \ldots, r_{n-1}$ have been defined. For the definition of r_n, we consider the following two cases:

Case I. $r_{n-1} \in \alpha$, say $r_{n-1} = j_3(t_x, y, 0), 0 \leqslant y \leqslant a_x - 1$.

Subcase (i) $y \neq a_x - 1$. Set $r_n = j_3(t_x, y+1, 0)$.

Subcase (ii) $y = a_x - 1$. Set $r_n = s_z$ where $p(s_z) = j_3(t_x, 0, 1)$.

Case II. $r_{n-1} \in \beta$, say $r_{n-1} = j_3(t_x, y, 1), 0 \leqslant y \leqslant b_x - 1$.

Subcase (i) $y \neq 0$. Set $r_n = s_z$ where $p(s_z) = r_{n-1}$.

Subcase (ii) $y = 0$. Set $r_n = j_3(t_x, 0, 0)$.

This completes the definition of r_n. It can be seen that r_n ranges over $\alpha \cup \beta$. Further, consider the function q_n defined on ρr_n by

$$q_n(r_n) = \begin{cases} j_3(t_x, y-1, 0) & \text{for } r_n = j_3(t_x, y, 0),\ y \neq 0 \\ j_3(t_x, 0, 1) & \text{for } r_n = j_3(t_x, 0, 0) \\ p(r_n) & \text{for } r_n = j_3(t_x, y, 1),\ k_{32}p(r_n) \neq 0 \\ j_3(k_{31}p(r_n), fk_{31}p(r_n) - 1, 0) & \text{for } r_n = j_3(t_x, y, 1),\ k_{32}p(r_n) = 0 \end{cases}$$

Then q has a partial recursive extension, say q^*, and $q^*(r_n) = r_{n-1}$. Therefore r_n is a regressive function, $\alpha \cup \beta$ is a regressive set, and $\sum_T a_n + \sum_T b_n \in \Lambda_R$.

Lemma 3 *If* $T \leqslant^* a_n$ *and for all* n, $b_n \geqslant 1$, *then* $\sum_T a_n + \sum_T b_n \in \Lambda_R$.

Proof: Let $\sum_T a_n + \sum_T b_n = A$. Then

$$\begin{aligned} & \textstyle\sum_T a_n + \sum_T b_n + T = A + T \\ \Rightarrow\ & \textstyle\sum_T a_n + \sum_T b_n + \sum_T 1 = A + T && (\text{since } T = \textstyle\sum_T 1) \\ \Rightarrow\ & \textstyle\sum_T (a_n + 1) + \sum_T b_n = A + T && (\text{by Corollary 1, since } T \leqslant^* a_n) \\ \Rightarrow\ & A + T \in \Lambda_R && (\text{by Lemma 2, since } T \leqslant^* a_n + 1) \end{aligned}$$

Because $A \leqslant A + T$, it follows that $A \in \Lambda_R$.

Actually, the argument of Lemma 2 can easily be modified to take care of the possibility of the function a_n having zero values, but this does not seem as elegant an approach as the proof of Lemma 3!

Theorem 3 *If* $T \leqslant^* a_n$ *and there exists a number* m *such that for* $n \geqslant m$, $b_n \geqslant 1$, *then* $\sum_T a_n + \sum_T b_n \in \Lambda_R$.

Proof: $\sum_T a_n + \sum_T b_n$

$$\begin{aligned} &= (a_0 + \ldots + a_{m-1}) + \textstyle\sum_{T-m} a_{n+m} + (b_0 + \ldots + b_{m-1}) + \sum_{T-m} b_{n+m} \\ &= k + \textstyle\sum_{T-m} a_{n+m} + \sum_{T-m} b_{n+m} \end{aligned}$$

where $k \in E$. By the assumption that $\sum_T a_n, \sum_T b_n \in \Lambda_R$, it follows that $\sum_{T-m} a_{n+m}, \sum_{T-m} b_{n+m} \in \Lambda_R$. Also, since $T \leqslant^* a_n$, we have that $T - m \leqslant^* a_{n+m}$. Thus, by Lemma 3, $\sum_{T-m} a_{n+m} + \sum_{T-m} b_{n+m} \in \Lambda_R$ and hence $\sum_T a_n + \sum_T b_n \in \Lambda_R$.

Remark: Theorem 3 of [5] provides an example of an infinite regressive isol T and a function b_n with $b_n \geqslant 1$ for all n, $\sum_T b_n \in \Lambda_R$, and $T \not\leqslant^* b_{n-1}$. We can use Theorem 3 above to generate a whole class of such examples from this one. Let a_n be any function with $\mathrm{T} \leqslant^* a_n$, and let c_n be the function defined by $c_n = a_n + b_n$. Then $\sum_{\mathrm{T}} a_n + \sum_{\mathrm{T}} b_n = \sum_{\mathrm{T}} c_n \in \Lambda_{\mathrm{R}}$ but $\mathrm{T} \not\leqslant^* c_{n-1}$.

What happens in the case of b_n functions that do not fit Theorems 2 or 3 above, that is, $\mathrm{T} \not\leqslant^* b_{n-1}$ and $b_n = 0$ at infinitely many places? The following Lemma, due to Professor M. Hassett, shows that such functions do exist.

Lemma 4 (Hassett) *Let* $\mathrm{T} \in \Lambda_{\mathrm{R}} - E$. *Then there exists a function* $b_n: E \to E$ *such that for all* n, $0 \leqslant b_n \leqslant 1$, $b_n = 0$ *at infinitely many values of* n, $\sum_{\mathrm{T}} b_n \in \Lambda_{\mathrm{R}}$, *and* $\mathrm{T} \not\leqslant^* b_{n-1}$.

Proof: Let t_n be a retraceable function ranging over a set in T. Let a_n be any retraceable function such that $a_0 > 0$ and $\rho t(a_n)$ is not a separated subset of ρt_n. This is possible because there are c retraceable functions, hence c subsets of the form $\rho t(a_n)$, but ρt_n has only $\aleph_0$ separated subsets. Let $\alpha = \rho t(a_n)$. We define a function b_n by

$$b_n = \begin{cases} 0 \text{ if } t_{n+1} \notin \alpha \\ 1 \text{ if } t_{n+1} \in \alpha. \end{cases}$$

Then $0 \leqslant b_n \leqslant 1$ for all n. Also $b_n = 0$ at infinitely many values of n, because if $b_n = 1$ from some point on, then α would be a separated subset of ρt_n.

The function $t(a_n - 1)$ is the composition of two retraceable functions, hence is retraceable, and $\mathrm{Req}\ \rho t(a_n - 1) = \mathrm{Req} \sum_0^\infty j(t_{a_n-1}, 0) = \sum_{\mathrm{T}} b_n$. Thus $\sum_{\mathrm{T}} b_n \in \Lambda_{\mathrm{R}}$. Finally, if $\mathrm{T} \leqslant^* b_{n-1}$, then given t_n we could compute b_{n-1} and hence decide whether or not $t_n \in \alpha$. This would contradict the fact that α is not a separated subset of ρt_n.

Lemma 5 *Let* $\mathrm{T} \in \Lambda_{\mathrm{TR}}$. *Let* $c_n: E \to E$ *be such that there exists a number* M *with* $1 \leqslant c_n \leqslant M$ *for all* n. *If* $\sum_{\mathrm{T}} c_n \in \Lambda_{\mathrm{R}}$, *then* $\mathrm{T} \leqslant^* c_{n-1}$.

Proof: Let t_n be a T-retraceable function ranging over a set in T. Let $\sigma = \sum_0^\infty j(t_n, \nu(c_n))$, and let $\sum_{\mathrm{T}} c_n \in \Lambda_{\mathrm{R}}$. Then σ is an infinite regressive set. Let $\sigma = \rho s_n$ where s_n is a regressive function and let $p(x)$ be a regressing function for s_n. For $0 \leqslant i \leqslant M - 1$, we define functions $q_i(x)$ by $q_i(x) = pj(x, i)$. Then each q_i is a partial recursive function. Because t_n is a T-retraceable function, it follows that for each $q_i(x)$, $0 \leqslant i \leqslant M - 1$, there exists a number m_i such that for $n \geqslant m_i$, $q_i(t_n) < t_{n+1}$. Let $m = \max_{0 \leqslant i \leqslant M-1} m_i$, and consider the finite set

$$j(t_0, 0), \ldots, j(t_0, c_0 - 1), j(t_1, 0), \ldots, j(t_1, c_1 - 1), \ldots, j(t_m, 0), \ldots, j(t_m, c_m - 1)\ .$$

Let q be the maximum index of s_n represented in this set, and consider the finite set

$$k(s_q),\ k(s_{q-1}),\ \ldots,\ k(s_0).$$

Let k be the maximum index of t_n occurring in this set. We can now describe an effective procedure for computing c_{n-1} from t_n for $n \geqslant k + 1$. Thus, assume $n \geqslant k + 1$. Then it follows that $j(t_n, 0) = s_r$ with $r > q$.

Suppose that a term of the form $j(t_{n-1}, y)$ with $0 \leqslant y < c_{n-1} \leqslant M$ has an index in s of r_1 with $r_1 > r$, say $r_1 = r + b$, $b \geqslant 1$. Then

$$s_{r_1 - 1} = p(s_{r_1}) = pj(t_{n-1}, y) = q_y(t_{n-1}) < t_n$$

since $n - 1 \geqslant k \geqslant m \geqslant m_y$. The term s_{r_1-1} has the following properties:

(i) $s_{r_1-1} = j(t_p, y_p)$ with $0 \leqslant y_p \leqslant c_p - 1$
(ii) $p \leqslant n - 1$.

Property (i) follows from the definition of the s_n function. For (ii), note that $t_p \leqslant j(t_p, y_p) = s_{r_1-1} < t_n$ and since t_n is a strictly increasing function, $p < n$. Also, $r_1 - 1 \geqslant r > q$ so that $p > m$. Therefore this argument may be repeated on the term s_{r_1-2}, etc. The result is that each term below s_{r_1} in the ordering s_n has for the t-index of its first component a number $\leqslant n - 1$. After b times, however, $j(t_n, 0)$ is reached and a contradiction is obtained. Hence every term of the form $j(t_{n-1}, y)$, $0 \leqslant y < c_{n-1}$, has an index in s which is less than r.

Now let t_n be given, with $n \geqslant k + 1$. We may then compute the index n and the term $j(t_n, 0) = s_r$. We can then effectively generate the list

$$s_r, s_{r-1}, \ldots, s_0$$

and compute the t-indices of all the first components of this list. The number of t-indices with value $n - 1$ is equal to c_{n-1}. We can easily patch up the finite number of points with index below $k + 1$ and thus conclude that $\mathrm{T} \leqslant^* c_{n-1}$.

Combining Lemmas 4 and 5, we will see that even with a very strong condition on the function a_n, namely a_n equal to the constant function 1, we can produce a case where the answer to Question (1) is negative.

Theorem 4 *There exists* $\mathrm{T} \in \Lambda_{\mathrm{R}} - E$ *and functions* a_n, b_n: $E \to E$ *such that* $\sum_{\mathrm{T}} a_n, \sum_{\mathrm{T}} b_n \in \Lambda_{\mathrm{R}}$ *but* $\sum_{\mathrm{T}} a_n + \sum_{\mathrm{T}} b_n \notin \Lambda_{\mathrm{R}}$.

Proof: Let $\mathrm{T} \in \Lambda_{\mathrm{TR}}$ and let b_n be the function guaranteed by Lemma 4. Then $0 \leqslant b_n \leqslant 1$ for all n, $\sum_{\mathrm{T}} b_n \in \Lambda_{\mathrm{R}}$ and $\mathrm{T} \not\leqslant^* b_{n-1}$. Let a_n be the constant function 1. By Corollary 1,

$$\textstyle\sum_{\mathrm{T}} a_n + \sum_{\mathrm{T}} b_n = \sum_{\mathrm{T}} (a_n + b_n) = \sum_{\mathrm{T}} (1 + b_n).$$

Now $1 \leqslant 1 + b_n \leqslant 2$ and $\mathrm{T} \not\leqslant^* (1 + b_{n-1})$, so by Lemma 5, $\sum_{\mathrm{T}} (1 + b_n) \notin \Lambda_{\mathrm{R}}$.

Remark: Theorem 4 above provides still another example of the non-closure of Λ_{R} under addition.

3 *An open question* For $\mathrm{T} \in \Lambda_{\mathrm{R}} - E$, we know that $\mathrm{T} \leqslant^* a_{n-1}$ and $\mathrm{T} \leqslant^* b_{n-1}$ implies $\sum_{\mathrm{T}} a_n, \sum_{\mathrm{T}} b_n \in \Lambda_{\mathrm{R}}$. This is certainly an obvious way to pursue Questions (1) and (2). Under these conditions we of course have $\mathrm{T} \leqslant^* (a_{n-1} + b_{n-1})$ so that $\sum_{\mathrm{T}} (a_n + b_n) \in \Lambda_{\mathrm{R}}$, and an affirmative answer to Question (2) for this case is mentioned by Barback in Lemma 3 of [1]. However, for $\mathrm{T} \leqslant^* a_{n-1}$, $\mathrm{T} \leqslant^* b_{n-1}$, it remains an open question whether $\sum_{\mathrm{T}} a_n + \sum_{\mathrm{T}} b_n = \sum_{\mathrm{T}} (a_n + b_n)$ or even whether $\sum_{\mathrm{T}} a_n + \sum_{\mathrm{T}} b_n$ is regressive at all.

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Indiana University-Purdue University at Indianapolis
Indianapolis, Indiana