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# TWO NOTES ON RECURSIVELY ENUMERABLE VECTOR SPACES

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1 *A Characterization of Recursive Spaces* We retain the terminology of [4] of which this note is a continuation.

Definition Let $\sigma_0 = \sigma - (0)$. Let $n \subset \overline{U}_0$. Define $\mathsf{m}(n)$ to be the (largest prime divisor of $n + 1$).

Obviously:

(i) $\mathsf{m}(n)$ *is a partial recursive function,*
(ii) $\mathsf{m}(V_0)$ *infinite iff* $\dim V$ *infinite,*
(iii) $\mathsf{m}$ *respects inclusion on sets,*
(iv) $\mathsf{m}$ *maps (proper) subspaces to (proper) subsets,*
(v) *for* $\beta \subset \overline{U}_0$, $\mathsf{m}(x)$ 1 - 1 *on* $\beta$ *implies* $\beta$ *is a repère.*

Definition Let $V$ be a subspace of $\overline{U}$. $\gamma$ is a cobasis for $V$ if $\gamma$ is a basis for a complementary space for $V$. $\eta_V$ is the canonical cobasis for $V$ if $\eta_V$ is a cobasis for $V$ and $\eta_V \subset \eta$.

Remark The canonical cobasis for a space $V$ is defined to be the set $\gamma$ such that $\gamma = (e_i \text{ in } \eta | e_i \text{ is not in } (\mathsf{L}(e_j)\ j < i) + V)$.

Proposition F *The canonical cobasis for a recursive space is recursive.*

*Proof:* Let $f(i)$ list the canonical cobasis in increasing order. If $f$ is a finite function, then its range is recursive. Otherwise $f$ is a recursive function by the Corollary to Proposition C.

Proposition G *For any space* $V$, $e_n \in \mathsf{m}(V_0)$ *iff* $e_n$ *is not in the canonical cobasis for* $V$.

*Proof:* It suffices to show that

(19) $e_n \in \mathsf{m}(V_0)$ iff $e_n \in (\mathsf{L}(e_i)\ i < n) + V)$.

Assume the left hand side. Then there is an element $e$ in $V$ such that:

$$e = r_0 e_0 + \ldots + r_n e_n, \text{ where } r_n \neq 0.$$

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So $e_n = -1/r_n(r_0e_0 + \ldots + r_{n-1}e_{n-1}) + 1/r_n(e)$. So the right hand side holds. Assume the right hand side. Suppose $e_n = (r_0e_0 + \ldots + r_{n-1}e_{n-1}) + s_0e_0 + \ldots + s_ne_n + \ldots + s_ke_k$, where $s_0e_0 + \ldots + s_ke_k$ is in $V$. Then: $r_0 + s_0 = 0, \ldots, r_{n-1} + s_{n-1} = 0$, $s_n = 1$, $s_{n+1} = 0, \ldots, s_k = 0$. So $e_n \in \mathrm{m}(V_0)$ and the left hand side holds.

Proposition H *If $V$ is any space, $\eta$ - $\mathrm{m}(V_0)$ is a cobasis for $V$.*

*Proof:* It is the canonical cobasis for $V$ by Proposition G.

We now characterize recursive spaces by their images under m.

Proposition I *Let $V$ be a* r.e. *space. Then $\mathrm{m}(V_0)$ is* r.e. *and*

(20) *$V$ recursive space iff $\mathrm{m}(V_0)$ recursive set.*

*Proof:* Suppose $V$ is r.e. Then $\mathrm{m}(V_0)$ is r.e. as a consequence of the definition of $\mathrm{m}(x)$. Retaining the assumption that $V$ is r.e. throughout, we have the following chain of reversible implications: $\mathrm{m}(V_0)$ recursive iff $\eta$ - $\mathrm{m}(V_0)$ is recursive iff the canonical cobasis for $V$ is recursive iff $V$ is recursive.

We now refine our characterization by investigating the effect of m on repères.

Proposition J *Let $V$ be any space and $\beta$ any basis of $V$. If m is 1 - 1 on $\beta$, $\mathrm{m}(\beta) = \mathrm{m}(V_0)$.*

*Proof:* $\mathrm{m}(\beta) \subset \mathrm{m}(V_0)$ by (i). Suppose $e_n \in \mathrm{m}(V_0)$. Then there is an element $v \in V$ such that $\mathrm{m}(v) = e_n$ and

(21) $$v = s_0b_0 + \ldots + s_kb_k \text{ where no } s_i \text{ is } 0,$$

and the $b_i$ are all distinct elements of $\beta$. Since $\mathrm{m}(x)$ is 1 - 1 on $\beta$, $\mathrm{m}(v) = \mathrm{m}(b_0)$ or ... or $\mathrm{m}(b_k)$, so $e_n = \mathrm{m}(b_0)$ or ... or $\mathrm{m}(b_k)$.

Corollary K *If $V$ is a* r.e. *space and $\gamma$ is a generating set for $V$ on which $\mathrm{m}(x)$ is 1 - 1, then $V$ is a recursive space iff $\mathrm{m}(\gamma_0)$ is a recursive set.*

*Proof:* $\gamma_0$ is a repère in $V$ so by Proposition J, $\mathrm{m}(\gamma_0) = \mathrm{m}(V_0)$. Now apply Proposition I.

**2** *Another Type of* r.e. *Vector Space* We assume some familiarity with the content of [3].

Definition Let $S$ be a non-empty md (mutually disjoint) class of r.e. sets. Then $S$ is called r.e. if there is a recursive function $g_n(x)$ such that $S = (\rho g_0, \rho g_1, \ldots)$.

Definition Let $V$ be a subspace of $\overline{U}$. Then $(\overline{U}/V)$ is the md-class of cosets of $V$ in $\overline{U}$. If this class is considered as a vector space over $F$ it is assumed that addition and scalar multiplication are defined in the usual way.

Definition Let $V$ be a subspace of $\overline{U}$ and $\gamma$ a choice set of the md-class

$(\overline{U}/V)$ such that $0 \in \gamma$. Let $c(x)$ be the choice set of $(\overline{U}/V)$ associated with $\gamma$. Then $C = (\gamma, +, \cdot)$ is the vector space over $F$ determined by $\gamma$ and $V$, where

$$x + y = c(x + y)\ x,\ y \in \gamma,$$
$$r \cdot x = c(r \cdot x) \text{ for } x \in \gamma,\ r \in F.$$

The following proposition depends, for its proof, on the fact that $\overline{c}(x + V) = c(x)$ is an isomorphism of vector spaces taking $(\overline{U}/V)$ onto $C$; whence the properties of $(\overline{U}/V)$ are transferred to $C$. The assertions are well known properties of the quotient space.

Proposition L *Let $V$ be a subspace of $\overline{U}$, $0 \in \gamma$ a choice set for $(\overline{U}/V)$, $c(x)$ the choice function associated with $\gamma$, and $C$ the associated vector space. Then:*

(a) *if $\beta$ is a repère in $\overline{U}$ whose span is disjoint from $V$, then $c(\beta)$ is a repère in $C$,*
(b) *if $\gamma$ is a repère in $C$ then $\gamma$ is a repère in $\overline{U}$ and its span is disjoint from $V$,*
(c) *if $\beta_0$ is a cobasis for $V$ in $\overline{U}$ then $c(\beta_0)$ is a basis for $C$,*
(d) *if $\gamma$ is a basis for $C$ then it is a cobasis for $V$ in $\overline{U}$.*

Proposition M *If $V$ is a* r.e. *space, $(\overline{U}/V)$ is a* r.e. *class of* r.e. *sets.*

*Proof:* Let $a(n)$ and $v(x)$ be recursive functions ranging over $\varepsilon_F$ and $V$ respectively. Put

$$g_n(x) = a(n) + v(x), \text{ for } n, x \in \varepsilon.$$

Then $g_n$ is a recursive function of two variables such that $(\overline{U}/V) = (\rho g_0, \rho g_1, \ldots)$, and the proposition is proved.

Proposition N *For a* r.e. *space $V$, $V$ decidable iff the* md*-class $(\overline{U}/V)$ has a partial recursive choice set.* (A space $V$ is a gc-subspace of $\overline{U}$ by definition if this condition is satisfied.)

*Proof:* Assume $V$ decidable. Since every finite md-class of non-empty r.e. sets has a r.e. choice set, we may assume that $V$ has infinitely many cosets in $\overline{U}$. Put $c_n = x(x \in \varepsilon_F \ \&\ (i < n \rightarrow x - c_i \in V)$. Then $c(x)$ is a strictly increasing function, and it is recursive since $V$ is decidable and so a recursive set. So $V$ is a gc-subset of $\overline{U}$. Assume $V$ a gc-subspace of $\overline{U}$. Let $0 \in \gamma$ be a recursive choice set for $(\overline{U}/V)$ with associated choice function $c(x)$. Then $c^{-1}(0) = V$ and $c^{-1}(\gamma - 0) = \overline{U} - V$ are r.e. sets. So $V$ is decidable.

Definition Let $F$ satisfy our requirements for a field. A r.e. space over $F$ is an ordered triple $(\rho, +, \cdot) = R$ such that

(a) $R$ is a vector space over $F$,
(b) $\rho$ is a r.e. set,
(c) $0 \in \rho$ and 0 is the zero-element of $R$,
(d) the following functions are partial recursive:

$$f(x, y) = x + y, \text{ for } x, y \in \rho,$$
$$g(n, x) = \phi^{-1}(n) \cdot x \text{ for } n \in \phi(F), x \in \rho.$$

Proposition O *Let $V$ be a decidable space, $\gamma$ a* r.e. *choice set of $(\overline{U}/V)$ with $0 \in \gamma$ and $C$ the associated vector space. Then:*

(a) *$C$ is a* r.e. *space,*
(b) *$C$ has a* r.e. *basis iff $V$ is a recursive space.*

*Proof:* (a) $C$ is obviously a r.e. space (in the sense of the immediately preceding definition).
(b) Use Proposition L. Let $V$ be recursive and let $\delta$ be a r.e. cobasis for $V$. Then by part (a) of Proposition L, $c(\delta)$ is a r.e. basis for $C$. So $C$ has a r.e. basis. On the other hand, if $C$ has a r.e. basis $\delta$, then by part (d) of Proposition L, $\delta$ is a cobasis for $V$ in $\overline{U}$. So $V$ is a recursive space.

We believe that the definition we have given is the intuitively obvious definition of r.e. space. We now present the following theorem.

Proposition P *There exists a* r.e. *space (in the sense of our definition) which has no* r.e. *basis.*

*Proof:* By Proposition E, there exists a decidable space that is not recursive. Choosing such a space for our $V$ in Proposition O, we have as associated vector space $C$, a space that is r.e. by part (a) and that has no r.e. basis by part (b).

The intuitive content of Proposition P is that linear independence is not a recursive property, since all we have changed in our definition is the relation of linear independence.

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