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A TABLEAU SYSTEM FOR PROPOSITIONAL S5

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We present an amusing semantic tableau system for propositional S5 which is actually quite efficient in practice. We assume the reader is familiar with the propositional tableau system using unsigned formulas as presented in [4], Ch. II. We continue the α, β classification of formulas, and add two new categories, necessaries (ν) and possibles (π). These, together with their respective components ν_0 and π_0 are defined by the following tables:

ν	ν_0
$\Box X$	X
$\sim\Diamond X$	$\sim X$

π	π_0
$\Diamond X$	X
$\sim\Box X$	$\sim X$

We begin with a tableau system for propositional S4. To the α and β rules of [4] we add the following two rules:

Rule ν: $\dfrac{\nu}{\nu_0}$

(i.e., if a ν formula occurs on a branch, ν_0 may be added to the end of the branch).

Rule π: $\dfrac{\pi}{\pi_0}$ proviso

(interpreted similarly) where the proviso reads: before adding π_0 to the end of a branch on which π occurs, *cross out all formulas on that branch which are not ν formulas*. (Note: a given occurrence of a non-ν formula X may be common to several branches, and it may be desired to cross it out on only one branch. If this happens, simply add fresh occurrences of X to the ends of all branches on which it should remain undeleted.) Now a branch is called closed if it contains X and $\sim X$, both un-crossed out. The above system is propositional S4. There is a completeness proof for the corresponding first order system in [2]. To modify the above into a propositional S5 system we add one more rule:

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Rule S: If a formula X occurs crossed out on a branch, a new, un-crossed out occurrence may be added to the end of the branch, but with the same proviso as above, namely: first cross out all formulas on the branch that are not ν formulas.

For example, the following is a closed tableau proving $(\Diamond P \wedge Q) \supset \Diamond(P \wedge \Diamond Q)$. For printing ease, instead of crossing formulas out, we have placed asterisks after them.

$$\sim[(\Diamond P \wedge Q) \supset \Diamond(P \wedge \Diamond Q)]^* \qquad (1)$$
$$(\Diamond P \wedge Q)^* \qquad (2)$$
$$\sim\Diamond(P \wedge \Diamond Q) \qquad (3)$$
$$\Diamond P^* \qquad (4)$$
$$Q^* \qquad (5)$$
$$P^* \qquad (6)$$
$$\sim(P \wedge \Diamond Q)^* \qquad (7)$$

$\sim P$	(8)	$\sim\Diamond Q$	(9)
closure		Q	(10)
		$\sim Q$	(11)
		closure	

where the reasons for steps are as follows: 2 and 3 are from 1 by α; 4 and 5 are from 2 by α; 6 is from 4 by π (at this point 1, 2, 4, and 5 are crossed out); 7 is from 3 by ν; 8 and 9 are from 7 by β (at this point the left branch is closed); 10 is from 5 by rule S (at this point 6 and 7 are also crossed out); 11 is from 9 by ν (at this point the right branch also closes).

The correctness of this system may be shown by standard tableau methods as follows. Call a set of formulas *satisfiable* if there is a possible world in a Kripke propositional S5 model [3] in which every member of the set is true. It is easy to show: if the origin of a tableau is satisfiable, then at each stage of the tableau construction the set of (un-crossed out) formulas on at least one branch is satisfiable. Now, if X is provable, there is a closed tableau with $\sim X$ at the origin, $\sim X$ cannot be satisfiable because if it were the set of formulas on one of the closed branches of the finished tableau would also be satisfiable, which is not possible. Since $\sim X$ is not satisfiable, X is true in all possible worlds of all Kripke S5 models.

Completeness follows easily from the completeness of the S4 tableau system together with the result ([1], page 177) that X is a propositional S5 theorem iff $\Diamond\Box X$ is a propositional S4 theorem (a result easily established by induction on the lengths of axiom system proofs). Then if X is valid in all Kripke S5 models, $\Diamond\Box X$ is provable in the S4 tableau system given above. It is easy to convert this into a proof of X in the S5 system. In fact, we get the stronger result that Rule S need only be applied to the formula at the origin of the tableau.

Open problem: If γ and δ rules from [4] are added, is the resulting tableau system complete for first order S5? The corresponding first order S4 system is complete [2], but the above S5 completeness proof based on the reduction to S4 does not extend.

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