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SIMPLIFIED FORMALIZATIONS OF FRAGMENTS OF THE PROPOSITIONAL CALCULUS

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Henkin has given [1] a general method of formalizing 2-valued propositional calculi whose primitive functors are such that material implication is definable in terms of them. Let the primitive functors, other than implication if implication is a primitive functor, be the functors F_i of n_i arguments $(i = 1, \ldots, b)$ and let the formulae $P_1, \ldots, P_{n_i}, F_iP_1 \ldots P_{n_i}$ take the truth-values $x_1, \ldots, x_{n_i}, f_i(x_1, \ldots, x_{n_i})$ respectively $(i = 1, \ldots, b)$. The axiom schemes are

A1 $CPCQP$,
A2 $CCPQCCPCQRCPR$,
A3 $CCPRCCCPQRR$,
A4 $CV_{x_1}P_1Q \ldots CV_{x_{n_i}}P_{n_i}QV_yF_iP_1 \ldots P_{n_i}Q(y = f_i(x_1, \ldots, x_{n_i});$
$x_1 = \mathbf{T}, \mathbf{F}; \ldots; x_{n_i} = \mathbf{T}, \mathbf{F}; i = 1, \ldots, b)$,

A4 denoting $\sum_{i=1}^{b} 2^{n_i}$ axiom schemes and the functors $V_{\mathbf{T}}$, $V_{\mathbf{F}}$ being defined by the equations

$$V_{\mathbf{T}}PQ =_{df} CCPQQ,$$
$$V_{\mathbf{F}}PQ =_{df} CPQ.$$

The only primitive rule of procedure is

R1 *If P and CPQ then Q.*

We shall show how to reduce[1] the number and lengths of the axiom schemes.

It follows at once from a result of Łukasiewicz [3] that *A1-3* may be replaced by the axiom scheme

B1 $CCCPQRCCRPCSP$.

1. The axiom schemes C are similar to those obtained by using a general method of Shoesmith [5], but his completeness proof is non-constructive.

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Since the axiom schemes *A4* are used in Henkin's completeness proof only to establish the hypothetical deductions

$$V_{x_1}P_1Q, \ldots, V_{x_{n_i}}P_{n_i}Q \vdash V_yF_iP_1 \ldots P_{n_i}Q$$

it follows at once that if there exist integers $\alpha_{i1}, \ldots, \alpha_{ik_i}$ of the set $\{1, \ldots, n_i\}$ and truth-values $x'_{\alpha_{i1}}, \ldots, x'_{\alpha_{ik_i}}$ such that $f_i(x_1, \ldots, x_{n_i})$ has the constant value y' in all the $2^{n_i-k_i}$ cases where $x_{\alpha_{i1}} = x'_{\alpha_{i1}}, \ldots, x'_{\alpha_{ik_i}} = x_{\alpha_{ik_i}}$ then the corresponding $2^{n_i-k_i}$ of the axiom schemes *A4* may be replaced by the single axiom scheme

$$B2 \quad CV_{x'_{\alpha_{i1}}}P_{\alpha_{i1}}Q \ldots CV_{x'_{\alpha_{ik_i}}}P_{\alpha_{ik_i}}QV_{y'}F_iP_1 \ldots P_{n_i}Q$$

(for any of the $k_i!$ ways of assigning values to $\alpha_{i1}, \ldots, \alpha_{ik_i}$). Thus we may replace *A1-A4* by *B1* and all the[2] axiom schemes *B2*.

We may assume, without real loss of generality, that

$$f_i(\mathbf{T}, \ldots, \mathbf{T}) = \mathbf{T}(i = 1, \ldots, b)$$

since, if this is not so, we may, for some integer i ($1 \leqslant i \leqslant b$), make the definition

$$NP =_{df} F_iP \ldots P$$

if f_i $(\mathbf{F}, \ldots, \mathbf{F}) = \mathbf{T}$ or the definition

$$NP =_{df} CPF_iP \ldots P$$

if $f_i(\mathbf{F}, \ldots, \mathbf{F}) = \mathbf{F}$. Functional completeness of the propositional calculus would then follow at once, making the use of the method of Henkin, rather than that of Kalmár [2], unnecessary. One or more of the axiom schemes *B2* will then be of the form

$$B2A \quad CV_{\mathbf{T}}P_{\alpha_{i1}}Q \ldots CV_{\mathbf{T}}P_{\alpha_{ik_i}}QV_{\mathbf{T}}F_iP_1 \ldots P_{n_i}Q$$

and, for the remaining axiom schemes, $x'_{\alpha_{i1}}, \ldots, x'_{\alpha_{ik_i}}$ will not all be $\mathbf{T}$. Thus they will be of one of the forms

$$B2B \quad CV_{x'_{\alpha_{i1}}}P_{\alpha_{i1}}Q \ldots CV_{x'_{\alpha_{ik_i}}}P_{\alpha_{ik_i}}QV_{\mathbf{T}}F_iP_1 \ldots P_{n_i}Q,$$

$$B2C \quad CV_{x'_{\alpha_{i1}}}P_{\alpha_{i1}}Q \ldots CV_{x'_{\alpha_{ik_i}}}P_{\alpha_{ik_i}}QV_{\mathbf{F}}F_iP_1 \ldots P_{n_i}Q$$

and we may assign values to $\alpha_{i1}, \ldots, \alpha_{ik_i}$ in such a way that, for some integer j ($1 \leqslant j \leqslant \alpha_{k_i}$),

$$x'_{\alpha_{i1}}, \ldots, x'_{\alpha_{ij}} = \mathbf{F};$$
$$x'_{\alpha_{i,j+1}}, \ldots, x'_{\alpha_{ik_i}} = \mathbf{T}.$$

We shall show that the axiom schemes *B2* may be replaced by the axiom schemes

2. Two or more axiom schemes *B2* may replace two or more overlapping groups of some of the axiom schemes *A4*.

C2A $CP_{\alpha_{i1}} \ldots CP_{\alpha_{ik_i}} F_i P_1 \ldots P_{n_i}$,

C2B $C^{2j} CP_{\alpha_{i,j+1}} \ldots CP_{\alpha_{i,k_i}} F_i P_1 \ldots P_{n_i} P_{\alpha_{i1}} P_{\alpha_{i1}} \ldots P_{\alpha_{ij}} P_{\alpha_{ij}}$,

C2C $C^{2j-2} CP_{\alpha_{i,j+1}} \ldots CP_{\alpha_{i,k_i}} CF_i P_1 \ldots P_{n_i} P_{\alpha_{i1}} P_{\alpha_{i2}} P_{\alpha_{i2}} \ldots P_{\alpha_{ij}} P_{\alpha_{ij}}$.

It will be sufficient to establish that *C2A* (*C2B*, *C2C*) follows from *B1*, *B2A* (*B2B*, *B2C*) and *R1*. We shall sometimes abbreviate formulae of the form *CCPQQ* by *APQ*.

By *B1* and *R1*

$$\vdash CCP_1 \ldots CP_n RCAP_1 Q \ldots CAP_n QARQ \; (n = 1, 2, \ldots).$$

Thus, by *R1*,

$$CP_1 \ldots CP_n R \vdash CAP_1 Q \ldots CAP_n QARQ \; (n = 1, 2, \ldots)$$

and *B2A* then follows from *C2A*.

By *B1* and *R1*

$$C^{2m} CP_1 \ldots CP_n RQ_1 Q_1 \ldots Q_m Q_m \vdash CV_{\mathsf{F}} Q_1 S \ldots CV_{\mathsf{F}} Q_m SCV_{\mathsf{T}} P_1 S \ldots CV_{\mathsf{T}} P_n SV_{\mathsf{T}} RS$$
$$(m = 1, 2, \ldots; n = 0, 1, \ldots)$$

and *B2B* then follows from *C2B*.

By *B1* and *R1*

$$C^{2m-2} CP_1 \ldots CP_n CRQ_1 Q_2 Q_2 \ldots Q_m Q_m \vdash CV_{\mathsf{F}} Q_1 S \ldots$$
$$CV_{\mathsf{F}} Q_m SCV_{\mathsf{T}} P_1 S \ldots CV_{\mathsf{T}} P_n SV_{\mathsf{F}} RS$$
$$(m = 1, 2, \ldots; n = 0, 1, \ldots)$$

and *B2C* then follows from *C2C*.

As an example of the above simplifications we shall consider the case where the primitive functors are implication and the functor *F* of 6 arguments whose truth-table is defined by the equation

$$FPQRSUV =_{\mathsf{T}} KAPQEERSEUV.$$

8 of the 64 axiom schemes *A4* are

$$CV_{\mathsf{T}}PWCV_{\mathsf{T}}QWCV_{\mathsf{T}}RWCV_{\mathsf{T}}SWCV_{\mathsf{T}}UWCV_{\mathsf{T}}VWV_{\mathsf{T}}FPQRSUVW,$$
$$CV_{\mathsf{T}}PWCV_{\mathsf{F}}QWCV_{\mathsf{T}}RWCV_{\mathsf{T}}SWCV_{\mathsf{T}}UWCV_{\mathsf{T}}VWV_{\mathsf{T}}FPQRSUVW,$$
$$CV_{\mathsf{T}}PWCV_{\mathsf{T}}QWCV_{\mathsf{F}}RWCV_{\mathsf{T}}SWCV_{\mathsf{F}}UWCV_{\mathsf{T}}VWV_{\mathsf{T}}FPQRSUVW,$$
$$CV_{\mathsf{F}}PWCV_{\mathsf{T}}QWCV_{\mathsf{F}}RWCV_{\mathsf{T}}SWCV_{\mathsf{F}}UWCV_{\mathsf{T}}VWV_{\mathsf{T}}FPQRSUVW,$$
$$CV_{\mathsf{T}}PWCV_{\mathsf{T}}QWCV_{\mathsf{F}}RWCV_{\mathsf{F}}SWCV_{\mathsf{T}}UWCV_{\mathsf{F}}VWV_{\mathsf{F}}FPQRSUVW,$$
$$CV_{\mathsf{T}}PWCV_{\mathsf{F}}QWCV_{\mathsf{F}}RWCV_{\mathsf{F}}SWCV_{\mathsf{T}}UWCV_{\mathsf{F}}VWV_{\mathsf{F}}FPQRSUVW,$$
$$CV_{\mathsf{F}}PWCV_{\mathsf{T}}QWCV_{\mathsf{F}}RWCV_{\mathsf{F}}SWCV_{\mathsf{T}}UWCV_{\mathsf{F}}VWV_{\mathsf{F}}FPQRSUVW,$$
$$CV_{\mathsf{F}}PWCV_{\mathsf{F}}QWCV_{\mathsf{F}}RWCV_{\mathsf{F}}SWCV_{\mathsf{T}}UWCV_{\mathsf{F}}VWV_{\mathsf{F}}FPQRSUVW.$$

These give rise[3] to

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3. These three axiom schemes are not the only axiom schemes *B2*. The total number of such axiom schemes is 25.

B2A $CV_{\mathsf{T}}PWCV_{\mathsf{T}}RWCV_{\mathsf{T}}SWCV_{\mathsf{T}}UWCV_{\mathsf{T}}VWV_{\mathsf{T}}FPQRSUVW$,
B2B $CV_{\mathsf{T}}QWCV_{\mathsf{F}}RWCV_{\mathsf{T}}SWCV_{\mathsf{F}}UWCV_{\mathsf{T}}VWV_{\mathsf{T}}FPQRSUVW$,
B2C $CV_{\mathsf{F}}RWCV_{\mathsf{F}}SWCV_{\mathsf{T}}UWCV_{\mathsf{F}}VWV_{\mathsf{F}}FPQRSUVW$.

Alternative forms of *B2B*, *B2C* are

B2B′ $CV_{\mathsf{F}}RWCV_{\mathsf{F}}UWCV_{\mathsf{T}}QWCV_{\mathsf{T}}SWCV_{\mathsf{T}}VWV_{\mathsf{T}}FPQRSUVW$,
B2C′ $CV_{\mathsf{F}}RWCV_{\mathsf{F}}SWCV_{\mathsf{F}}VWCV_{\mathsf{T}}UWV_{\mathsf{F}}FPQRSUVW$.

B2A, *B2B′*, *B2C′* may, in turn, be simplified as follows:

C2A *CPCRCSCUCVFPQRSUV*,
C2B *CCCCCQCSCVFPQRSUVRRUU*,
C2C *CCCCCUCFPQRSUVRSSVV*.

In some cases we may use methods somewhat similar to those used above to replace some of the axiom schemes *C* by simpler axiom schemes *D*. For example, in the propositional calculus with the single primitive functor *G* of 4 arguments whose truth-table is defined by the equation

$$GPQRS =_{\mathsf{T}} ACPQERS,$$

we may make the definition

$$CPQ =_{df} GPQPQ$$

and the methods used above lead us to adopt as some of the axiom schemes,

C2A *CQGPQRS*,
CRCSGPQRS;
C2B *CCGPQRSPP*,
CCCCGPQRSRRSS.

We may replace the first (second) of the axiom schemes *C2A*, *C2B* by the axiom scheme *D2* (*D3*) given below.

D2 *CCPQGPQRS*,
D3 *CCRSCCSRGPQRS*.

This follows at once from the hypothetical deductions

$$CCPQR \vdash CQR,\ CCRPP;$$
$$CCPQCCQPR \vdash CPCQR,\ CCCCRPPQQ;$$

which we can establish by means of *B1* and *R1*.

If the truth-table of the functor *H* of 5 arguments is defined by the equation

$$HPQRSU =_{\mathsf{T}} ECPQAKRSU$$

we may, similarly, in the *C*-*H* propositional calculus, replace the axiom schemes

C2C *CCCQCHPQRSURUU*,
CCCCCHPQRSUPRRUU

by the axiom scheme

D2 $CCCCPQCHPQRSURUU$

using the hypothetical deductions

$$CCCCPQCSRUU \vdash CCCQCSRUU,\ CCCCCSPRRUU.$$

The related axiom scheme

D3 $CCCCPQCHPQRSUSUU$

may be replaced by the axiom scheme

$$CHPQRSUHPQSRU$$

since, by *B1* and *R1*, we may derive the hypothetical deduction

$$CCCPCSRUU,\ CVS \vdash CCCPCVRUU.$$

An alternative approach to the problem of replacing the $\sum_{i=1}^{b} 2^{n_i}$ axiom schemes *A4* by simpler axiom schemes is provided by replacing the 2^{n_i} axiom schemes describing the truth-table of the functor F_i by two longer axiom schemes $(i = 1, \ldots, b)$. Since the *C-N* propositional calculus is functionally complete, there exists a formula $\Phi_i(P_1, \ldots, P_{n_i})$ of this propositional calculus such that, for all formulae $P_1, \ldots, P_{n_i}$,

$$\Phi_i(P_1, \ldots, P_{n_i}) =_{\mathsf{T}} F_i P_1 \ldots P_{n_i}\ (i = 1, \ldots, b).$$

Let $\Psi_i(P_1, \ldots, P_{n_i}, Q)$ denote the formula obtained from $\Phi_i(P_1, \ldots, P_{n_i})$ by replacing[4] each sub-formula of the form NP, starting from the innermost, by CPQ. We shall show that the $\sum_{i=1}^{b} 2^{n_i}$ axiom schemes *A4* may be replaced by the $2b$ axiom schemes

E1 $ACF_iP_1 \ldots P_{n_i}\Psi_i(P_1, \ldots, P_{n_i}, Q)Q\ (i = 1, \ldots, b)$,
E2 $AC\Psi_i(P_1, \ldots, P_{n_i}, Q)F_iP_1 \ldots P_{n_i}Q\ (i = 1, \ldots, b)$.

For example, if F_i is K then suitable choices for $\Phi_i(P_1, P_2), \Psi_i(P_1, P_2, Q)$ are

$$NCP_1NP_2,\ CCP_1CP_2QQ$$

respectively and the corresponding two axiom schemes are

$$ACKP_1P_2CCP_1CP_2QQQ,\ ACCCP_1CP_2QQKP_1P_2Q.$$

We note that, by *B1* and *R1*,

$$\vdash CCP_1 \ldots CP_nV_{\mathsf{T}}RQCACRSQCP_1 \ldots CP_nV_{\mathsf{T}}SQ\ (n = 1, 2, \ldots),$$
$$\vdash CCP_1 \ldots CP_nV_{\mathsf{F}}RQCACSRQCP_1 \ldots CP_nV_{\mathsf{F}}SQ\ (n = 1, 2, \ldots).$$

Hence, by *R1*,

4. In some cases we must, of course, replace occurrences of the primitive symbol *C* by the corresponding abbreviations.

$$CP_1 \ldots CP_nV_{\mathsf{T}}RQ,\ ACRSQ \vdash CP_1 \ldots CP_nV_{\mathsf{T}}SQ\ (n = 1, 2, \ldots),$$
$$CP_1 \ldots CP_nV_{\mathsf{F}}RQ,\ ACSRQ \vdash CP_1 \ldots CP_nV_{\mathsf{F}}SQ\ (n = 1, 2, \ldots).$$

Thus, by *E1* and *E2*

$$CV_{x_1}P_1Q \ldots CV_{x_{n_i}}P_{n_i}QV_y\Psi_i(P_1, \ldots, P_{n_i}, Q)Q \vdash$$
$$CV_{x_1}P_1Q \ldots CV_{x_{n_i}}P_{n_i}QV_yF_iP_1 \ldots P_{n_i}Q\ (y = f_i(x_1, \ldots, x_{n_i});$$
$$x_1 = \mathsf{T}, \mathsf{F}; \ldots; x_{n_i} = \mathsf{T}, \mathsf{F}; i = 1, \ldots, b).$$

Since the assumption formula of the last hypothetical deduction contains no functors other than C (with the possible exceptions of functors occurring in $P_1, \ldots, P_{n_i}, Q$) it is derivable from *B1* and *R1*. In all the $\sum_{i=1}^{b} 2^{n_i}$ cases *A4* then follows at once.

In some cases (such as, for example, that where F_i is K, discussed above) the formula scheme

E1' $CF_iP_1 \ldots P_{n_i}\Psi_i(P_1, \ldots, P_{n_i}, Q)$

will have the property that every instance of it is a tautology. In all these cases it may replace *E1* since, by *B1* and *R1*

$$P \vdash APQ.$$

Similarly we may, in some cases, replace the axiom scheme *E2* by

E2' $C\Psi_i(P_1, \ldots, P_{n_i}, Q)F_iP_1 \ldots P_{n_i}$.

If the symbol N does not occur in the formula $\Phi_i(P_1, \ldots, P_{n_i})$ (for example, if $b = 1$, $n_1 = 3$ and $F_1PQR =_{\mathsf{T}} CPCQR$, when we may make the definition $CPQ =_{df} F_1PPQ$) both simplifications are, of course, always permissible.

Corresponding to the replacement of some of the axiom schemes *A4* by the corresponding axiom schemes *C2B*, we may replace the axiom scheme *A3* (i.e., $CV_{\mathsf{F}}PRV_{\mathsf{T}}CPQR$) by

A3' $CCCPQPP$.

In order to prove[5] this we first note that, since *A1*, *A2*, and *R1* are unchanged, the Deduction Theorem remains valid.

By *R1*

$$P,\ CPQ,\ CQR \vdash R. \tag{1}$$

By (1) and the Deduction Theorem

$$CPQ,\ CQR \vdash CPR. \tag{2}$$

By (2)

$$CCPQQ,\ CQP \vdash CCPQP. \tag{3}$$

5. For the case where $b = 0$ this result has already been proved by Schumm [4], but his proof is entirely different from that given here.

By $A3'$ and $R1$

$$CCPQP \vdash P. \tag{4}$$

By (3), (4) and the Deduction Theorem

$$CCPQQ \vdash CCQPP \tag{5}$$

By (2)

$$CPR,\ CRCPQ \vdash CPCPQ \tag{6}$$

By $R1$

$$P,\ CPCPQ \vdash Q. \tag{7}$$

By (7) and the Deduction Theorem

$$CPCPQ \vdash CPQ. \tag{8}$$

By (6), (8) and the Deduction Theorem

$$CPR \vdash CCRCPQCPQ. \tag{9}$$

By (9) and (5)

$$CPR \vdash CCCPQRR. \tag{10}$$

By (10) and the Deduction Theorem

$$\vdash CCPRCCCPQRR.$$

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