Notre Dame Journal of Formal Logic
Volume XVIII, Number 1, January 1977
NDJFAM

FIRST DEGREE FORMULAS IN CURRY'S LD

ROBERT K. MEYER

In [1], Belnap provided an algebraic semantics for the first-degree fragments of the relevant logics **E** and **R**, i.e., the sets of formulas A such that no implication signs $\rightarrow$ themselves occur within the scope of implication signs. First-degree formulas, also studied from a Kripke-style semantic point of view in Routley's [2], are particularly important because only on such can implication be taken in its natural sense as a *relation* between sentences,[1] which either holds or does not hold, rather than as a *connective* to be applied to sentences to yield further sentences. Arguments against using implication as a connective seem to be losing force as years go by, but both for those who continue to take them seriously and on considerations of general simplicity, independent characterizations of the first-degree fragments of familiar logics are important and interesting. Accordingly, in the present note Belnap's methods will be adapted to give a very simple characterization of the set of valid first-degree formulas in Curry's **D**, as presented in [3].[2] The intuitionist logic **J** comes along to some extent, so it is included in the characterization. And I note here that although I am indebted to Belnap for root insights in the relevant contexts, I am equally indebted to Dunn for his penetrating algebraic analyses and explications of these insights in [6] and [7].

1 I shall take as an underlying language $\mathcal{L}$ one with denumerably many sentential variables, positive connectives &, v, $\supset$, and sentential constants f, F. Formulas $A, B,$ etc., are built up as usual, and the following definitions are entered.

D0. $A \equiv B =_{df} (A \supset B) \,\&\, (B \supset A)$

D1. $\sim A =_{df} A \supset \mathsf{f}$ (**D**-negation)

D2. $\neg A =_{df} A \supset \mathsf{F}$ (intuitionist (**J**) negation)

The following axiom schemes and rule produce a system **DJ**.[3]

A1. $A \supset . B \supset C : \supset : A \supset B . \supset . A \supset C$

A2. $A \supset . B \supset A$

A3. $A \,\&\, B \supset A$

Received October 5, 1971

A4. $A \,\&\, B \supset B$
A5. $A \supset . B \supset A \,\&\, B$
A6. $A \supset . A \vee B$
A7. $B \supset . A \vee B$
A8. $A \supset C . \supset : B \supset C . \supset . A \vee B \supset C$
A9. $A \vee \sim A$
A10. $\mathsf{F} \supset A$
R1. *From A and $A \supset B$, infer B.*

I have formulated **DJ** with two false constants, and two negations, so that **D** and **J** might be dealt with together. A lemma, however, is indicated to show that it is indeed **D** and **J** that we deal with.

Lemma 1 **DJ** *is a conservative extension of each of* **D**, **J**.

Proof: The F-formulations of [3] will do for **D** and **J**, whence it is obvious that all theorems of each of these systems are theorems of **DJ**, when F is taken as the **J**-false constant and f as the **D**-false constant. Proof of the lemma is complete when it is shown (1) that each f-free theorem of **DJ** is already a theorem of **J**, and (2) that each F-free theorem of **DJ** is already a theorem of **D**.

Let A be an f-free theorem of **DJ**. Replace each occurrence of f in a proof of A in **DJ** with $\mathsf{F} \supset \mathsf{F}$. Only A9 might cause trouble, and it clearly does not, in showing each step of the transformed proof a theorem of **J**, whence A, being f-free, is a theorem of **J**, establishing (1).

Let A be an F-free theorem of **DJ**. Let B be the conjunction of all of the sentential variables which occur in a proof of A in **DJ**; transform the proof by replacing each occurrence of F therein by $B \,\&\, \mathsf{f}$. Only A10 might cause trouble, and by induction it does not, in showing each step of the transformed proof a theorem of **D**, when A, being F-free, is a theorem of **D**, establishing (2) and ending the proof of Lemma 1.

So all of our work may be done in **DJ**. That is nice for **J**, which we examine classically, since we may use excluded middle in the form A9 without assuming that principle for the **J**-negation $\neg$. (**J**-negation, in fact, is almost ignored in this paper, though some few formulas in which it occurs will count for us as first-degree; in view of the lemma, positive formulas of **J** may be counted the same in **J**, **D**, and **DJ**.)

2 In this section I give in passing an algebraic semantics for **DJ**, constructed along well-known lines and identical in essential respects to that given for **D** in [5].[4]

A Curry lattice $\mathfrak{L}$ is a structure $\langle L, \wedge, \vee, \supset, 0, \mathsf{f}, 1\rangle$, where

(i) L is a set, and $0, \mathsf{f}, 1$ are elements of L.
(ii) $\mathfrak{L}$ is a pseudo-Boolean algebra under $\wedge, \vee, \supset$, with least element 0 and greatest element 1, i.e., $\mathfrak{L}$ is a distributive lattice under $\wedge$, $\vee$ and is *residuated* with respect to $\supset$, in the sense $a \wedge b \leqslant c$ iff $a \leqslant b \supset c$.
(iii) f is the unique *counteratom* of $\mathfrak{L}$, i.e., $\mathsf{f} \neq 1$, and for all a in L, if $a \neq 1$ then $a \leqslant \mathsf{f}$.

Our definitions do not require $0 \neq \mathsf{f}$, though both must be distinguished from 1. But the only Curry lattice for which $0 = \mathsf{f}$ is the two-element Boolean algebra $\mathbf{2}$. I call the first nontrivial Curry lattice $\mathfrak{C}$, and because it plays a crucial role in subsequent developments enclose a snapshot.

Note that if $a \leqslant b$, $a \supset b = 1$ in $\mathfrak{C}$; otherwise $a \supset b = b$.[5]

Let $\mathfrak{L}$ be a Curry lattice. An *assignment* in $\mathfrak{L}$ is a function from the set of sentential variables of **DJ** into L. Each assignment α in $\mathfrak{L}$ is uniquely extended to a function I_α defined on all formulas of **DJ** by setting $I_\alpha(\mathsf{F}) = 0$, $I_\alpha(\mathsf{f}) = \mathsf{f}$, and otherwise requiring that $I_\alpha(A \,\&\, B) = I_\alpha(A) \wedge I_\alpha(B)$, $I_\alpha(A \vee B) = I_\alpha(A) \vee I_\alpha(B)$, and $I_\alpha(A \supset B) = I_\alpha(A) \supset I_\alpha(B)$; I_α is called the *interpretation* associated with α. A formula A of **DJ** is *true* in $\mathfrak{L}$ on assignment α (or on the associated interpretation) if and only if $I_\alpha(A) = 1$; otherwise A is *false* on α. A is *valid* in $\mathfrak{L}$ if and only if A is true on all assignments in $\mathfrak{L}$. A is **DJ**-*valid* iff A is valid in all Curry lattices.

Theorem 1 *A is a theorem of* **DJ** *if and only if A is* **DJ**-*valid.*

Proof as in [5]: I give here only main ideas. Semantic consistency is as usual, i.e., the axioms are **DJ**-valid and the rules preserve this property. For the converse, let a *normal a-filter* for a pseudo-Boolean algebra $\mathfrak{K}$ be a prime filter in the sense of [8] which does not contain a. Define an equivalence relation on the set of formulas of **DJ** by setting A eq B just in case $A \equiv B$ is a theorem of **DJ**; the resulting equivalence classes $[A]$, $[B]$, etc. form in a natural way as in [8] a pseudo-Boolean algebra, the so-called *Lindenbaum algebra* of **DJ**.

Let $\mathfrak{L}$ be the Lindenbaum algebra of **DJ**. We note that $\mathfrak{L}$ is *not* a Curry lattice, since there is no unique counteratom. Let A be a given non-theorem of **DJ**. Prove as in [5] the non-trivial result that $A \supset \mathsf{f}. \supset \mathsf{f}$ is a non-theorem of **DJ** and hence that there is a prime filter in $\mathfrak{L}$ containing $[A \supset \mathsf{f}]$ but not containing $[\mathsf{f}]$. This filter, call it F, is a normal $[\mathsf{f}]$-filter in the sense just introduced; define an equivalence relation on $\mathfrak{L}$ by setting a eq b just in case both $a \supset b$ and $b \supset a$ are in F. The resulting equivalence classes $[[A]]$, $[[B]]$, etc., again form as in [8] a pseudo-Boolean algebra $\mathfrak{L}'$. Moreover this one is a Curry lattice, since $[[\mathsf{f}]]$ is easily shown to be the unique counteratom. Taking $[[F]]$ as 0, $[[\mathsf{f}]]$ as f, and $[[\mathsf{f} \supset \mathsf{f}]]$ as 1 and defining operations on equivalence classes by operations on representatives in $\mathfrak{L}$, that $\mathfrak{L}'$ has been well-defined and that it is a Curry lattice is again shown as in [5]; the chosen non-theorem A is falsified in $\mathfrak{L}'$ by assigning the element $[[p]]$ to each sentential variable p, ending the sketch of the proof of Theorem 1.

The proof of Theorem 1 was not immediate from [8] only because we wanted to keep f distinct from 1 but bigger than everything else. The reason for this is that we want Curry lattices to furnish a normal semantics that respects the intuitions behind **D**, namely that each sentence is either true or false but not both. Similar motivation lay behind work done on the relevant logics in [1], [7], and [9]. But **D** partially resists paradoxes of implication, so that alternative possibilities that some sentences are *both* true and false, for example, are open; the algebraic analogue, not pursued here, is to lift the restriction $\mathsf{f} \neq 1$.[6]

3 In defense of the claim that $\mathfrak{C}$ is for present purposes the *key* Curry lattice, I play in this section the prime filter game of [7]. To do so, I modify the semantics above along the lines of [1] so that it takes an appropriate first-degree form.

An *elementary Curry lattice* (henceforth, elementary lattice) will be a Curry lattice with $\supset$ chopped out, i.e., the **D**-negation $\sim$ is thrown in to take up some of the slack. Specifically, $\mathfrak{L} = \langle L, \wedge, \vee, \sim, 0, \mathsf{f}, 1\rangle$ will be an elementary lattice provided that $\mathfrak{L}$ is a distributive lattice under $\wedge$ and $\vee$, 0 is lattice zero, 1 is lattice unit, and f is a unique counteratom as above, and finally $\sim 1 = \mathsf{f}$ and, for $a \neq 1$, $\sim a = 1$. It is readily seen that defining $\sim a$ as $a \supset \mathsf{f}$ every Curry lattice is an elementary lattice; the converse may fail.

We assume henceforth that $\sim$ is primitive for **DJ**, taking the equivalence corresponding to D1 as a new axiom, and making the obvious adjustments above. Then A shall be a *zero-degree formula* of **DJ** (*zdf*) if it is a sentential variable or constant or of the form $B \mathbin{\&} C$, $B \vee C$, or $\sim B$, where B and C are zdf. A shall be a *first-degree implication* (*fdi*) if it is of the form $B \supset C$, where B and C are zdf. Finally, A shall be a *first-degree formula* (*fdf*) if it is a zdf or an fdi, or of the form $B \mathbin{\&} C$, $B \vee C$, or $\sim B$, where B and C are fdf.

Unlike the relevant logics, for which nested implications are hard to break down, the fdf of **D** and, so far as we consider them, of **J** embrace for practical purposes a wider class than is at first evident, viz. $p \supset. q \supset r$, though not legally an fdf, is by an obvious equivalence as good as one, and so forth.

Still, the *operation* of implication on propositions has been lifted from our semantics, leaving as in [1] the corresponding *relation* $\leqslant$, defined lattice-theoretically as usual. Accordingly, the notion of an *interpretation* I_α determined by an assignment α as characterized above makes no sense for *elementary* lattices beyond the level of zdf. (If necessary, we may use I_α^0 to show the distinction.) Another notion, again as in [1], takes up the semantic job. The *truth-valuation* V_α is a function from the fdf of **DJ** to $\{\mathsf{T}, \mathsf{F}\}$ determined thus: if A is a zdf, $V_\alpha(A) = \mathsf{T}$ iff $I_\alpha(A) = 1$; if A is an fdi $B \supset C$, $V_\alpha(B \supset C) = \mathsf{T}$ iff $V_\alpha(A) \leqslant V_\alpha(B)$; finally, on composition of fdf under $\&$, $\vee$, $\sim$, V_α works truth-functionally. Earlier semantic notions are redefined appropriately in the present context; in particular, the fdf A of **DJ** is *elementarily valid* iff A is true on all *truth-valuations* in all elementary lattices. In fact, the notions of validity and elementary validity coincide. For we have on essentially lattice-theoretic grounds

Lemma 2 *Every Curry lattice can be isomorphically embedded in a* *complete Curry lattice; every elementary lattice can be isomorphically embedded in a complete lattice that satisfies the infinite distributive law* $a \wedge \bigvee_{i \in I} b_i = \bigvee_{i \in I} a \wedge b_i$. *So every elementary Curry lattice can be isomorphically embedded in a Curry lattice.*

Proof: Let $\mathfrak{L}$ be a Curry lattice (elementary Curry lattice), and let S be the set of prime filters in $\mathfrak{L}$. A subset J of S is *closed upward* if for all $P, Q \in S$, $P \in J$ and $P \subseteq Q$ imply $Q \in J$. Let U be the set of all closed upward subsets of S. U is clearly a complete lattice satisfying the infinite distributive law. Furthermore, on the Stone isomorphism $h{:}L \rightarrow U$ taking each element a of L into the set of prime filters P such that $a \in P$, $\mathfrak{L}$ is isomorphically embedded in U, preserving lattice operations $\wedge$ and $\vee$ and with zero $\emptyset$ and unit S, by Stone's theorem. (*Cf.* [8] for references.) We must show moreover that $h(\mathsf{f})$ is the unique counteratom of U and that h preserves $\sim$; moreover, if $\mathfrak{L}$ is a Curry lattice we must show that h preserves $\supset$.

First, $h(\mathsf{f})$ is a counteratom, since it is $S - \{\{1\}\}$ since f is a unique counteratom. Moreover since 1 is in every filter, by closure upward S is the only member of U to which $\{1\}$ belongs; so $h(\mathsf{f})$ is unique. $h(\sim 1) = h(\mathsf{f}) = \sim S = \sim h(1)$; if $a \neq 1$, $h(\sim a) = h(1) = S = \sim h(a)$ on straightforward definition of $\sim$ on U; so h preserves $\sim$. Demonstration that $h(a \supset b) = h(a) \supset h(b)$, defining $J \supset K = \bigcup\{L\colon J \cap L \subseteq K\}$ on U, is not particularly to our point and is left to the reader, ending the proof of the first sentence of the theorem. The second sentence follows since it is well-known (*cf.* [10]) that defining $\supset$ as just above on a complete lattice satisfying the infinite distributive law produces a pseudo-Boolean algebra.

Theorem 2 *Let A be a first-degree formula of* **DJ**. *Then the following conditions are equivalent:*

(i) *A is a theorem of* **DJ**.
(ii) *A is* **DJ**-*valid.*
(iii) *A is elementarily valid.*

Proof: The equivalence of (i) and (ii) was Theorem 1. Since as noted every Curry lattice is on trivial reconstruction an elementary lattice, (iii) implies (ii); one need only show that for each assignment α in a Curry lattice $\mathfrak{L}$ and for each fdf B, $I_\alpha(B) = 1$ iff $V_\alpha(B) = \mathbf{T}$ when $\mathfrak{L}$ is reconstrued as an elementary lattice. Proof is by induction. Finally, since by Lemma 2 every elementary lattice is isomorphically embeddable in a Curry lattice, (i) implies (iii), for since all fdf are valid in the embedding Curry lattice they are *a fortiori* valid in the embedded elementary lattice—whenever, in both cases, they are theorems. To illustrate the simplicity of these methods, the following (known) corollary is immediate.

Corollary 2.1 *Let A be a zero-degree formula of* **DJ**. *Then A is a theorem of* **DJ** *if and only if A is a classical tautology.*

Proof: Necessity is immediate, since truth-tables are a Curry lattice. So is sufficiency, since taking the zdf A to be a non-theorem, it has by the theorem a falsifying truth-valuation V_α which clearly suffices to make A a non-tautology.

We turn now to the promised algebra, which produces among other things an efficient decision method for fdf's. Let $\mathfrak{L}$ and $\mathfrak{M}$ be elementary lattices (which may be Curry lattices). A *first-degree homomorphism* from $\mathfrak{L}$ to $\mathfrak{M}$ is a function that preserves constants and the elementary operations $\wedge$, $\vee$, $\sim$. A *fundamental homomorphism from* $\mathfrak{L}$ is a first-degree homomorphism to one of $\mathit{2}$, $\mathfrak{C}$, the two and three element Curry lattices introduced above. A *prime filter in* $\mathfrak{L}$ is to be understood as above in the usual lattice-theoretic sense; as usual, neither $\emptyset$ nor L counts here as a prime filter. By borrowing in the present context techniques from [7] and [1], we put the prime filters of an elementary lattice in 1-1 correspondence with its fundamental homomorphisms getting thereby a structural decomposition of $\mathfrak{L}$ into *special* products of $\mathit{2}$ and replicas of $\mathfrak{C}$.

Lemma 3 *Let $\mathfrak{L}$ be a Curry lattice; S the set of prime filters in $\mathfrak{L}$, and H the set of fundamental homomorphisms from $\mathfrak{L}$. Then the mapping which sends P in S to h_P as immediately defined is a* 1-1 *correspondence from S onto H: $h_P(1) = 1$; if $a \notin P$, $h_P(a) = 0$; otherwise $h_P(a) = \mathsf{f}$; the target set of h_P is $\mathit{2}$ or $\mathfrak{C}$ as $P = \{1\}$ or $P \neq \{1\}$ respectively.*

Remark For any elementary lattice, the requirement that homomorphisms preserve constants implies that there is at most one fundamental homomorphism to $\mathit{2}$, i.e., the one that identifies the minimally and maximally false constants 0 and f, with everything in between. Furthermore, every first-degree homomorphism takes 1 and only 1 into 1, i.e., $h(1) = 1$ on preservation of constants; moreover 1 must be distinguished by the normality of our semantics from its neighbor f, so that nothing but 1 gets 1 under h. $h(\mathsf{f}) = \mathsf{f}$ and $h(0) = 0$ are distinct unless $\mathit{2}$ is the target lattice; if $\mathit{2}$ is the domain, it must be the target.

Proof: Clearly $h_{\{1\}}$ is the unique fundamental homorphism from $\mathfrak{L}$ to $\mathit{2}$ and satisfies the stated conditions; that $h_{\{1\}}$ is a lattice homomorphism is obvious; furthermore $h(\sim 1) = h(\mathsf{f}) = 0 = \mathsf{f} = \sim h(1)$ in this case, while if $a \neq 1$ then here $h(\sim a) = h(1) = 1 = \sim 0 = \sim h(a)$, proving that $h_{\{1\}}$ is a homomorphism; $\{1\}$ is of course always a prime filter.

All other prime filters P in $\mathfrak{L}$ contain f but not 0, so that h_P: $L \to C$. Clearly given P, h_P is determined uniquely; moreover, given any homomorphism from $\mathfrak{L}$ to $\mathfrak{C}$, the set of a in L such that $\mathsf{f} \leq h(a)$ in $\mathfrak{C}$ will constitute a prime filter in $\mathfrak{L}$, since $\mathfrak{C}$ is a chain; so any such h will be one of the h_P which moreover determines P uniquely; so the stated mapping from S to H is indeed a bijection, provided that *all* of the h_P are indeed homomorphisms. The reader is assured that they are and, since he may easily check, this ends the proof of Lemma 3.

Lemma 3 sets up an embedding theorem in *special products* of $\mathit{2}$ and $\mathfrak{C}$. We cannot take direct products of Curry lattices in the usual way, defining operations pointwise on Cartesian products, without escaping from the class of Curry lattices, i.e., the problem is that there would not be in general a unique counteratom to serve as f. Belnap handled an analogous problem in [1] subdirectly by disallowing elements that are true on some coordinates and false on others; we adopt his solution in principle.

Let $\{\mathfrak{L}_i\}_{i \in I}$ be an indexed set of elementary lattices. Let $\mathfrak{D}\mathfrak{L} = \bigtimes_{i \in I} \mathfrak{L}_i$ be the direct product of the $\mathfrak{L}_i$—i.e., members of $\mathfrak{D}\mathfrak{L}$ are all functions g defined on I such that $f(i) \in L_i$ for each i in I, constants are corresponding constant functions, and operations are defined pointwise. It is well-known that $\mathfrak{D}\mathfrak{L}$ is a distributive lattice. *Remove* all elements of $\mathfrak{D}\mathfrak{L}$ between f = ffff . . . and 1 = 1111 . . . Then we are left with a set SL, consisting of *one* element 1 which is *true* on *all* coordinates together with *all* members of $\mathfrak{D}\mathfrak{L}$ which are true on *no* coordinates. (**D** contests Quine's dictum that there are as many truths as falsehoods—for **D**, as in life, there are many more ways to go wrong than to be right.) Let $\mathfrak{S}\mathfrak{L} = \langle SL, \wedge, \vee, \sim, 0, \mathsf{f}, 1\rangle$ be the subalgebra of $\mathfrak{D}\mathfrak{L}$ got by restricting all operations to SL. Clearly $\mathfrak{S}\mathfrak{L}$ *is* closed under the operations and contains the constants of $\mathfrak{D}\mathfrak{L}$. Moreover, we have:

Lemma 4 *Let* $\mathfrak{S}\mathfrak{L}$ *be as just defined, a* special product *of elementary lattices. Then* $\mathfrak{S}\mathfrak{L}$ *is an elementary lattice.*

Proof: By straightforward verification. The important point is that f is a *counteratom*, settled drastically by amputation.

We can now embed. A special product is a *special power* if all component lattices are the same. A *fundamental lattice* is either $\mathit{2}$, $\mathfrak{C}$, or a special power of $\mathfrak{C}$. We index $\mathit{2}$, $\mathfrak{C}$, and its powers by ordinals (and possibly in other ways), letting $\mathfrak{C}_0$ be $\mathit{2}$, $\mathfrak{C}_1$ be $\mathfrak{C}$, and in general for each ordinal λ setting $\mathfrak{C}_\lambda$ equal to the special power of copies of $\mathfrak{C}$, one for each ordinal ν such that $\nu < \lambda$. Note that without loss of generality we may pretend that we have multiplied in a copy of $\mathit{2}$ where desired, since *special* multiplication by $\mathit{2}$ produces an elementary lattice isomorphic to the multiplicand.

Theorem 3 *Let* $\mathfrak{L}$ *be an elementary lattice. Then* $\mathfrak{L}$ *is first-degree embeddable in some fundamental lattice* $\mathfrak{C}_\lambda$, i.e., *there is a* 1-1 *first-degree homomorphism from* $\mathfrak{L}$ *to* $\mathfrak{C}_\lambda$. *If* $\mathfrak{L}$ *is of finite cardinality* k, $\mathfrak{L}$ *is first-degree embeddable in* $\mathfrak{C}_j$ *for some* $j < k$.

Proof: Let S be the set of all prime filters in $\mathfrak{L}$. For each $P \in S$, there is a fundamental homomorphism h_P to $\mathit{2}$ or $\mathfrak{C}$ by Lemma 3. Well-ordering S and indexing by corresponding ordinals, where the ordinal number of S can be taken to be $\lambda + 1$ without loss of generality, taken as the set of prior ordinals. (If S turned out to have a limit ordinal, shuffle.) If $\lambda = 0$, clearly $\mathfrak{L} = \mathit{2}$, so that $\mathfrak{L}$ being identical with $\mathfrak{C}_\lambda$ is certainly embeddable therein. Let P_λ be the filter $\{1\}$. Then λ itself indexes, shuffling if required, the filters that contain f—there are some, the contrary case having been disposed of. Letting h_ν be the homomorphism from $\mathfrak{L}$ to $\mathfrak{C}$ corresponding to P_ν by Lemma 3 for each $\nu < \lambda$, define a function h from $\mathfrak{L}$ to $\mathfrak{C}_\lambda$ pointwise as follows: for each $a \in L$ and $\nu < \lambda$, let $[h(a)](\nu) = h_\nu(a)$. We show h so defined is a first-degree embedding of $\mathfrak{L}$ in $\mathfrak{C}_\lambda$.

Clearly h is 1-1. For by Stone's prime filter theorem for distributive lattices, any two elements of L are separated by a prime filter P_ν and hence, if $\nu < \lambda$, they take different values under h on the ν'th coordinate and are hence separated in $\mathfrak{C}_\lambda$; in the last case, they are separated by P_λ, which means that one of the elements is 1 and the other is not; but by definition of the h_ν, 1 differs under h on every coordinate from any other element, so P_λ is not needed when $\lambda > 0$. Furthermore the values of h are really in $\mathfrak{C}_\lambda$, since, as just noted, $h(1)$ is 1 everywhere and h (anything else) is 1 nowhere, so that the amputation to get special products is respected. Finally, it follows immediately from the fact that h is a pointwise homomorphism from $\mathfrak{L}$ to $\mathfrak{C}$ that it is a homomorphism from $\mathfrak{L}$ to $\mathfrak{C}_\lambda$, inasmuch as operations are defined pointwise in special powers. This ends the proof of Theorem 3.

4 The algebraic interlude simplifies the logical work. One thing that a Belnap semantics like the above is good for is in providing efficient decision methods for fdf. Decision methods for logics in the intuitionistic family exist, of course, but they tend to be extremely impracticable (a bound which turns up regularly is 2^{2^n} (e.g., in [8]), which goes up pretty quickly); Gentzen methods, if anything, are worse. Accordingly, it is nice to know that it is the complexities of nested implications, which we do not really understand that well anyway, which are responsible for those high numbers. The reader no doubt knows about *positive* and *negative* occurrences of formulas. I recall here only that when an occurrence of B in A is *truth-functional*, i.e., within the scope of no connectives but &, $\vee$, $\sim$ at most, that occurrence is *negative in A* if B is within the scope of an *odd* number of negation signs and is otherwise *positive*. I recall too that in familiar systems, including **DJ**, if $A \supset B$ is a theorem then replacement of positive occurrences of A by B and negative occurrences of B by A in a theorem C yields a theorem. (Replacement Theorem for $\supset$)

The Pleasant Lemma *Let A be built up truth-functionally from formulas $B, C_1, \ldots, C_n$. Let each occurrence of B be negative in A. Let v assign truth-values to the C_i, and let $v(B) = \mathsf{F}$; suppose further that, by truth-tables, $v(A) = \mathsf{F}$. Let v' agree with v on the C_i, but let $v'(B) = \mathbf{T}$. Then $v'(A)$ remains* F.

Proof: Apply the Replacement Theorem. Suppose for *reductio* that $v'(A) = \mathbf{T}$. Substitute the sentential constant $\mathbf{T}$ for B, and correspondingly substitute the constant $v'(C_i)$ for C_i in A. On these substitutions A is a theorem. But B occurred only in negative parts, whence since F classically implies $\mathbf{T}$ putting $\mathbf{F}$ in where B was produces another theorem, corresponding exactly to the valuation v on which A is false, which is absurd. So $v'(A) = \mathsf{F}$, proving the lemma.

The next is our decision method for fdf.

Theorem 4 *Let A be an fdf of* **DJ**. *Let there be exactly k positive occurrences of first-degree implications in A. Then a necessary and sufficient condition that A be a theorem of* **DJ** *is that A be valid in both $\mathfrak{C}_0$ and $\mathfrak{C}_k$.*

Remark $\mathfrak{C}_0$ is sort of along for the ride, except if $k = 0$; otherwise it blocks the logical falsehood of $f \supset \mathsf{F}$. And, for what it is worth, note that $\mathfrak{C}_k$ has $2^k + 1$ elements.

Proof: If A is a theorem, it is by Theorem 2 valid in all elementary lattices, including the $\mathfrak{C}_i$. So suppose A is a non-theorem; our job is to refute it in the proper $\mathfrak{C}_k$.

At any rate, A is invalid in some elementary lattice $\mathfrak{L}$, by Theorem 2. $\mathfrak{L}$ is by Theorem 3 isomorphically embeddable in some $\mathfrak{C}_\lambda$; A remains invalid, extra elements in the embedding lattice being irrelevant to its first-degree falsification. Let ν be the least ordinal such that A is invalid in $\mathfrak{C}_\nu$; if $\nu = 0$ we are through, validity in $\mathfrak{C}_0$ having been part of the condition that A be a theorem. Furthermore, if $0 < \nu \leqslant k$, it is easy to concoct from a falsifying assignment α in $\mathfrak{C}_\nu$ a corresponding falsifying assignment in $\mathfrak{C}_k$, since $\alpha(p)$ is for practical purposes a ν-tuple, construct a k-tuple $\alpha'(p)$ like $\alpha(p)$ in each of the first ν places and constant from the ν'th place on; the extra coordinates do no work on the resulting assignment α', falsifying A as before in $\mathfrak{C}_k$. We may assume accordingly that $\nu > k$, finishing the proof of the theorem by deriving a contradiction from this assumption. Let α be the assignment that falsifies A in $\mathfrak{C}_\nu$, I_α the corresponding interpretation, and V_α the resultant truth-valuation. I_α, we recall, is defined in the elementary lattice $\mathfrak{C}_\nu$ only on zero-degree formulas, and that, the coordinates of $I_\alpha(B)$ all being the same in $\mathfrak{C}_\nu$ with respect to *truth-value* for each *zdf* B, inspection of any one of them suffices to determine truth-value for any zdf on V_α.

ν is, by convention, a set of ordinals. An ordinal β is *critical* in ν iff for some *fdi* $B \supset C$ that occurs *positively* in A it is the first ordinal γ such that $[I_\alpha(B)](\gamma) \nleqslant [I_\alpha(C)](\gamma)$. Let the critical ordinals, if any, be $\beta_0, \ldots, \beta_j$; if it exists, $j < k$, there being k positive occurrences of fdi and clearly at most one β_i for each such occurrence. Suppose first that there are no critical ordinals in ν. Then, recalling the definition of *truth* on V_α for fdi, all *positive* fdi $B \supset C$ in A are true on α, $I_\alpha(B) \leqslant I_\alpha(C)$ holding in $\mathfrak{C}_\nu$ on all coordinates. Define an assignment α' in $\mathfrak{C}_0$ by setting $\alpha'(p) = 1$ iff $V_\alpha(p) = \mathbf{T}$ and $\alpha'(p) = 0$ otherwise. An easy induction shows that $V_{\alpha'}$ agrees with V_α on zdf; moreover $V_{\alpha'}$ is determined truth-functionally on fdi $B \supset C$; since V_α respects truth-functionality to the extent that it never makes conditionals with true antecedents and false consequents true, this means that $V_{\alpha'}$ differs from V_α only in sometimes making $B \supset C$ *true* where V_α makes it false, at the level of fdi. Since V_α already made all the positive fdi true, so does $V_{\alpha'}$. This sets things up for our Pleasant Lemma, A being compounded truth-functionally from fdi and those zdf not occurring within the scope of any horseshoe. For since $V_{\alpha'}$ differs at most on these subformulas of A from V_α in changing F at some places to $\mathbf{T}$ in *negative* parts, by (possibly repeated) application of the *Pleasant Lemma* we see that since A is false on V_α it is also false on $V_{\alpha'}$. But this means that A is *invalid* in $\mathfrak{C}_0$ [= truth-tables], contradicting $0 \leq k < \nu$ and the fact that ν is the least ordinal such that $\mathfrak{C}_\nu$ invalidates A.

So there are critical ordinals in ν, $\beta_0, \ldots, \beta_j$, $j < k$. Define an assignment α' in $\mathfrak{C}_{j+1}$ using α as follows: for each sentential variable p, define $\alpha'(p)$ pointwise by setting, for $0 \leq i \leq j$, (i) $[\alpha'(p)](i) = [\alpha(p)](\beta_i)$. Thus, in effect, α' is like α on critical coordinates while taking no account of the others. Prove by induction that (i) holds for all zdf B for I_α; thus, truth-values being determined by any coordinate, $V_\alpha = V_{\alpha'}$ on restriction of both to zdf. Moreover, if $B \supset C$ is an fdi that occurs *positively* in A, it has the same values under V_α and $V_{\alpha'}$; for if $B \supset C$ is *false* on V_α, then there is some coordinate γ such that $[I_\alpha(B)](\gamma) \nleq [I_\alpha(C)](\gamma)$; the first such γ being critical is one of the β_i, assuring the corresponding falsehood of $B \supset C$ on $V_{\alpha'}$; and if on the other hand $B \supset C$ is *true* on V_α, then on *all* coordinates including critical ones $I_\alpha(B) \leq I_\alpha(C)$, implying the corresponding truth of $B \supset C$ on $V_{\alpha'}$. The *Pleasant Lemma* may now be applied as above; A is built up truth-functionally from fdi and zdf such that $V_{\alpha'}$ differs maybe from V_α in making more *negative* components true; by the cited lemma, this does not affect the falsification of A. But α' falsifies A in $\mathfrak{C}_{j+1}$, $j < k < \nu$, again contradicting the leastness of ν, exhausting the cases, and completing the proof of Theorem 4.

We note some easy corollaries to Theorem 4.

Corollary 4.1 *$\mathfrak{C}$ is characteristic for the first-degree implications of* **D**, *i.e., a first-degree implication of* **D**, *if falsifiable at all, is falsifiable in the 3-point Curry lattice.*

Proof: It suffices, in view of Theorem 4, to note that all formulas *of* **D** valid in $\mathfrak{C}$ are classically valid.

Corollary 4.2 *Let A be an fdf of* **D**, *and let k fdi occur positively in A. Then A is a theorem of* **D** *iff A is valid in $\mathfrak{C}_k$.*

Proof: By Theorem 4 and the Corollary 4.1.[7]

NOTES

1. If one prefers, between propositions.

2. Called there **HD**, **LD** (as in the title), etc., depending on the style of formulation. Henceforth, I call the system simply **D**. The methods of [1] are particularly appropriate for the analysis of **D**, since **D** has been shown in [4] and [5] to be the logic in the intuitionistic family most closely related to the relevant logics of Anderson and Belnap, being in fact an exact and clearly defined subsystem of the calculus **R** of relevant implication. But the first-degree analysis of **D**, as we shall see, is much simpler than that of **R**—central is the 3-point chain $\mathfrak{C}$ characterized below, which plays for **D** the role of the 8-point lattice $\mathfrak{M}_0$ of [1].

3. Binary connectives are ranked &, v, ⊃, ≡ in order of increasing scope for ease in reading formulas; otherwise we (i) associate to the left and (ii) use dots for parentheses in accordance with the conventions of Curry's [3]. I have not run across the system **DJ**, but that is probably an oversight, since the idea is too simple to have gone unnoticed—on the motivation of [3] the negations of **D** and **J** have nothing to do with one another, so that one might as well have them both together; Lemma 1 shows that there is no harm in doing so; since *this* paper is mainly about **D**, negation here almost always means **D**-negation.

4. To take account of the absurd constant F, the characterization of Curry lattices here differs from that of [5] in that we require here a least element 0; the point is insubstantial in the present context.

5. $\mathfrak{C}$ has a distinguished history in the study of intuitionist logic, having been used by such scholars as Gödel and Jaśkowski in their pioneer research into intuitionist logic; it is important.

6. Yielding a semantics for non-trivial inconsistent theories.

7. The results of this paper are essentially spin-off from hard work done for the relevant logics by Belnap and by Dunn, to whom thanks are due in particular, into more familiar terrain. Thanks are due also to the National Science Foundation for partial support through grant GS2648.

REFERENCES

[1] Belnap, N. D., Jr., "Intensional models for first degree formulas," *The Journal of Symbolic Logic*, vol. 32 (1967), pp. 1-22.

[2] Routley, R. and V., "The semantics of first degree entailment," *Noûs*, vol. 6 (1972), pp. 335-359.

[3] Curry, H. B., *Foundations of Mathematical Logic*, McGraw-Hill, New York (1963).

[4] Meyer, R. K., "Some problems no longer open for *E* and related logics," abstract, *The Journal of Symbolic Logic*, vol. 35 (1970), p. 353.

[5] Meyer, R. K., "Entailment, intuitionism, negation," in H. Leblanc, ed., *Truth, Syntax, Modality*, North-Holland, Amsterdam (1973).

[6] Dunn, J. M., *The Algebra of Intensional Logics*, doctoral dissertation, University of Pittsburgh (1966).

[7] Dunn, J. M., and N. D. Belnap, Jr., "Homomorphisms of intensionally complemented distributive lattices," *Mathematische Annalen*, vol. 176 (1968), pp. 28-38.

[8] Rasiowa, H., and R. Sikorski, *The Mathematics of Metamathematics*, Pânstwowe Wydawnu, Warsaw (1963).

[9] Meyer, R. K., and J. M. Dunn, "E, R and γ," *The Journal of Symbolic Logic*, vol. 34 (1969), pp. 460-474.

[10] Birkhoff, G., *Lattice Theory*, 3rd ed., American Mathematical Society, Providence (1967).

Australian National University
Canberra, Australia