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NEXT $\mathcal{P}$ ADMISSIBLE SETS ARE OF COFINALITY ω

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The first and most direct generalization of the Barwise compactness theorem to the uncountable case was the cofinality ω compactness theorem of Barwise and Karp [1], [2]—a power set admissible set which can be written as a union of countably many of its elements is Σ_1 (in the graph of the power set) compact. Thus, in order to directly generalize the many situations in which the Barwise compactness theorem is applied to a next admissible set, we need to know that all next power set admissible sets can be written as appropriate countable unions. In this paper* we show, using elementary methods, that they can. A modification of the proof of Theorem 5.3 of [1] can also be used but involves higher order predicates.

We assume familiarity with the notion of power set admissibility, presented for example in [2], and the fact that any power set admissible set can be written as a $V(\kappa)$. We also will use the obvious fact that there are only countably many formulas which are Δ_0 in the graph of the power set and abuse notation slightly by calling these Δ_0 in $\mathcal{P}$ formulas. For each cardinal λ we let $\beth_0(\lambda) = \lambda$, $\beth_{n+1}(\lambda) = 2^{\beth_n(\lambda)}$, and $\beth_\omega(\lambda) = \bigcup\limits_{n \in \omega} \beth_n(\lambda)$.

Theorem *Every next power set admissible set is of cofinality ω.*

Proof: Suppose $V(\kappa)$ is the smallest power set admissible set containing the set A and $\kappa_0 = \beth_\omega(\rho)$ where ρ is the cardinality of the rank of A. Clearly $A \in V(\kappa)$ and $V(\kappa)$ $\mathcal{P}$ admissible implies $\kappa_0 < \kappa$. Starting with κ_0 we construct a sequence of cardinals, each of cofinality ω, such that for each n, $\kappa_{n-1} < \kappa_n \leqslant \kappa$. If at any time we find $\kappa_n = \kappa$ we are done so we may assume $\kappa_0 < \kappa_1 < \ldots < \kappa_n < \kappa$ and for $j \leqslant n$, $\kappa_j = \bigcup\limits_{m \in \omega} \kappa_{j,m}$. Since we will eventually want to show that $V\left(\bigcup \kappa_n\right)$ is $\mathcal{P}$ admissible $\left(\text{and hence } \kappa = \bigcup \kappa_n\right)$ we want to construct the sequence to satisfy:

if Q is any $k + 2$ place Δ_0 in $\mathcal{P}$ formula and $a, b_1, \ldots, b_k \in V\left(\bigcup \kappa_n\right)$ then $\forall x \in a \exists y \in V\left(\bigcup \kappa_n\right) Q(x, y, b_1, \ldots, b_k)$ implies there is an $n \in \omega$ such that $\forall x \in a \exists y \in V(\kappa_n)\, Q(x, y, b_1, \ldots, b_k)$.

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Let $Q_{k,0}$, $Q_{k,1}, \ldots$ be a list of the $k+2$ place Δ_0 in $\mathcal{P}$ formulas and let $B_{k,i}(n, m) = \{(x, z_1, \ldots, z_k) \in V(\kappa_{n,m}) \mid \exists y \in V(\kappa)\, Q_{k,i}(x, y, z_1, \ldots, z_k)\}$. Since each $B_{k,i}(n, m) \in V(\kappa)$ we may use Δ_0 in $\mathcal{P}$ collection on $\forall (x, z_1, \ldots, z_k) \in B_{k,i}(n, m) \exists \zeta \in V(\kappa) \exists y \in V(\zeta) Q_{k,i}(x, y, z_1, \ldots, z_k)$ to show $\exists \xi \in \kappa \forall (x, z_1, \ldots, z_k) \in B_{k,i}(n, m) \exists \zeta \in \xi \exists y \in V(\zeta)\, Q_{k,i}(x, y, z_1, \ldots, z_k)$. But then $\alpha_{k,i}(n, m) = \bigcup \{\min \zeta \exists y \in V(\zeta) Q_{k,i}(x, y, z_1, \ldots, z_k) \mid (x, z_1, \ldots, z_k) \in B_{k,i}(n, m)\} < \kappa$ so that $\lambda_{n,m} = \bigcup\limits_{k,i \leq n} \alpha_{k,i}(n, m) < \kappa$. We now let $\lambda_n = \bigcup\limits_{m \in \omega} (\kappa_{n,m} \cup \lambda_{n,m})$. If $\lambda_n = \kappa$ we are done so assume $\lambda_n < \kappa$ and $\kappa_{n+1,m} = \beth_m(\lambda_n)$. Thus $\kappa_{n+1} = \bigcup \kappa_{n+1,m}$ is constructed as desired.

To show $V\left(\bigcup \kappa_n\right)$ is $\mathcal{P}$ admissible we need only show it satisfies Δ_0 in $\mathcal{P}$ collection. So suppose $Q_{k,i}$ is a $k+2$ place Δ_0 in $\mathcal{P}$ formula and $a, b_1, \ldots, b_k$ are elements of $V\left(\bigcup \kappa_n\right)$ such that $\forall x \in a \exists y \in V\left(\bigcup \kappa_n\right) Q_{k,i}(x, y, b_1, \ldots, b_k)$. Since $a, b_1, \ldots, b_k$ are elements of $V\left(\bigcup \kappa_n\right)$ there are $n, m \in \omega$ such that $a, b_1, \ldots, b_k$ are elements of $V(\kappa_{n,m})$ and hence $\forall x \in a\left((x, b_1, \ldots, b_k) \in B_{k,i}(n, m)\right)$. But then $\forall x \in a \exists y \in V(\alpha_{k,i}(n, m))\, Q_{k,i}(x, y, b_1, \ldots, b_k)$ and therefore $\forall x \in a \exists y \in V(\kappa_{n+k+i})\, Q_{k,i}(x, y, b_1, \ldots, b_k)$. Thus $V\left(\bigcup \kappa_n\right)$ satisfies Δ_0 in $\mathcal{P}$ collection.

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