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# REGRESSIVE ORDER-TYPES

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It is of course well-known that with respect to addition, ordinals possess the property of left-cancellation, but that this property is not shared by order-types in general. In this note* we introduce the property of regressiveness, show that an order-type is regressive if and only if it cannot be additively left-cancelled in general, and give a simple canonical form for regressive order-types. We conclude by giving criteria for the regressiveness of an order-type possessing a (nontrivial) ordinal right divisor.

Lower-case Greek letters are used to denote order-types, and upper-case Latin letters to denote (ordered) sets, and we usually suppress mention of the order relation on a set. $A$ is called a "representative" set for $\alpha$ if $\alpha$ is the isomorphism type of $A$. The ordered union and ordered product of sets $A$, $B$ are respectively denoted by "$A \dot{+} B$" and "$A \dot{\times} B$". If $A = B \dot{+} C \dot{+} D$, then $B(D)$ is called an "initial (final) segment" of $A$, and $C$ is sometimes called an "interval" of $A$. The same terminology is used for order-types, although we sometimes refer to a final segment of an order-type as a remainder. A (strict) order-preserving map $f : A \to B$ is called an "isomorphism", and if $B = f''A$, then we write "$f : A \simeq B$".

The first transfinite ordinal is denoted by "$\omega$": in general we use "$\sigma$", "$\tau$", "$\rho$" for ordinals and "$\alpha$", "$\beta$", "$\gamma$", . . . for general order-types. "$i$", "$j$", "$k$", . . . are used to denote finite ordinals. The converse order-type of $\alpha$ is denoted by "$\alpha^*$".

An elementary property of ordinals is that of left-cancellation: given ordinals $\sigma$, $\tau$, $\rho$, if $\sigma + \tau = \sigma + \rho$, then $\tau = \rho$. This property is not shared by all order-types, since for example if $\eta$ is the order-type of the rationals under their usual ordering, we have.

$$\eta + (1 + \eta) = \eta = \eta + \eta, \text{ but not } 1 + \eta = \eta.$$

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We can see that even if we restrict the right-hand components to the class of ordinals, the left-cancellation law is still not universally valid: for example, we have $\omega\omega^* + \omega = \omega\omega^* = \omega\omega^* + \omega 2$, but not $\omega = \omega 2$.

The left-cancellation law for ordinals in no way depends, however, upon the right-hand components being ordinals. We give the proof of the general case, not because it is novel or at all difficult, but because it gives us the basic idea of "regressiveness".

Theorem 1 *For any order-types* $\alpha$, $\beta$, *and any ordinal* $\sigma$, *if* $\sigma + \alpha = \sigma + \beta$, *then* $\alpha = \beta$.

*Proof:* Let $A$, $B$, $S$ be representative sets for $\alpha$, $\beta$, $\sigma$ respectively, and let $f : S \dot{+} A \simeq S \dot{+} B$ be an isomorphism. We claim that $f''A = B$, which of course shows that $A \simeq B$, i.e., $\alpha = \beta$. Suppose that $f''A \neq B$. Then either $f(a) \in S$ for some $a \in A$, or $f^{-1}(b) \in S$ for some $b \in B$. In the first case, the restriction $g = f|S$ gives an isomorphism $S \simeq T$, where $T$ is some proper initial segment of $S$, an obvious impossibility. In the second case, the inverse $g^{-1}$ gives a similar isomorphism.

Let us now define a few terms.

Definition 1 An order-type $\alpha$ is said to be left-cancellable (l.c.) if for any order-types $\beta$, $\gamma$ such that $\alpha + \beta = \alpha + \gamma$, we have $\beta = \gamma$.

At this point we note that an order-type may be l.c. amongst the ordinals, i.e., when the right-hand components are ordinals, but still not l.c. A case in point is $\eta$; as was seen above, $\eta$ is not l.c., but if $\sigma$, $\tau$ are any ordinals such that $\eta + \sigma = \eta + \tau$, then $\sigma = \tau$. The proof of this last assertion is an easy modification of the proof of Theorem 1 above.

Definition 2 Given two order-types $\alpha$, $\beta$, we write "$\alpha \downarrow \beta$" if $\alpha$ is a proper initial segment of $\beta$, i.e., if $\beta = \alpha + \gamma$ for some nonzero order-type $\gamma$.

If either $\alpha \downarrow \beta$ or $\alpha = \beta$, then we write "$\alpha \underline{\downarrow} \beta$". An order-type $\alpha$ is said to be regressive if $\alpha \downarrow \alpha$.

Although the binary relation $\downarrow$ has a superficial resemblance to an order relation, and in fact when we restrict the field of $\downarrow$ to the class of ordinals, $\downarrow$ becomes identical with the standard well-ordering of the ordinals, it is not one. For firstly, since we have $\eta \downarrow \eta$ (as $\eta + \eta = \eta$), if $\downarrow$ were an order relation, it would have to be a reflexive order relation, and thus be antisymmetric. However, since we have $\eta \downarrow \eta + 1$ (obviously) and $\eta + 1 \downarrow \eta$ (from $\eta + 1 + \eta = \eta$), antisymmetry would give the contradiction $\eta = \eta + 1$. Thus $\downarrow$ is not an order relation. In practice, trying to determine whether or not a given order-type is l.c. directly from the definition can be a little cumbersome, and it is easier to work with the related concept of regressiveness.

The connection between these two concepts is very straightforward:

Theorem 2 *An order-type is* l.c. *if and only if it is not regressive.*

*Proof:* Let $\alpha$ be an order-type, and assume that $\alpha$ is regressive. Thus there is some order-type $\beta \neq 0$ such that $\alpha + \beta = \alpha$. Since of course $\alpha = \alpha + 0$, it follows immediately that $\alpha$ is not l.c. Now suppose that $\alpha$ is not l.c.; then there are order-types $\beta, \gamma$ such that $\beta \neq \gamma$ but $\alpha + \beta = \alpha + \gamma$. Let $A$, $B$, $C$ be representative sets for $\alpha, \beta, \gamma$ respectively, and let $f : A \dot{+} B \simeq A \dot{+} C$ be an isomorphism. Since $\beta \neq \gamma$, we cannot have $f''B = C$. Thus either there is $a \in A$ with $f(a) \in C$, or else there is $a \in A$ with $f^{-1}(a) \in B$: without loss of generality, we may assume the latter. But now $D = f''A$ is a proper initial segment of $A$, from which it follows at once that $\alpha$ is regressive.

The simplest example of a regressive order-type is of course $\omega^*$: $\omega^* + 1 = \omega^*$. We now show that $\omega^*$ is in fact a paradigm example, which can be said (in a certain sense) to generate the class of regressive order-types.

Theorem 3 *Let $\alpha$ be an order-type. Then $\alpha$ is regressive if and only if $\alpha = \beta + \gamma\omega^*$ for some order-types $\beta, \gamma$ with $\gamma \neq 0$.*

*Proof:* If $\alpha = \beta + \gamma\omega^*$ for some order-types $\beta, \gamma$ with $\gamma \neq 0$, then we have $\alpha + \gamma = \beta + \gamma\omega^* + \gamma = \beta + \gamma\omega^* = \alpha$, and thus $\alpha \downarrow \alpha$. Now suppose that $\alpha \downarrow \alpha$, i.e., $\alpha + \gamma = \alpha$ for some order-type $\gamma \neq 0$, and let $A$, $C$ be representative sets for $\alpha, \gamma$ respectively, and let $f : A \dot{+} C \simeq A$ be an isomorphism.

Now for any positive integer $n$, the $(n+1)$st iterate $f^{n+1}$ of $f$ has domain $A$, and thus for any $c \in C$ and any $n > 0$, $f^n(c)$ is defined and is an element of $A$. Put $D = \{f^n(c) : c \in C, n > 0\}$. We claim that $D$ (under the ordering induced by that of $A$) is a nonempty final segment of $A$ with order-type $\text{o}(D) = \gamma\omega^*$. Since $C \neq \emptyset$, clearly $D \neq \emptyset$. Take any $d \in D$, $a \in A$ with $d \leqslant a$; then $d = f^n(c)$ for some $n > 0$ and some $c \in C$. Let $k$ be the smallest positive integer such that $f^k(c') \leqslant a$ for some $c' \in C$. Then certainly $f^{-k}(a)$ is defined, and $f^{-k}(a) \geqslant c'$, whence $f^{-k}(a) \in C$, and thus $a \in D$. Hence $D$ is a nonempty final segment of $A$. It remains to show that $\text{o}(D) = \gamma\omega^*$. Let $N = \{1, 2, 3, \ldots\}$, under the order $<^*$ defined by $m <^* n$ if $n < m$, be a representative set for $\omega^*$, and consider the map $g : D \to C \dot{\times} N$ defined by $g(f^n(c)) = (c, n)$. Since for any $f^n(c), f^{n'}(c') \in D$ we have $f^n(c) < f^{n'}(c')$ if either $n > n'$ or else $n = n'$ and $c < c'$, it is clear that $g$ is an isomorphism: $D \simeq C \dot{\times} N$. Thus $\text{o}(D) = \text{o}(C \dot{\times} N) = \gamma\omega^*$.

It is obvious from this last result that for any order-type $\alpha \neq 0$ and any ordinal $\sigma \geqslant \omega$, the order-type $\alpha\sigma^*$ is regressive. It is perhaps not quite so obvious that for each ordinal $\sigma \neq 0$, there is an order-type $\alpha$ such that $\alpha\sigma$ is regressive. We demonstrate this via a more restricted result of the same kind.

Theorem 4 *For each prime component $\sigma$, there is an order-type $\alpha$ such that $\alpha\sigma$ is regressive.*

*Proof:* For $\sigma = 1$ the claim is obvious; thus take $\sigma > 1$, and let $S$ be a representative set for $\sigma$. We need to construct a set $A$ and an isomorphism $f : A \dot{\times} S \simeq B$, where $B$ is some proper initial segment of $A \dot{\times} S$. We define

$A$ to be the (lexicographically ordered) set of all $\sigma$-sequences $a = (a_\xi)_{\xi<\sigma}$ such that $a_\xi \in \{0, 1\}$ for each $\xi < \sigma$. For each $\zeta < \sigma$, we define $A_\zeta \subseteq A$ by

$$A_\zeta = \{a \in A;\ a_\zeta = 0 \text{ and } a_\xi = 1 \text{ for } \xi < \zeta\}.$$

Clearly $A_\zeta$ is an interval of $A$; we claim that $A_\zeta \simeq A$. To show this, we define a map $g : A_\zeta \to A$ by setting, for each $a \in A_\zeta$, $g(a)$ to be the sequence $(a_{\zeta+\xi})_{\xi<\sigma}$. Since $\sigma$ is a prime component and thus $\zeta + \xi < \sigma$ for each $\xi < \sigma$, $g$ is well-defined. To show that $g$ is surjective, we simply take $a \in A$ and construct the $\zeta + \sigma$-sequence $b$ by $b_\xi = 1$ for $\xi < \zeta$, $b_\zeta = 0$, and $b_\xi = a_{\xi-(\zeta+1)}$ for $\zeta < \xi < \zeta + \sigma$. Again the fact that $\sigma$ is a prime component tells us that we have a $\sigma$-sequence, and so $b \in A_\zeta$. But obviously $g(b) = a$. Finally, it is clear that $g$ does not disturb the orderings, and is thus an isomorphism. This establishes our claim that $A_\zeta \simeq A$.

For our representative set $S$ for $\sigma$, we may simply take the set of all ordinals $\tau < \sigma$. Now for each $\tau \in S$, we let $h_\tau$ be the inverse $g^{-1}$ of the isomorphism $g : A_\tau \simeq A$ defined above, and we define $f : A \dot{\times} S \to A$ by $f(a, \tau) = h_\tau(a)$. Since the $A_\tau$ form a partitioning of $A$, $f$ is bijective; and clearly $f$ is order-preserving. Thus $A \dot{\times} S \simeq A$, and as $S$ has a first element and $\sigma > 1$, $A$ is (isomorphic to) a proper initial segment of $A \dot{\times} S$, thus showing that $\alpha\sigma$ is regressive.

Theorem 5 *For any ordinal $\sigma \neq 0$, there is an order-type $\alpha$ such that $\alpha\sigma$ is regressive.*

*Proof:* Express $\sigma$ as the sum $\tau_0 + \tau_1 + \ldots + \tau_n$ of a finite nonascending sequence of prime components, and let $\alpha$ be an order-type such that $\alpha\tau_n \dotplus \alpha\tau_n$. Then we have

$$\alpha\sigma = \alpha\tau_0 + \alpha\tau_1 + \ldots + \alpha\tau_n \dotplus \alpha\tau_0 + \alpha\tau_1 + \ldots + \alpha\tau_n = \alpha\sigma,$$

and thus $\alpha\sigma$ is regressive.

We continue looking at order-types of the form $\alpha\sigma$, $\sigma$ an ordinal.

Theorem 6 *Let $\alpha$ be an order-type, and $\sigma$ an ordinal. Then:*

(a) *If $\alpha\sigma$ is regressive, then $\alpha$ is regressive.*

and

(b) *If $\sigma$ is a successor ordinal, then $\alpha\sigma$ is regressive if and only if $\alpha$ is regressive.*

*Proof:* (a) Assume that $\alpha\sigma$ is regressive but that $\alpha$ is not, and let $f : A \dot{\times} S \simeq B$ be an isomorphism, where $A$, $S$ are representative sets for $\alpha$, $\sigma$ respectively and $B$ is a proper initial segment of $A \dot{\times} S$. Clearly we have $S \neq \emptyset$. We define a map $g : S \to S$ as follows:

$$g(s) = \min\{t \in S;\ f''(A \dot{\times} \{s\}) \cap (A \dot{\times} \{t\}) \neq \emptyset\},\ s \in S.$$

Since $S$ is well-ordered and $B \subseteq A \dot{\times} S$, $g$ is certainly well-defined. We claim that $g(s) < s$ for some $s \in S$.

Firstly, if $\sigma$ is limit, then we must have $B$ an initial segment of $A \dot{\times} R$ for some proper initial segment $R$ of $S$, whence it follows of course that $g(s) < s$ for all $s \in S - R$.

Secondly, if $\sigma$ is successor, let $s^0$ be the last element of $S$: obviously $g(s^0) \leqslant s^0$, and if we had $g(s^0) = s^0$, then $f''(A \dot{\times} \{s^0\})$ would be a proper initial segment of $A \dot{\times} \{s^0\}$, from which it follows at once that $\alpha$ is regressive. This establishes our claim.

Put $r = \min\{s \in S;\ g(s) < s\}$. Then $g(r) < r$, and $f(a, r) \in A \dot{\times} \{g(r)\}$ for some $a \in A$. But then the non-regressiveness of $\alpha$ implies that $f''(A \dot{\times} \{g(r)\}) \cap A \dot{\times} \{t\} \neq \emptyset$ for some $t \in S$ with $t < g(r)$. That is, $g(g(r)) < g(r)$, contradicting the minimality of $r$.

(b) In view of (a), it suffices to show that if $\sigma$ is successor, then $\alpha \downarrow \alpha$ implies $\alpha\sigma \downarrow \alpha\sigma$. But this is easy, for if $\sigma = \rho + 1$, then we have, for $\alpha$ regressive,

$$\alpha\sigma = \alpha\rho + \alpha \downarrow \alpha\rho + \alpha = \alpha\sigma.$$

Theorem 6 is "best possible" in the sense that the converse to Theorem 6 (a) is false. We show this by demonstrating that for each limit ordinal $\sigma \geqslant \omega$, the order-type $\omega^*\sigma$ is not regressive, although of course $\omega^*$ is regressive. Firstly, let us prove that $\omega^*\omega$ is not regressive. Suppose the contrary; then we would have $\omega^*\omega = \beta$ for some proper initial segment $\beta \neq 0$ of $\omega^*\omega$. It is easy to see, however, that each such $\beta$ is of the form $\omega^*n$ for some positive integer $n$, and obviously there is no $n$ such that $\omega^*\omega = \omega^*n$. Now let $\sigma$ be an arbitrary nonzero limit ordinal; thus $\sigma = \omega\tau$ for some ordinal $\tau > 0$. But now, if $\omega^*\sigma = (\omega^*\omega)\tau$ were regressive, then $\omega^*\omega$ would be regressive by Theorem 6 (a), contradicting the result just established.

We wish now to give a simple but rather interesting criterion for the regressiveness of $\alpha\sigma$, $\alpha$ an order-type and $\sigma$ a (nonzero) limit ordinal. A preliminary result is required.

Theorem 7 *Let $A$ be a set, and suppose that $A \simeq I$, $I$ being some interval of $A$. Then $A \simeq S$, where $S$ is the initial segment $\{a \in A;\ a \leqslant x$ for some $x \in I\}$ of $A$.*

*Proof:* Let $f : A \simeq I$ be an isomorphism, and define a map $g : A \to A$ by $g(a) = a$ for all $a \in \{b \in A;\ a \leqslant f(a)\}$, and $g(a) = f(a)$ otherwise. Putting $B = g''A$, we see easily that $g : A \simeq B$ is an isomorphism, and that $B \subseteq S$. It thus suffices to show that $S \subseteq B$. Take $x \in S$. If $x \notin I$, then $x < f(x) \in I$, and so $x = g(x) \in B$. Therefore it remains to show that $I \subseteq B$, and so we now take $x \in I$. But then $y = f^{-1}(x)$ is a well-defined member of $A$. If $y > x$, then $g(y) = f(y) = x$, and so $x \in B$. If, on the other hand, $y \leqslant x$, then $x = f(y) \leqslant f(x)$, and once again $x = g(x) \in B$. This proves our result.

Theorem 8 *Let $\alpha$ be any order-type, $\sigma$ any nonzero limit ordinal, and let the smallest positive remainder of $\sigma$ be $\rho$. Then $\alpha\sigma \downarrow \alpha\sigma$ if and only if $\alpha\rho \,\underline{\downarrow}\, \alpha$.*

*Proof:* Assume firstly that $\alpha\rho \,\underline{\downarrow}\, \alpha$, and let $\tau$ be any ordinal such that

$\sigma = \tau + \rho$. Then $\alpha\sigma = \alpha\tau + \alpha\rho \underline{\downarrow} \alpha\tau + \alpha \downarrow \alpha\tau + \alpha\rho = \alpha\sigma$. Now suppose that $\alpha\sigma$ is regressive, and let $A$, $S$, $R$ be representative sets for $\alpha$, $\sigma$, $\rho$ respectively. Then there is an isomorphism $f : A \dot{\times} S \simeq B$, where $B$ is some proper initial segment of $A \dot{\times} S$. Since $\sigma$ is limit, there is a proper initial segment $S^0$ of $S$ such that $B$ is an initial segment of $A \dot{\times} S^0$, and we may assume $o(S^0)$ to be minimal with respect to this property. We now consider two cases.

(1) $B \neq A \dot{\times} S^0$. From the minimality of $o(S^0)$, we conclude that $S^0 = S^{\#} \dotplus \{s\}$ for some initial segment $S^{\#}$ of $S$ and some $s \in S$, and that $B = A \dot{\times} S^{\#} \dotplus C$ for some proper initial segment $C$ of $A$. Of course $C \neq \emptyset$. Put $T = S - S^0$: then $T$ is a nonempty final segment of $S$, and so $o(T) \geqslant \rho$; thus we may without loss of generality assume that $R$ is a final segment of $T$. But then there is a nonempty final segment $R'$ of $R$ such that $f''(A \dot{\times} R')$ is an interval of $C$: put $C = H \dotplus I \dotplus J$, where $I = f''(A \dot{\times} R')$, and let $r$ be the first element of $R'$. Then Theorem 7 tells us that $f''(A \dot{\times} \{r\}) \simeq H \dotplus f''(A \dot{\times} \{r\})$, whence $A \dot{\times} R' \simeq H \dotplus I$, which gives us $\alpha\rho \underline{\downarrow} \alpha$.

(2) $B = A \dot{\times} S^0$. We thus have an isomorphism $f : A \dot{\times} S \simeq A \dot{\times} S^0$, where $S^0$ is a proper initial segment of $S$, and we may assume $f$ to have been chosen so that $o(S^0)$ is minimal in this respect. However, $g = f^2$ is certainly an isomorphism: $A \dot{\times} S \simeq B'$, with $B'$ a proper initial segment of $B = A \dot{\times} S^0$, and so we cannot have $B' = A \dot{\times} S'$ for any initial segment $S'$ of $S$. Therefore we are in the situation of (1), and can proceed as there to show that $\alpha\rho \underline{\downarrow} \alpha$.

At first glance, it may appear plausible to try to strengthen Theorem 8 by replacing "$\alpha\rho \underline{\downarrow} \alpha$" with "$\alpha\rho = \alpha$". This, however, cannot be done, as we show by an example.

Take $\alpha = \eta + 1$, $\sigma = \omega$. Thus we have $\rho = \omega$. By considering the semi-open intervals $(n/(n+1), (n+1)/(n+2)]$ of the set $(0, 1)$ of rationals, we see that $(\eta + 1)\omega = \eta$. Hence $(\eta + 1)\omega$ is regressive and (in accordance with Theorem 8) $(\eta + 1)\omega = \eta \underline{\downarrow} \eta + 1$. But obviously $\eta \neq \eta + 1$.

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