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NEGATION, MATERIAL EQUIVALENCE, AND CONDITIONED NONCONJUNCTION: COMPLETENESS AND DUALITY

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1 *Object of paper* The object of this paper is threefold: to prove the functional incompleteness of $\{\sim, \equiv\}$ without appeal to a tedious analysis of cases; to give a proof simpler than the one in [2], p. 284 f., of the non-existence of indigenous Sheffer connectives for $\{\sim, \equiv\}$; and to furnish self-dual (in Church's sense) ternary Sheffer connectives for propositional logic.

2 *Functional incompleteness of* $\{\sim, \equiv\}$

Lemma 1 *Let A be a propositional wff containing no connectives other than $\sim$ and $\equiv$, and let A' be the wff that results when all occurrences of $\sim$ in A are deleted. Then A is equivalent to A' or to $\sim A'$.*

Proof: Let $\Leftrightarrow$ signify (semantic) equivalence. Since $B \Leftrightarrow \sim\sim B$ and since $\sim(B \equiv C) \Leftrightarrow \sim B \equiv C \Leftrightarrow B \equiv \sim C$, Lemma 1 follows from the substitutivity of equivalents by induction.

Theorem 1 $\{\sim, \equiv\}$ *is functionally incomplete.*

Proof: We call a truth-value α a *fixed point for* a wff B just in case the value of B is α when all its variables are assigned the value α, and we say that α is a *fixed point for* an n-ary connective $\otimes$ just in case α is a fixed point for $\otimes(p_1, \ldots, p_n)$. Notice that $\mathbf{t}$ is a fixed point for $\equiv$ and for any wff that contains no connectives other than $\equiv$. Suppose that & is definable from $\{\sim, \equiv\}$. Then by Lemma 1 there is a wff $A(p, q)$ containing no connectives other than $\equiv$ such that $p \,\&\, q$ is equivalent to $A(p, q)$ or to $\sim A(p, q)$. But since $\mathbf{t}$ is a fixed point for both $p \,\&\, q$ and $A(p, q)$, $p \,\&\, q$ cannot be equivalent to $\sim A(p, q)$. So, $p \,\&\, q \Leftrightarrow A(p, q)$. Therefore, $p \,\&\, \sim q \Leftrightarrow A(p, \sim q)$. By Lemma 1 again, $A(p, \sim q)$ is equivalent to $A(p, q)$ or to $\sim A(p, q)$. Since $\mathbf{t}$ is not a fixed point for $p \,\&\, \sim q$, we have $A(p, \sim q) \Leftrightarrow p \,\&\, \sim q \Leftrightarrow \sim A(p, q)$. So, $\sim(p \,\&\, \sim q) \Leftrightarrow A(p, q) \Leftrightarrow p \,\&\, q$, which is a contradiction. So & is not definable from $\{\sim, \equiv\}$. Theorem 1 follows.

3 *Indigenous Sheffer connectives* An n-ary connective $\otimes$ is said to be a

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Sheffer connective for a set Δ of truth-functional connectives just in case every member of Δ is definable from $\otimes$. We say that $\otimes$ is *indigenous* to Δ just in case $\otimes$ is definable from Δ. We call **t** and **f** *opposites*. Let Σ and Σ' be valuations, i.e., assignments of truth-values to all variables. Then Σ' is said to be *opposite to* Σ just in case, for every variable v, $\Sigma'(v)$ is opposite to $\Sigma(v)$. A wff A is said to be (semantically) *dual to* a wff B just in case, for every valuation Σ, $\Sigma(A)$ is opposite to $\Sigma'(A)$, where Σ' is the valuation opposite to Σ. We say that A is (semantically) *self-dual* just in case A is dual to A.

Lemma 2 *Where $\otimes$ is an n-ary connective, if $\otimes(p_1, \ldots, p_n)$ is self-dual, then any wff B containing no connectives other than $\otimes$ is self-dual.*

Proof: Induction on the number of occurrences of $\otimes$ in B.

Theorem 2 *No connective indigenous to $\{\sim, \equiv\}$ is a Sheffer connective for $\{\sim, \equiv\}$.*

Proof: Suppose that $\otimes$ is an n-ary connective indigenous to $\{\sim, \equiv\}$ and that $\otimes$ is a Sheffer connective for $\{\sim, \equiv\}$. Since $\otimes$ is indigenous to $\{\sim, \equiv\}$, by Lemma 1 there is a wff $A(p_1, \ldots, p_n)$ containing no connectives other than $\equiv$ such that $\otimes(p_1, \ldots, p_n)$ is equivalent to $A(p_1, \ldots, p_n)$ or to $\sim A(p_1, \ldots, p_n)$. If $\otimes(p_1, \ldots, p_n)$ were equivalent to $A(p_1, \ldots, p_n)$, **t** would be a fixed point for $\otimes$. Then $\sim$ would not be definable from $\otimes$, and so $\otimes$ would not be a Sheffer connective for $\{\sim, \equiv\}$. So $\otimes(p_1, \ldots, p_n)$ must be equivalent to $\sim A(p_1, \ldots, p_n)$. Let **k** be the number of *occurrences* of variables in $A(p_1, \ldots, p_n)$. Suppose that **k** is odd. Then by Lemma 1, $A(\sim p_1, \ldots, \sim p_n) \Leftrightarrow \sim A(p_1, \ldots, p_n)$. But then

$$\sim\otimes(\sim p_1, \ldots, \sim p_n) \Leftrightarrow A(\sim p_1, \ldots, \sim p_n) \Leftrightarrow \sim A(p_1, \ldots, p_n) \Leftrightarrow \otimes(p_1, \ldots, p_n).$$

But $\otimes(p_1, \ldots, p_n)$ is self-dual iff $\otimes(p_1, \ldots, p_n) \Leftrightarrow \sim\otimes(\sim p_1, \ldots, \sim p_n)$. So $\otimes(p_1, \ldots, p_n)$ is self-dual. Hence, by Lemma 2, any wff containing no connectives other than $\otimes$ is also self-dual. But $p \equiv p$ is not self-dual. So $\equiv$ is not definable from $\otimes$, which contradicts our assumption that $\otimes$ is a Sheffer connective for $\{\sim, \equiv\}$. Therefore, **k** must be even. So, $A(\sim p_1, \ldots, \sim p_n) \Leftrightarrow A(p_1, \ldots, p_n)$. So **f** is a fixed point for $\sim A(p_1, \ldots, p_n)$, and hence also for $\otimes(p_1, \ldots, p_n)$. But then $\sim$ is not definable from $\otimes$, since **f** is not a fixed point for $\sim p$. Again, this contradicts our assumption that $\otimes$ is a Sheffer connective for $\{\sim, \equiv\}$. Theorem 2 follows by indirect proof.

Several corollaries can be drawn from the above about self-dual connectives. Let $\otimes_1$ and $\otimes_2$ be n-ary connectives. We say that $\otimes_1$ is (semantically) *dual to* $\otimes_2$ just in case $\otimes_1(p_1, \ldots, p_n)$ is dual to $\otimes_2(p_1, \ldots, p_n)$. We call $\otimes_1$ (semantically) *self-dual* just in case $\otimes_1$ is dual to $\otimes_1$.

Theorem 3 *No self-dual connective is functionally complete in propositional logic.*

Theorem 4 *If Δ is a set of self-dual connectives, then Δ is functionally incomplete.*

4 *Conditioned nonconjunction* In his treatment of primitive connectives for propositional logic in [1], pp. 129-139, Church emphasizes the value of a "self-dual" set of functionally complete connectives. Church's notion of "self-duality" is somewhat more liberal than ours. For Church, $\otimes$ is *self-dual*$_C$ just in case $\otimes(p_1, \ldots, p_n)$ is dual to at least one of the wffs that result when the operands of $\otimes$ are permuted (the identity permutation is not excluded). Church's set of truth, falsity, and conditioned disjunction is a set of self-dual$_C$ independent connectives for propositional logic. A set of self-dual$_C$ primitives allows one to define connectives in such a way that, for any primitive or defined n-ary connective $\otimes$, the dual of $\otimes(A_1, \ldots, A_n)$ is $\otimes'(A_1', \ldots, A_n')$, where $\otimes'$ is the connective dual to $\otimes$ and where $A_1', \ldots, A_n'$ are the respective duals of $A_1, \ldots, A_n$. Church states that "For certain purposes there are advantages in a complete system of primitive connectives which consists of one connective only, although to obtain such a system it is necessary to make a rather artificial choice of the primitive connective (and also to abandon any requirement of self-duality if the primitive connective is to be no more than binary)," *cf.* [1], p. 133. The parenthetical statement suggests that there may be self-dual$_C$ single primitives for propositional logic of degree greater than 2. This suggestion is shown to be true by Theorem 5.

Let $⦶$ be the ternary connective defined by the table below. Note that $⦶(p, q, r)$ may be read as "not p or not r according as q or not q". Hence we call $⦶$ *conditioned nonconjunction* in analogy to Church's conditioned disjunction which we will symbolize by $\ominus$; one may read $\ominus(p, q, r)$ as "p or r according as q or not q".

p	q	r	$⦶(p, q, r)$	$⦶(r, q, p)$	$\ominus(p, q, r)$
t	t	t	f	f	t
t	t	f	f	t	t
t	f	t	f	f	t
t	f	f	t	f	f
f	t	t	t	f	f
f	t	f	t	t	f
f	f	t	f	t	t
f	f	f	t	t	f

Theorem 5 *Conditioned nonconjunction is a self-dual*$_C$ *functionally complete primitive for propositional logic.*

Proof: That $⦶$ is self-dual$_C$ is evident from the table above since $⦶(p, q, r)$ is (semantically) dual to $⦶(r, q, p)$. And $⦶$ is functionally complete since $⦶(p, q, q)$ is equivalent to $p|q$ (nonconjunction). Actually, it takes only a glance at the above table to recognize that $⦶$ is a Sheffer connective for propositional logic, since $⦶$ satisfies Post's criterion for Sheffer connectives. Let $\mathfrak{T}$ be the truth-table for a connective $\otimes(p_1, \ldots, p_n)$ such that the value assignment portions of the rows of $\mathfrak{T}$ are ordered lexicographically, with **t** before **f**, as in the table above. (Note that when its rows are so ordered, the value assignment to the variables given by the $\mathfrak{i}$'th row from the top of $\mathfrak{T}$ is dual to the value assignment given by the $\mathfrak{i}$'th row from the bottom, $1 \leqslant \mathfrak{i} \leqslant 2^{n-1}$.) Post's criterion is this: $\otimes$ is a Sheffer connective for propositional logic iff, in the table $\mathfrak{T}$, (a) the top and bottom entries under $\otimes$ are respectively **f** and **t**, and (b) for at least one positive integer $\mathfrak{i}$, $1 < \mathfrak{i} < 2^{n-1}$, the $\mathfrak{i}$'th entry from the top under $\otimes$ is the same truth-value as the $\mathfrak{i}$'th entry from the bottom under $\otimes$. Clearly the table above satisfies Post's criterion, showing that $\oplus$ is a Sheffer connective. Here, incidentally, is an especially simple proof that Post's criterion works. The criterion is obviously necessary, for if the top entry under $\otimes$ were **t** or the bottom entry were **f**, $\otimes$ would have fixed points. And if (a) is satisfied but (b) is not, then $\otimes$ is (semantically) self-dual, and so not functionally complete. To see that Post's criterion is sufficient, suppose it is satisfied by a truth-table $\mathfrak{T}$ for $\otimes$, and let $\mathfrak{i}$ be an integer satisfying (b). Let α be the truth-value that occurs beneath $\otimes$ on the $\mathfrak{i}$'th row (from top or bottom). Let $\alpha_1, \ldots, \alpha_n$ be the values assigned to $p_1, \ldots, p_n$ respectively on row $\mathfrak{i}$. Let v_k be p or q according as α_k is **t** or **f**. Then $\otimes(v_1, \ldots, v_n)$ is equivalent to $p | q$ or $p \downarrow q$ (nondisjunction) according as α is **t** or **f**. Hence $\otimes$ is a Sheffer connective.

Like {**t**, **f**, $\ominus$}, the set {$\oplus$} makes possible a simple and elegant treatment of duality: for any n-ary connective $\otimes$, one can so define $\otimes$ that the dual of $\otimes(A_1, \ldots, A_n)$ is $\otimes'(A_1', \ldots, A_n')$, where $A_1', \ldots, A_n'$ are the respective duals of $A_1, \ldots, A_n$ and where $\otimes'$ is the connective dual to $\otimes$. Furthermore, extremely simple definitions of all the singulary and binary connectives can be given: each such connective can be defined by using only one occurrence of $\oplus$ in addition to non-prenex tildes. The following list provides such definitions for negation and for the non-degenerate binary connectives; dual connectives are paired.

$$\begin{aligned}
\sim A &=_{Df} \oplus(A, A, A)\\
A \downarrow B &=_{Df} \oplus(A, A, B)\\
A \mid B &=_{Df} \oplus(B, A, A)\\
A \vee B &=_{Df} \oplus(\sim A, A, \sim B)\\
A \,\&\, B &=_{Df} \oplus(\sim B, A, \sim A)\\
A \not\subset B &=_{Df} \oplus(A, A, \sim B)\\
A \supset B &=_{Df} \oplus(\sim B, A, A)\\
A \subset B &=_{Df} \oplus(\sim A, B, B)\\
A \not\supset B &=_{Df} \oplus(B, B, \sim A)\\
A \equiv B &=_{Df} \oplus(\sim B, A, B)\\
A \not\equiv B &=_{Df} \oplus(B, A, \sim B)
\end{aligned}$$

Sobociński [3] has proved that there are quaternary connectives $\otimes$, but no connectives of smaller degree, such that all the singulary and binary connectives can be defined from {$\otimes$, **t**, **f**} using just one occurrence of $\otimes$. Thus we know that the singularly and binary connectives cannot all be so defined from {$\oplus$, **t**, **f**}, although many of them can be. The only four that cannot be so defined are &, v, ≡, and ≢.[1]

1. I am indebted to John Tinnon for calling Post's criterion to my attention.

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[2] Hendry, H. E., and G. J. Massey, "On the concepts of Sheffer functions," in J. K. Lambert, ed., *The Logical Way of Doing Things*, Yale University Press, New Haven (1969), pp. 279-293.

[3] Sobociński, Bolesław, "On a universal decision element," *The Journal of Computing Systems*, vol. 1 (1953), pp. 71-80.

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