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THREE-VALUED FREE TENSE LOGIC

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1 *Introduction* In [11] Strawson suggested that sentences containing singular terms, i.e., proper names or definite descriptions, may be true at one time, false at another, and unvalued (neither true nor false) at a third due to a failure of reference. Both Van Fraassen [12] and Woodruff [13] have supplied semantic interpretations of standard quantificational logics which embody Strawson's theory. In this paper, we extend their results to a quantificational version of the tense logic $\mathbf{K}_t$.[1] Our semantics reflects the fact that in temporal contexts there are several ways a singular term may fail to refer. For example, both of the sentences

(1) The King of France is wise.
(2) Sherlock Holmes lives on Baker Street.

are (now) unvalued, whereas of

(3) The King of France was wise.
(4) Sherlock Holmes lived on Baker Street.

only the last is. Furthermore, taking issue with a point of Ryle's,[2] if "Junior" (timelessly) is the name of my yet unborn son, then the first *but not the second* of the following

(5) Junior (now) goes to school.
(6) Junior will go to school.

is unvalued. The point here is that the singular terms which do and do not refer may vary from time to time.

2 *Syntactical Preliminaries* Among the signs of $\mathbf{QK}_t^*$ are the usual denumerable infinities of sentence parameters, m-adic predicate parameters, individual variables, and individual parameters (doing duty for singular terms), plus the connectives '$\sim$' and '$\supset$', parentheses, the quantifier letter '$\forall$', and the tense operators '$\mathbf{F}$' (read, "It will be the case that") and '$\mathbf{P}$' (read, "It has been the case that"). Two additional operators are defined in terms of $\mathbf{F}$ and $\mathbf{P}$, '$\mathbf{G}$' (read, "It will always be the case

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that") as '$\sim \mathsf{F} \sim$' and 'H' (read, "It has always been the case that") as '$\sim \mathsf{P} \sim$'. The wffs of QK_t^* are defined in the standard way, with the extra case that if A is a wff so are $\mathsf{F}A$ and $\mathsf{P}A$. In what follows we shall employ 'X' and 'Y' to refer to the individual variables of QK_t^*, 'X' to refer to the individual parameters, 'A', 'B', and 'C' to refer to the wffs and those formulas like wffs except for having one or more individual variables where a wff would have individual parameters, and we shall understand by '$A(I'/I)$' the result of replacing every occurrence of I in A by I'. Lastly, any wff not having occurrences of '$\sim$', '$\supset$', '$\forall$', 'F', and 'P' shall be called an *atomic wff*, and any set of wffs to which $\aleph_0$ individual parameters are foreign shall be called *infinitely extendible.*

We shall count all those wffs of QK_t^* as *axioms* which have the following forms:

A1. $A \supset (B \supset A)$,
A2. $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$,
A3. $(\sim B \supset \sim A) \supset (A \supset B)$,
A4. $A \supset (\forall X)A$,
A5. $(\forall X)(A \supset B) \supset ((\forall X)A \supset (\forall X)B)$,
A6. $(\forall Y)((\forall X)A \supset A(Y/X))$,
A7. $(\forall Y)(\forall X)A \supset (\forall X)(\forall Y)A$,
A8. $(\forall X)A \supset A(\mathsf{X}/X)$, *where A contains no tense operators,*[3]
A9. $(\forall X)A$, *where—for any individual parameter X—$A(\mathsf{X}/X)$ is an axiom,*
A10. $\mathsf{G}(A \supset B) \supset (\mathsf{G}A \supset \mathsf{G}B)$,
A11. $\mathsf{PG}A \supset A$,
A12. $\mathsf{G}(\forall X)A \supset (\forall X)\mathsf{G}A$,
A13. $\mathsf{G}A$, *where A is an axiom,*
A14. $A(\mathsf{P}/\mathsf{F})(\mathsf{F}/\mathsf{P})$, *where A is an axiom.*[4]

In addition, Modus Ponens will serve as the rule of inference.

A set S of wffs will be said to be *syntactically inconsistent* in QK_t^* if any wff of the sort $\sim(A \supset A)$ is derivable from S, otherwise, S will be said to be *syntactically consistent.*

3 *The Semantics of* QK_t^*[5] By a *truth-value assignment* for QK_t^* we shall understand any function from the sentence parameters and zero or more of the remaining atomic wffs (i.e., those containing individual parameters) to $\{1, 0\}$ (where 1 is the truth-value "true" and 0 "false"). Those atomic wffs not receiving a truth-value at an assignment $\mathfrak{a}$ shall be called *unvalued* at $\mathfrak{a}$. (Note that only those atomic wffs having occurrences of individual parameters can go unvalued at any assignment.)

Corresponding to each truth-value assignment for QK_t^* is a set E of individual parameters which occur in those atomic wffs assigned a truth-value by the assignment. Intuitively, E is the set of singular terms which designate objects according to the assignment. Where $\mathfrak{a}$ is a truth-value assignment and E the aforementioned set, we shall refer to E as the *parametric associate* of $\mathfrak{a}$.

A *history* for QK_t^* shall be any pair of the sort $\langle \mathscr{S}, R\rangle$ where $\mathscr{S}$ is a set of *indexed*[6] truth-value assignments and R is a dyadic relation on $\mathscr{S}$. As the truth conditions below reveal, $\mathcal{S}$ is the formal counterpart to the states of the world through time and R is the earlier-later than relation which orders (partially) these states. The members of $\mathcal{S}$ shall be called the *moments* of $\langle \mathcal{S}, R\rangle$.[7]

We now turn to the truth conditions for the wffs of $\mathbf{QK}_t^*$. These are as follows, where A is a wff, $\mathfrak{a}_n$ a moment of a history $\langle \mathcal{S}, R\rangle$, and E_n the parametric associate of $\mathfrak{a}_n$:

1. If A is atomic; A is true at $\mathfrak{a}_n$ iff $\mathfrak{a}_n(A) = 1$, false iff $\mathfrak{a}_n(A) = 0$, and unvalued otherwise.[8]
2. If A is a negation $\sim B$; A is true at $\mathfrak{a}_n$ iff B is false, false iff A is true, and unvalued otherwise.
3. If A is a conditional $B \supset C$; A is unvalued at $\mathfrak{a}_n$ iff either B or C is unvalued at $\mathfrak{a}_n$, false iff B is true and C is false, and true otherwise.
4. If A is a quantification $(\forall X)B$; A is unvalued at $\mathfrak{a}_n$ iff $A(\mathbf{X}/X)$ is unvalued for any member $\mathbf{X}$ of E_n, true iff $A(\mathbf{X}/X)$ is true for every member $\mathbf{X}$ of E_n, and false otherwise.
5. If A is of the sort $\mathbf{F}B$; A is true at $\mathfrak{a}_n$ iff B is true at some $\mathfrak{a}_p'$ such that $R(\mathfrak{a}_n, \mathfrak{a}_p')$, unvalued iff B is unvalued at some $\mathfrak{a}_p'$ such that $R(\mathfrak{a}_n, \mathfrak{a}_p')$ *and* B is not true at any $\mathfrak{a}_r''$ such that $R(\mathfrak{a}_n, \mathfrak{a}_r'')$, and A is false otherwise.[9]
6. If A is of the sort $\mathbf{P}B$; A is true at $\mathfrak{a}_n$ iff B is true at some $\mathfrak{a}_k'$ such that $R(\mathfrak{a}_k', \mathfrak{a}_n)$, unvalued iff B is unvalued at some $\mathfrak{a}_k'$ such that $R(\mathfrak{a}_k', \mathfrak{a}_n)$ *and* B is not true at any $\mathfrak{a}_j''$ such that $R(\mathfrak{a}_j'', \mathfrak{a}_n)$, and A is false otherwise.

Given these truth conditions, which for the connectives are essentially Bochvar's,[10] it is easily shown that the converse of A12, $(\forall X(\mathbf{G}A \supset \mathbf{G}(\forall X)A$, which is similar to the well-known Barcan formula of modal logic, is falsifiable. Suppose there is an assignment $\mathfrak{a}_n$ at which only two individual parameters 'p_1' and 'p_2' are members of its parametric associate. Further suppose $A(p_1/X)$ and $A(p_2/X)$ are true at every other moment. Then $(\forall X)\mathbf{G}A$ is true at $\mathfrak{a}_n$. But also suppose $(\forall X)A$ to be false at all of these assignments because, e.g., their parametric associates contain other parameters in addition to p_1 and p_2. Then $\mathbf{G}(\forall X)A$ is sure to be false at $\mathfrak{a}_n$, and, hence, the entire conditional.

An infinitely extendible set of wffs S shall be said to be *semantically consistent* in $\mathbf{QK}_t^*$ if there is a moment of a history at which all of the members of S are true, and a noninfinitely extendible set shall be said to be semantically consistent if it is *isomorphic*[11] to an infinitely extendible set which meets the above condition. Otherwise, the set will be said to be semantically inconsistent.

As for validity, a slight alteration is required here of the standard account. We shall declare a wff to be *valid* in $\mathbf{QK}_t^*$ if it is *not false* at any moment of any history of $\mathbf{QK}_t^*$. Finally, a set S will be said to *entail* a wff A if $S \cup \{\sim A\}$ is semantically inconsistent.

4 *Soundness and Completeness* That our axiomatization is sound under our interpretation is trivially verified and need not concern us here. The interested reader will find the full proof in [6]. As for (strong) completeness, we will provide a sketch of a proof based on results of Henkin, Makinson, Hughes and Cresswell, and Leblanc.[12] The proof turns on the construction of a set of indexed relatively maximally consistent and omega-complete sets together with a dyadic relation on the membership of the set. The steps of this construction are as follows:

(1) Beginning with a syntactically consistent, infinitely extendible set S, we form its maximally consistent and omega-complete extension relative to a set S^E which contains all the wffs of $\mathbf{QK}_t^*$ which do not exhibit individual parameters and all those having individual parameters from the set E. E minimally contains all the individual parameters occurring in the members of S, and is the parametric associate of the extended set, S^∞.
(2) The individual parameters foreign to E are sorted into $\aleph_0$ cells—E^1, E^2, E^3, etc.
(3) For every wff of the sort $\mathbf{F}A$ in S^∞ a new set is formed consisting of A and every wff B such that $\mathbf{G}B$ is in S^∞. These sets are extended into maximally consistent and omega-complete sets relative to the set $S^{E'}$ which is like S^E above except for containing all those wffs exhibiting parameters from E^1 and those parameters exhibited in the membership of $\{A\} \cup \{B: \mathbf{G}B \in S^\infty\}$. A second batch of sets is similarly formed for each wff of the sort $\mathbf{P}A$ in S^∞ containing A and every wff B such that $\mathbf{H}B$ is in S^∞.
(4) Further sets are formed in the same manner for the results of step (3), except with E^2 in place of E^1, and sets are likewise formed for these with E^3 in place of E^2, etc.
(5) Each relatively maximally consistent and omega-complete set is (arbitrarily) assigned an index as it is formed.
(6) A relation R' is defined on the resulting set of indexed sets as follows: For any two sets $\mathbf{S}$ and $\mathbf{S}'$, if $\mathbf{S}'$ contains A for every $\mathbf{G}A$ in $\mathbf{S}$, then $R'(\mathbf{S}, \mathbf{S}')$.

Note that each indexed set has a parametric associate containing just those individual parameters occurring in the atomic wffs in the set and the negated atomic wffs in the set. Hence, corresponding to each indexed set is a truth-value assignment which assigns 1 to all the atomic wffs in the set, 0 to all the atomic wffs such that their negation is in the set, and leaves the rest of the atomic wffs of $\mathbf{QK}_t^*$ unvalued. And the parametric associates of these assignments are exactly the parametric associates of the indexed sets. Hence a set $\mathfrak{F}$ of indexed truth-value assignments can be formed which contains the corresponding truth-value assignment to each of the indexed sets. From the relation R' defined above, a relation R on $\mathfrak{F}$ is defined as follows:

Where $\mathfrak{a}_n$ and $\mathfrak{a}'_p$ are the corresponding truth-value assignments to $\mathbf{S}$ and $\mathbf{S}'$, respectively, if $R'(\mathbf{S}, \mathbf{S}')$ then $R(\mathfrak{a}_n, \mathfrak{a}'_p)$. The result, of course, is a history $\langle \mathfrak{F}, \mathbf{R}\rangle$.

A lengthy, but straightforward, induction shows that a wff A is a member of an indexed set $\mathbf{S}$ in the constructed set if and only if A is true at the corresponding moment $\mathfrak{a}_n$ of $\langle \mathfrak{F}, R\rangle$. A consequence of this is the semantical consistency of each of the indexed sets. But, the original set S is a subset of one of these indexed sets, and hence it too is semantically consistent. Therefore we have shown that every syntactically consistent set, which is infinitely extendible, is semantically consistent as well. And as for those sets which are not infinitely extendible, they are isomorphic to ones which are, and the result holds also for them. Consequently, by the standard argument, we have our strong completeness theorem.

Theorem *If S entails A in* $\mathbf{QK}_t^*$, *then* $S \vdash A$.[13]

NOTES

1. The propositional formulation of $\mathbf{K}_t$ was originally given by E. J. Lemmon. See Prior [7], Appendix I, for details.

2. Ryle claims there is a reference failure no matter what the tense is. The point is made on pp. 25-27 of [9].

3. The inclusion of this schema results in a restriction of substitution that proves tricky when doing proofs. It also blocks the converse of A12 from being provable. The absence of the converse of A12—sometimes called the Barcan formula—when A12 is an axiom is one of the notable features of this system.

4. The substitution of **P** for **F** and vice versa is meant to be simultaneous. This is sometimes called the Mirror Image Rule. It guarantees symmetry between past and future.

5. This semantics is based on [2], [10], and [3]. For the model-theoretic semantics for tense logic see [1] and [5].

6. This indexing serves to permit the same truth-value assignment to occur more than once in $\mathfrak{F}$ (albeit with a different index).

7. The terms "history" and "moment" in this sense stem from [1].

8. $\mathbf{a}_n(A) = 1$ is short for "the value of A at $\mathbf{a}_n$ is 1."

9. Since **G** is given a "Boolean" interpretation, $\mathbf{G}A$ is true at a moment $\mathbf{a}_n$ whenever there is no $\mathbf{a}'_p$ such that $R(\mathbf{a}_n, \mathbf{a}'_p)$ (the same, mutatis mutandis, holds for '**H**'). On such occasions, any wff of the sort $\mathbf{F}A$ is automatically false.

10. For more about Bochvarian three-valued logic see [8].

11. To illustrate the point. The set $\{f(p_1), f(p_2), \ldots, \sim(\forall x)f(x)\}$ is not infinitely extendible and there is no truth-value assignment on which all the members of the set come out true. Yet the set *is* semantically consistent and syntactically consistent in the standard (i.e., model-theoretical) semantics. To compensate, we say that two sets are isomorphic if there is a one-to-one map from the individual parameters of the first to the second. A set, isomorphic to the one above, is $\{f(p_2), f(p_4), \ldots, \sim(\forall x) f(x)\}$. Note that the second *is* infinitely extendible, and there is a truth-value assignment on which all of its members come out true. This move comes from [3].

12. See bibliography. The full version of the completeness proof is found in [6].

13. I should like to thank Hugues Leblanc for his helpful comments and criticism.

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