Notre Dame Journal of Formal Logic
Volume XVII, Number 4, October 1976
NDJFAM

A PROOF OF SOBOCIŃSKI'S CONJECTURE CONCERNING A CERTAIN SET OF LATTICE-THEORETICAL FORMULAS

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In [1], J. Ričan has proven that any algebraic system

$$\mathfrak{A} = \langle A, \cup, \cap \rangle$$

where $\cup$ and $\cap$ are two binary operations defined on the carrier set A, which satisfies the following two postulates

A1 $\quad [abc] : a, b, c \in A \;.\supset.\; (a \cap b) \cup (a \cap c) = ((c \cap a) \cup b) \cap a$

and

A2 $\quad [abc] : a, b, c \in A \;.\supset.\; a = (c \cup (b \cup a)) \cap a$

is a modular lattice. In [2] (*cf.* pp. 311-312, Remark I; pp. 313-314, section **2.2**; and p. 315, section **5**, Remark II), B. Sobociński has proven that if in Ričan's postulate-system we substitute *A2* by

B1 $\quad [ab] : a, b \in A \;.\supset.\; a = (b \cup a) \cap a$

then the resulting system $\{A1; B1\}$ satisfies the conditions of a modular lattice with the probable exception, he conjectures, that the associative laws for $\cup$ and $\cap$ fail to hold.

In this note I shall prove this conjecture using the following algebraic table

$\mathfrak{M}1$

$\cup$	0	α	β	γ	δ	1
0	0	α	β	γ	δ	1
α	α	α	β	1	1	1
β	β	β	β	γ	δ	1
γ	γ	1	γ	γ	1	1
δ	δ	1	δ	1	δ	1
1	1	1	1	1	1	1

$\cap$	0	α	β	γ	δ	1
0	0	0	0	0	0	0
α	0	α	α	0	0	α
β	0	α	β	β	β	β
γ	0	0	β	γ	β	γ
δ	0	0	β	β	δ	δ
1	0	α	β	γ	δ	1

which verifies the axioms *A1* and *B1*, but falsifies *A2*,

Received April 15, 1976

P1 $[abc]: a, b, c \in A \supset. (a \cup b) \cup c = a \cup (b \cup c)$

and

R1 $[abc]: a, b, c \in A \supset. (a \cap b) \cap c = a \cap (b \cap c)$

Namely:

(a) $\mathfrak{M}1$ falsifies *A2* for $a|\alpha$, $b|\beta$, and $c|\gamma$: (i) $a = \alpha$, (ii) $(\gamma \cup (\beta \cup \alpha)) \cap \alpha = (\gamma \cup \beta) \cap \alpha = \gamma \cap \alpha = 0$.
(b) $\mathfrak{M}1$ falsifies *P1* for $a|\alpha$, $b|\beta$, and $c|\gamma$: (i) $(\alpha \cup \beta) \cup \gamma = \beta \cup \gamma = \gamma$, (ii) $\alpha \cup (\beta \cup \gamma) = \alpha \cup \gamma = 1$.
(c) $\mathfrak{M}1$ falsifies *R1* for $a|\alpha$, $b|\beta$, and $c|\gamma$: (i) $(\alpha \cap \beta) \cap \gamma = \alpha \cap \gamma = 0$, (ii) $\alpha \cap (\beta \cap \gamma) = \alpha \cap \beta = \alpha$.

Thus the system $\{A1; B1\}$ is not a modular lattice and, therefore, the conjecture of Sobociński is proved.

REFERENCES

[1] Ričan, J., "Zu der Axiomatic der modulären Verbände," *Acta Facultatiae Nationalis Universitatis Comenianensis*, Mathematica, vol. 2 (1958), pp. 257-262.

[2] Sobociński, B., "A short equational axiomatization of modular ortholattices," *Notre Dame Journal of Formal Logic*, vol. XVII (1976), pp. 311-316.

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