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VARIATIONS OF ZORN'S LEMMA, PRINCIPLES OF COFINALITY, AND HAUSDORFF'S MAXIMAL PRINCIPLE
PART I: SET FORMS

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1 *Introduction* When is a maximal principle equivalent to the axiom of choice, **AC**? While we are not able to give a non-tautological answer to this question, we do study a large variety of maximal principles, some of which are equivalent to **AC**, some are weaker but do not follow from the other axioms of set theory, some are provable from the other axioms, and some have the property that their negations are provable from the other axioms.*

In section **2** we study variations of Zorn's Lemma and principles of cofinality. It is shown in most cases that a principle of cofinality is equivalent to a corresponding variation of Zorn's Lemma. Variations of Hausdorff's Maximal Principle are studied in section **3**. The results for class forms are similar to those for sets, but often the Axiom of Regularity is used to insure that we are dealing with sets and not proper classes at various stages of the proofs. These results will appear later in Part II of this paper. Section **4** is an appendix in which we list all the statements which are used in the paper along with their abbreviations, and in Figures 4.1 and 4.2, we summarize our results for the set forms of Zorn's Lemma and Hausdorff's Maximal Principle.

The following definitions and symbols are used throughout this paper:

1.1 Let $\mathbf{NBG}^{\circ}$ denote von Neumann-Bernays-Gödel set theory excluding the Axiom of Regularity, **AR**, and the Axiom of Choice, **AC**. Let $\mathbf{NBG} = \mathbf{NBG}^{\circ} + \mathbf{AR}$. *All results are in* $\mathbf{NBG}^{\circ}$ *unless it is explicitly stated otherwise.* Capital letters such as X, Y, and Z and these letters with subscripts, will be used for class variables, and lower case letters will denote set

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variables. Lower case Greek letters will be used to denote ordinal numbers. The power class of X, $\mathfrak{P}(X) = \{u: u \subseteq X\}$. If R is a relation on X, the domain of R, $\mathfrak{D}(R) = \{u \in X: (\exists v \in X)\ uRv\}$, and the range of R, $\mathfrak{R}(R) = \{v \in X: (\exists u \in X)\ uRv\}$. $\emptyset$ is the empty set, V is the universe, ω is the set of natural numbers, and On is the class of ordinal numbers.

1.2 A class X is *partially ordered* by a relation R iff R is a (binary) relation which is transitive, antisymmetric and reflexive on X.

1.3 A relation R is *connected* on a class X iff $(\forall u)(\forall v)(u, v \in X \ \&\ u \neq v \rightarrow uRv$ or $vRu)$.

1.4 A class X is linearly ordered by a relation R iff R is a connected partial ordering on X.

1.5 An element $u \in X$ is an *R-first* (*R-last*) element of X iff $(\forall v \in X)(v \neq u \rightarrow uRv)$. (*R-last* is defined similarly.)

1.6 An element $u \in X$ is an *R-maximal* element of X iff $(\forall v \in X)(uRv \rightarrow vRu)$. (*R-minimal* is defined similarly.)

1.7 If $Y \subseteq X$, $u \in X$ is an *R-upper bound* of Y in X iff $(\forall v \in Y)(v \neq u \rightarrow vRu)$. (*R-lower bound* is defined similarly.)

1.8 A class X is *well ordered* by a relation R iff R linearly orders X and every non-empty subclass of X has an R-first element.

1.9 A class X is *directed* (upwards) by a relation R iff R partially orders X such that every finite subset of X has an R-upper bound.

1.10 Let R partially order X. A subclass S of X is an *R-initial segment* of X iff

$$(\forall u)(\forall v)(v \in S \ \&\ u \in X \ \&\ uRv \rightarrow u \in S).$$

1.11 If S is an R-initial segment and there is a $y \in X$ such that $S = \{u: u \in X \ \&\ uRy\}$ then S will be called the *initial segment generated by y* and may be denoted by $\overleftarrow{y}$.

1.12 A class X is *ramified* by a relation R iff R partially orders X such that every initial segment $\overleftarrow{y}$ of X is linearly ordered by R.

1.13 A class X is a *forest* under the relation R iff R partially orders X such that every initial segment $\overleftarrow{y}$ of X is well ordered.

1.14 A class X is a *tree* under the relation R iff X is a forest and is directed downwards (i.e., every finite subset of X has an R-lower bound).

1.15 A subclass Q of a partially ordered class X is *quasi-cofinal* in X iff Q has no strict upper bound in X.

1.16 A subclass Q of a class X partially ordered by a relation R is *cofinal* in X iff Q is linearly ordered by R and

$$(\forall y)(y \in X \rightarrow (\exists z)(z \in Q \ \&\ yRz)).$$

1.17 A subclass A of a class X partially ordered by a relation R is called an *antichain* iff $(\forall y)(\forall z)(y \in A \;\&\; z \in A \;\&\; y \neq z \rightarrow \neg(yRz \text{ or } zRy))$. That is, no two distinct elements of A are related by R.

1.18 For a class X, let

$$W_X = \{\langle t, w\rangle : t \subseteq X \;\&\; w \subseteq t \times t \;\&\; w \text{ well orders } t\}.$$

Let I be the relation defined on ordered pairs such that

$$\langle t, w\rangle \, I \, \langle t', w'\rangle \text{ iff } t \subseteq t' \;\&\; w = w' \cap (t \times t) \;\&\; t \times (t' \sim t) \subseteq w'.$$

I will be called the *initial segment relation* and W_X as ordered by I will be called the *tree of well ordered subsets of X*. If x is a set, we may write "w_x" instead of "W_X".

Every set is a class and classes which are not sets are called proper classes. When considering only sets we use the preceding definitions with the word "class" replaced by "set". Also, the name of the relation may be omitted if no confusion arises. For example, we may write "maximal element" instead of "R-maximal element", if it is clearly understood which relation R is meant.

2 *Variations of Zorn's Lemma and Principles of Cofinality*

2.1 *Zorn's Lemma* Zorn's Lemma may be stated as follows:

Every non-empty partially ordered set x in which every linearly ordered subset has an upper bound, has a maximal element.

We vary two parts of the hypotheses: the type of order on the set x and the type of subset which always has an upper bound. To denote such a variation of Zorn's Lemma we write $\mathbf{Z}(Q, U)$ to stand for the statement:

Every non-empty Q-ordered set in which every U-ordered subset has an upper bound, has a maximal element.

As possibilities for Q and U we begin with the following:

A: arbitrary (any binary relation)
TR: transitive
AS: antisymmetric
C: connected
P: partially ordered
W: well ordered
L: linearly ordered (also called a chain)
D: directed
R: ramified
F: forest
T: tree

For instance, $\mathbf{Z}(\mathrm{D}, \mathrm{R})$ stands for the statement:

Every non-empty directed set in which every ramified subset has an upper bound, has a maximal element.

Zorn's Lemma has also been considered in the following form:

Every non-empty set x in which each subset which is linearly ordered by inclusion has an upper bound, has a maximal element.

However, this statement is equivalent to $\mathbf{Z}(\mathsf{P},\mathsf{L})$ because every partially ordered set x can be replaced by an isomorphic family of sets ordered by inclusion simply by mapping $y \epsilon x$ to $\overleftarrow{y}$. For this reason we shall not consider as special cases forms of Zorn's Lemma (or Hausdorff's Principle) in which inclusion is the ordering relation.

It is easy to see that $\mathbf{Z}(Q, U)$ is provable in **NBG°** if Q is a stronger relation than U; i.e., if x is Q-ordered implies x is U-ordered. Hence the following statements are all provable:

$\mathbf{Z}(Q, \mathsf{A})$ for all Q (so we may drop A from the U list),
$\mathbf{Z}(Q, U)$ for all $Q = U$,
$\mathbf{Z}(\mathsf{P}, U)$ for $U = \mathsf{TR}$ and AS,
$\mathbf{Z}(\mathsf{L}, U)$ for $U = \mathsf{TR}$, AS, C, P, D, and R,
$\mathbf{Z}(\mathsf{D}, U)$ for $U = \mathsf{TR}$, AS, and P,
$\mathbf{Z}(\mathsf{R}, U)$ for $U = \mathsf{TR}$, AS, and P,
$\mathbf{Z}(\mathsf{F}, U)$ for $U = \mathsf{TR}$, AS, P, and R,
$\mathbf{Z}(\mathsf{T}, U)$ for $U = \mathsf{TR}$, AS, P, R, and F,
$\mathbf{Z}(\mathsf{W}, U)$ for all U (so we may drop W from the Q list).

$\mathbf{Z}(\mathsf{F}, \mathsf{T})$ is also provable as we see by the following lemma.

Lemma 2.1 *Every forest is the union of a pairwise disjoint set of trees.*

Proof: Let f be a forest under the relation R. For each $x \epsilon f$, $\overleftarrow{x}$ is well ordered and hence has a least element, r_x. Now $\vec{r}_x = \{u \epsilon f\colon r_x R u\}$ is a tree, because as a subset of a forest, $\vec{r}_x$ is also a forest, and it has r_x as its least element. This proves that every element of f is a member of some tree in f which is generated by a minimal element of f. Now suppose r_0 and r_1 are two minimal elements of f and that $\vec{r}_0 \cap \vec{r}_1 \neq \emptyset$. Then $(\exists u \epsilon f)(r_0 R u \;\&\; r_1 R u)$. It follows that $r_0 = r_u = r_1$. Therefore, $\vec{r}_0 = \vec{r}_1$. Thus any two trees in f which are generated by a minimal element are either identical or disjoint. This completes the proof of the lemma. Q.E.D.

To show that certain of the maximal principles are false in **NBG°** (i.e., the negation is provable in **NBG°**), we give the following example. (See [21].)

Example 2.1:

$$x_1 = \{a, b, c, d\}$$
$$R_1 = \{\langle a, b\rangle, \langle b, c\rangle, \langle c, d\rangle, \langle d, a\rangle\} \cup \mathsf{Id}/x_1.$$

(Id is the identity relation.) R_1 is antisymmetric. R_1-connected subsets of x_1 have at most two elements and, therefore, have R_1-upper bounds. But x_1 has no R_1-maximal element.

Example 2.2:

$$x_2 = \{a, b, c\}$$

$$R_2 = \{\langle a, b\rangle, \langle b, c\rangle, \langle c, a\rangle\} \cup \mathsf{Id}/x_2.$$

R_2 is connected and antisymmetric. R_2-transitive subsets of x_2 have at most two elements and have R_2-upper bounds. But x_2 has no R_2-maximal element.

Example 2.3:

$$x_3 = x_1$$

$$R_3 = \breve{R}_1 \cup \mathsf{Id}/x_3$$

($\breve{R}_1 = x_1 \times x_1 \sim R_1$) R_3 is connected and R_3-antisymmetric subsets of x_3 have at most two elements, so have R_3-upper bounds, but x_3 has no R_3-maximal element.

Consequently, we see the following maximal principles are false in **NBG**°:

Z(A, C), **Z**(AS, C) (Example 2.1)
Z(A, TR), **Z**(AS, TR), **Z**(C, TR) (Example 2.2)
Z(A, AS), **Z**(C, AS) (Example 2.3)

Using the fact that if $U_1 \rightarrow U_2$ then **Z**(Q, U_1) $\rightarrow$ **Z**(Q, U_2) for all Q, and if $Q_1 \rightarrow Q_2$ then **Z**(Q_2, U) $\rightarrow$ **Z**(Q_1, U) for all U, we list in Figure 2.1 those maximal principles which are remaining after eliminating those which are known to be provable in **NBG**° and those which are false in **NBG**°.

Z(TR, AS)
Z(TR, C)
Z(TR, P)
Z(TR, W)
Z(TR, L)
Z(TR, D)
Z(TR, R)
Z(TR, F)
Z(TR, T)
Z(P, C)
Z(P, W)
Z(P, L)
Z(P, D)
Z(P, R)
Z(P, F)
Z(P, T)
Z(L, W)
Z(L, F)
Z(L, T)
Z(D, C)
Z(D, W)
Z(D, L)
Z(D, R)
Z(D, F)
Z(D, T)
Z(R, C)
Z(R, W)
Z(R, L)
Z(R, D)
Z(R, F)
Z(R, T)
Z(F, C)
Z(F, W)
Z(F, L)
Z(F, D)
Z(T, C)
Z(T, W)
Z(T, L)
Z(T, D)

Figure 2.1

In a connected set, W, F, and T are equivalent. Hence, **Z**(L, W) ≡ **Z**(L, F) ≡ **Z**(L, T). In a ramified set, L and D are equivalent. Hence, **Z**(R, L) ≡ **Z**(R, D). In both forests and trees, W, L, and D are equivalent. Moreover, every tree is a forest and to find a maximal element in a forest in which every U-ordered subset has an upper bound, select one of the constituent trees of the forest and use **Z**(T, U). Hence, **Z**(F, W) ≡ **Z**(F, L) ≡ **Z**(F, D) ≡ **Z**(T, W) ≡ **Z**(T, L) ≡ **Z**(T, D). Also, in a partially ordered set C and L are equivalent. Therefore, **Z**(P, C) ≡ **Z**(P, L), **Z**(D, C) ≡ **Z**(D, L),

$\mathbf{Z}(\mathrm{R}, \mathrm{C}) \equiv \mathbf{Z}(\mathrm{R}, \mathrm{L})$, $\mathbf{Z}(\mathrm{F}, \mathrm{C}) \equiv \mathbf{Z}(\mathrm{F}, \mathrm{L})$, and $\mathbf{Z}(\mathrm{T}, \mathrm{C}) \equiv \mathbf{Z}(\mathrm{T}, \mathrm{L})$. In a transitively ordered set AS and P are equivalent, so $\mathbf{Z}(\mathrm{TR}, \mathrm{AS}) \equiv \mathbf{Z}(\mathrm{TR}, \mathrm{P})$.

$\mathbf{Z}(\mathrm{P}, \mathrm{L})$ is Zorn's Lemma in its original form. We see that $\mathbf{Z}(\mathrm{TR}, \mathrm{W})$ is the strongest maximal principle in the list. It follows from the work of T. Szele [27] (see also [23], p. 12 ff) that $\mathbf{Z}(\mathrm{TR}, \mathrm{W})$ is equivalent to AC in **NBG°**. (Similarly for $\mathbf{Z}(\mathrm{P}, \mathrm{W})$.) Thus, AC implies each of the maximal principles in the list.

In order to prove additional implications we use the fact that $\mathbf{Z}(\mathrm{P}, \mathrm{L}) \rightarrow \mathbf{Z}(\mathrm{P}, \mathrm{W})$. The method of proof is similar to the usual proof that Zorn's Lemma implies the well ordering theorem. (For example, see Theorem 4.9, p. 16 in [23].) Similarly, $\mathbf{Z}(\mathrm{TR}, \mathrm{L}) \rightarrow \mathbf{Z}(\mathrm{TR}, \mathrm{W})$ and $\mathbf{Z}(\mathrm{R}, \mathrm{L}) \rightarrow \mathbf{Z}(\mathrm{R}, \mathrm{W})$. Moreover the proof that $\mathbf{Z}(\mathrm{P}, \mathrm{D}) \rightarrow \mathbf{Z}(\mathrm{P}, \mathrm{W})$ is similar to Felgner's proof [2] that $\mathbf{Z}(\mathrm{M}, \mathrm{D}, l) \rightarrow \mathbf{Z}(\mathrm{P}, \mathrm{W})$. (The l is to indicate that the hypotheses are modified to state that every directed set is assumed to have a least upper bound.) Since $\mathbf{Z}(\mathrm{P}, \mathrm{W}) \rightarrow \mathbf{Z}(\mathrm{P}, \mathrm{D})$, we have $\mathbf{Z}(\mathrm{P}, \mathrm{W}) \equiv \mathbf{Z}(\mathrm{P}, \mathrm{D})$. Using a proof which is similar to the proof that $\mathbf{Z}(\mathrm{P}, \mathrm{D}) \rightarrow \mathbf{Z}(\mathrm{P}, \mathrm{W})$. It can be shown that each of the following imply $\mathbf{Z}(\mathrm{P}, \mathrm{W})(\equiv \mathbf{Z}(\mathrm{TR}, \mathrm{W}) \equiv \mathbf{AC})$: $\mathbf{Z}(Q, U)$ for $Q = \mathrm{R}, \mathrm{F}$, or T, and $U = \mathrm{W}, \mathrm{L}, \mathrm{C}$, or D. Consequently, it follows that the 20 statements $\mathbf{Z}(Q, U)$ for $Q = \mathrm{TR}, \mathrm{P}, \mathrm{R}, \mathrm{F}$, or T, and $U = \mathrm{W}, \mathrm{L}, \mathrm{C}$, or D are all equivalent to **AC** in **NBG°**.

We prove next that $\mathbf{Z}(\mathrm{L}, \mathrm{W})$ and $\mathbf{Z}(\mathrm{D}, \mathrm{W})$ are equivalent to **AC** in **NBG°** if the order extension principle,

OE: *Every partial ordering can be extended to a linear ordering,*

holds. Moreover, the equivalence fails for $\mathbf{Z}(\mathrm{L}, \mathrm{W})$ if **OE** does not hold. Concerning the strength of **OE**, the following implications are known: **AC** → **BPI** → **OE** → **LO**, where **BPI** denotes the Boolean Prime Ideal Theorem and **LO** the linear ordering principle. That **LO** does not imply **OE** in **NBG** was proven by Mathias [17]. That **BPI** does not imply **AC** in **NBG** has been proven by Halpern and Levy [9], and it follows from this that **OE** does not imply **AC** in **NBG**. It was recently shown by Felgner that **OE** does not imply **BPI** in **NBG**.

To obtain the desired result we first use a result of Felgner [3] (as corrected by D. Morris) that in **NBG°** + **OE** the principle of cofinality,

Cof: *Every linearly ordered set has a well ordered cofinal subset,*

implies

LW: *Every linearly ordered set can be well ordered.*

However, it is clear that **OE** & **LW** implies **WO** (the Well Ordering Theorem) since every set can be partially ordered. Hence, from $\mathbf{NBG}^\circ + \mathbf{OE} \vdash \mathbf{Cof} \rightarrow \mathbf{LW}$, it follows that $\mathbf{NBG}^\circ + \mathbf{OE} \vdash \mathbf{Cof} \rightarrow \mathbf{AC}$.

Concerning the relative strength of **Cof**, the following implications hold in **NBG°**: **AC** → **KA** → **LW** → **Cof**, where **KA** is Kurepa's antichain principle: *Every partially ordered set has a maximal antichain.*

It has been shown by J. D. Halpern in [7], that **KA** does not imply **AC** in **NBG**°. However, it was shown by H. Rubin [22] that in **NBG**, **AC** is equivalent to **LW**. Clearly, **LW** & **Cof** does not imply **AC** in **NBG**°. It was shown by Morris [18] that **Cof** does not imply **OE** in **NBG**.

2.2 *Principles of Cofinality* We prove next that **NBG**° ⊢ **Cof** → **Z**(L, W). In fact, we prove a more general theorem of which this is a specific case.

Theorem 2.2 *Let P be a non-empty partially ordered set and let U be any property on subsets of P such that any singleton has the property U. Then the following are equivalent:*

(1) *If every subset of P with the property U has an upper bound, then P has a maximal element.*
(2) *P has a quasi-cofinal subset with the property U.*

Proof: (1) → (2). If P has a maximal element, m, then $\{m\}$ is a quasi-cofinal subset with the property U. If P has no maximal element, then P has a subset with the property U which has no upper bound. This set is clearly quasi-cofinal.
(2) → (1). Let c be a quasi-cofinal subset of P with the property U. If the hypotheses of (1) hold, then c has an upper bound, m. Clearly m is a maximal element of P. Q.E.D.

We note that we need not be restricted to partially ordered sets. The proof would go through if partially ordered were replaced by linearly ordered, directed, ramified, well ordered, forest, tree, or any order type which is at least a partial order. In fact, the theorem holds for any transitive order.

We will call a statement of the form (2) a principle of cofinality, and to denote variations we will write **C**(Q, U). The theorem implies that **Z**(Q, U) ≡ **C**(Q, U) for all Q and U as long as Q is at least a transitive ordering and U is a property which always holds for a singleton. For some properties U, (1) and (2) are both false. For example, if $U(x)$ is "x is finite". On the other hand, for many properties U either (1) or (2) has been shown to be equivalent to **AC**. **Cof** is the same as **C**(L, W). So as a corollary to the theorem we have that **Z**(L, W) ≡ **C**(L, W), and hence **NBG**° + **OE** ⊢ **Z**(L, W) ≡ **AC**, but not (**NBG**° ⊢ **Z**(L, W) → **AC**). Since **Z**(L, W) ≡ **Z**(L, F) ≡ **Z**(L, T) ≡ **C**(L, W) ≡ **C**(L, F) ≡ **C**(L, T), we have similar results for these variations. Moreover, **Z**(D, W) → **Z**(L, W) so that **NBG**° + **OE** ⊢ (**Z**(D, W) ≡ **AC**) & (**C**(D, W) ≡ **AC**).

It follows from our previous work that **Z**(Q, T) → **Z**(Q, F) because every tree is a forest. We show next that if Q satisfies certain properties then **Z**(Q, F) → **Z**(Q, T).

Theorem 2.3 *If Q is at least a transitive order and each set x which is Q-ordered by R has the property that for each $u \in x$, $\vec{u} = \{v \in x\colon uRv \;\&\; u \neq v \;\&\; \neg vRu\}$ is Q-ordered by R, then* **Z**(Q, F) → **Z**(Q, T).

Proof: Suppose x is a non-empty set Q-ordered by R in which each tree has an upper bound. Suppose $u \in x$. If $\vec{u} = \emptyset$ then u is a maximal element of x, so suppose $\vec{u} \neq \emptyset$. Let $y \subseteq \vec{u}$ be a non-empty forest. Let $y^* = y \cup \{u\}$. Then y^* is a tree, so by hypothesis y^* has an upper bound $b \in x$. Since $y \subseteq y^*$, b is also an upper bound for y. Suppose $v \in y$. Then uRv, vRb, and $u \neq v$. Moreover, since b is an upper bound of y^*, uRb holds. Now it follows from the transitivity of R that if either $u = b$ or bRu then vRu, but this contradicts the fact that $v \in \vec{u}$. Thus, we have shown that uRb, $u \neq b$, and $\neg bRu$, and this implies $b \in \vec{u}$. Now it follows that $\vec{u}$ satisfies the hypothesis of $\mathbf{Z}(Q, \mathsf{F})$. Consequently, $\vec{u}$ has a maximal element m, and since R is transitive, m must be a maximal element of x. Q.E.D.

Now, it follows from Theorem 2.3 and our previous results that

$$\mathbf{Z}(\mathsf{L}, \mathsf{F}) \equiv \mathbf{Z}(\mathsf{L}, \mathsf{T}) \equiv \mathbf{Z}(\mathsf{L}, \mathsf{W}) \equiv \mathbf{Z}(\mathsf{R}, \mathsf{F}) \equiv \mathbf{Z}(\mathsf{R}, \mathsf{T}),$$
$$\mathbf{Z}(\mathsf{TR}, \mathsf{F}) \equiv \mathbf{Z}(\mathsf{TR}, \mathsf{T}), \mathbf{Z}(\mathsf{P}, \mathsf{F}) \equiv \mathbf{Z}(\mathsf{P}, \mathsf{T}), \mathbf{Z}(\mathsf{D}, \mathsf{F}) \equiv \mathbf{Z}(\mathsf{D}, \mathsf{T}).$$

Moreover, we have that $\mathbf{Z}(\mathsf{D}, \mathsf{R}) \equiv \mathbf{Z}(\mathsf{P}, \mathsf{R})$ because a transitively ordered set in which every ramified subset has an upper bound is directed. Similarly, $\mathbf{Z}(\mathsf{D}, \mathsf{F}) \equiv \mathbf{Z}(\mathsf{P}, \mathsf{F})$. Also, a ramified set in which every forest has an upper bound is directed. Thus, such a set is linearly ordered, so $\mathbf{Z}(\mathsf{R}, \mathsf{F}) \equiv \mathbf{Z}(\mathsf{L}, \mathsf{F})$.

The result that $\mathbf{Z}(\mathsf{D}, \mathsf{W}) \rightarrow$ **AC** is due to U (Felgner (unpublished)). We include our proof of it here.

Theorem 2.4 $\mathbf{Z}(\mathsf{D}, \mathsf{W}) \rightarrow$ **WO**.

Proof: Let x be a non-empty set and assume x cannot be well ordered. Let $w = \{f : f \text{ is a 1-1 function } \& \ (\exists \alpha)\, \mathfrak{D}(f) = \alpha \in \mathrm{On}) \ \& \ \mathfrak{R}(f) \subseteq x\}$, and order w by S as follows: $f_1 S f_2$ iff $\mathfrak{R}(f_1) \subset \mathfrak{R}(f_2)$ or $f_1 = f_2$. S partially orders w and by the assumption that x cannot be well ordered, w is directed by S. Let c be a well ordered cofinal subset of w. One can easily construct a function which is an upper bound for c, and this upper bound, g, must be a maximal element of w. $\mathfrak{R}(g) = x$, for otherwise it is easy to construct a function which is S-greater than g. This contradicts c being cofinal. Hence x can be well ordered. Q.E.D.

We summarize the results we have so far in Figure 2.2. Let

A = $\{\mathbf{Z}(Q, U) : Q = \mathsf{TR}, \mathsf{P}, \mathsf{R}, \mathsf{F}, \text{or } \mathsf{T}, \text{and } U = \mathsf{W}, \mathsf{L}, \mathsf{C}, \text{or } \mathsf{D}\}$,
B = $\{\mathbf{Z}(\mathsf{L}, \mathsf{F}), \mathbf{Z}(\mathsf{L}, \mathsf{T}), \mathbf{Z}(\mathsf{L}, \mathsf{W}), \mathbf{Z}(\mathsf{R}, \mathsf{F}), \mathbf{Z}(\mathsf{R}, \mathsf{T})\}$,
C = $\{\mathbf{Z}(Q, \mathsf{R}) : Q = \mathsf{D} \text{ or } \mathsf{P}\}$,
D = $\{\mathbf{Z}(Q, U) : Q = \mathsf{P} \text{ or } \mathsf{D}, \text{and } U = \mathsf{F} \text{ or } \mathsf{T}\}$,
E = $\{\mathbf{Z}(\mathsf{TR}, \mathsf{P}), \mathbf{Z}(\mathsf{TR}, \mathsf{AS})\}$,
F = $\{\mathbf{Z}(\mathsf{D}, \mathsf{L}), \mathbf{Z}(\mathsf{D}, \mathsf{C})\}$.

It follows from our preceding remarks that each pair of statements in any one of the sets A - F is equivalent in **NBG°**. Each statement in A is equivalent to **AC** in **NBG°**, while each statement in B implies **AC** in **NBG°** + **OE** but does not imply **AC** in **NBG°**. In Figure 2.2 when we write, for example, A → F we mean any one of the equivalent statements in A implies any one of the equivalent statements in F in **NBG°**.

$$\begin{tikzcd}
\mathbf{Z}(\mathrm{D}, \mathrm{W}) \equiv \mathrm{A} \arrow[r] \arrow[d] & \mathrm{F} \arrow[r] & \mathrm{C} \\
\mathbf{Z}(\mathrm{TR}, \mathrm{F}) \arrow[d] \arrow[r, phantom, "\equiv"] & \mathbf{Z}(\mathrm{TR}, \mathrm{T}) \arrow[d] & \\
\mathbf{Z}(\mathrm{TR}, \mathrm{R}) \arrow[d] \arrow[dr] & \mathrm{D} \arrow[r] \arrow[d] & \mathrm{B} \\
\mathrm{E} & \mathrm{C} &
\end{tikzcd}$$

Figure 2.2

Let $\mathbf{AC}^{\mathbf{WO}}$ be the axiom of choice for a well ordered family of sets. That is:

For every non-empty, well ordered set x of non-empty sets there is a function f on x such that $f(u) \in u$ for all $u \in x$.

It is shown in [14] that E $\rightarrow$ $\mathbf{AC}^{\mathbf{WO}}$ but $\mathbf{AC}^{\mathbf{WO}}$ does not imply E in $\mathbf{NBG}^{\circ}$. We were also able to show that there is a variation of $\mathbf{Z}(\mathrm{D}, \mathrm{R})$ which does not imply **AC** in $\mathbf{NBG}^{\circ}$. Let $\mathbf{Z}(\mathrm{D}, \mathrm{R}, l)$ be the statement:

Every non-empty directed set x, in which every ramified subset has a least upper bound, has a maximal element.

We shall show that **KA**, Kurepa's antichain condition implies $\mathbf{Z}(\mathrm{D}, \mathrm{R}, l)$ in $\mathbf{NBG}^{\circ}$. First, Felgner [3] proves that **KA** is equivalent to a generalized antichain condition **GKA**:

For every x, if $x = \{\langle u, S_u \rangle : S_u$ is a partial order on $u\}$ is a non-empty set of non-empty partially ordered sets, then there is a function f on x such that $f(\langle u, S_u \rangle)$ is a maximal antichain in u for each $\langle u, S_u \rangle \in x$.

Theorem 2.5 $\mathbf{GKA} \rightarrow \mathbf{Z}(\mathrm{D}, \mathrm{R}, l)$.

Proof: Let x be a directed set under a relation S, in which every ramified subset has a least upper bound.

1. We note that for each $y \in x$, the set $\vec{y} = \{z \in x: ySz \ \& \ y \neq z\}$ is a directed subset of x. Call $\vec{y}$ the end segment generated by y.
2. An antichain in x is a ramified set (in fact, a forest), because initial segments in an antichain are singleton sets and hence are linearly ordered (in fact, well ordered).
3. Let f be a function on $\mathfrak{P}(x) \sim \{\emptyset\}$ such that $f(u)$ is a maximal antichain in u for $\emptyset \neq u \subseteq x$, where u is partially ordered by the restriction of S to u.
4. Define a function g on On recursively as follows: $g(0)$ = the least upper bound in x of $f(x)$. Since $f(x)$ is a maximal antichain in x, $f(x)$ is ramified (in fact, it is a forest) and by hypotheses has a least upper bound in x.

If $g(0)$ is a maximal element in x we are done essentially. To complete the definition of g, suppose $v \notin x$, then define $g(\alpha) = v$ and set $\overrightarrow{g(\alpha)} = \emptyset$ for all $\alpha > 0$. We note that if $g(0)$ is a maximal element in x then $\overrightarrow{g(0)} = \emptyset$. If $g(0)$ is not a maximal element define $g(1) =$ least upper bound of $f(\overrightarrow{g(0)})$. Inductively, if $g(\alpha)$ is defined, then let

$$\begin{aligned} g(\alpha + 1) &= \text{the least upper bound of } f(\overrightarrow{g(\alpha)}) \text{ if } \overrightarrow{g(\alpha)} \neq \emptyset \\ &= v \text{ if } \overrightarrow{g(\alpha)} = \emptyset. \end{aligned}$$

If α is a limit ordinal and $\overrightarrow{g(\beta)} \neq \emptyset$ for all $\beta < \alpha$ then $\{g(\beta): \beta < \alpha\}$ is a well ordered subset of x such that $g(\beta)\, S\, g(\gamma)$ for $\beta < \gamma < \alpha$. Hence, this set is ramified (in fact, it is well ordered so it is also a forest) and by hypothesis has a least upper bound in x. In this case, define $g(\alpha) =$ least upper bound of $\{g(\beta): \beta < \alpha\}$. If $\overrightarrow{g(\beta)} = \emptyset$ for some $\beta < \alpha$ set $g(\alpha) = v$.

We note that if $\overrightarrow{g(\beta)} = \emptyset$ then $g(\beta + 1) = v$. So we see that for a limit ordinal α, $g(\alpha) \neq v$ unless there is a $\beta < \alpha$ with $g(\beta) = v$.

If $g(\alpha) \neq v$ for all $\alpha \in \mathrm{On}$ then g is a 1-1 function from On into x. This is impossible because x is a set. Hence, there is some $\beta \in \mathrm{On}$ with $g(\beta) = v$. Let α be the least such ordinal number. By the above $\alpha = \gamma + 1$ for some $\gamma \in \mathrm{On}$. Thus, $g(\gamma + 1) = v$ but $g(\gamma) \neq v$. Hence $g(\gamma) \in x$ such that $\overrightarrow{g(\gamma)} = \emptyset$. This implies that $g(\gamma)$ is a maximal element of x. Q.E.D.

Since $\mathbf{Z}(\mathsf{P}, \mathsf{R}, l) \equiv \mathbf{Z}(\mathsf{D}, \mathsf{R}, l)$ it follows that **GKA** → $\mathbf{Z}(Q, \mathsf{R}, l)$ for $Q = \mathsf{P}$ or D. Also, as indicated in the proof, it is easy to see that **GKA** → $\mathbf{Z}(Q, \mathsf{F}, l)$ for $Q = \mathsf{P}$ or D. Consequently, since **KA** ≡ **GKA**, the following corollary follows from the Halpern result [7] that **KA** does not imply **AC** in **NBG°**.

Corollary 2.6 *Not* ($\mathbf{NBG}^\circ \vdash \mathbf{Z}(Q, U, l) \rightarrow \mathbf{AC}$) *for* $Q = \mathsf{P}$ *or* D, *and* $U = \mathsf{R}$ *or* F.

The proof of **GKA** → $\mathbf{Z}(\mathsf{D}, \mathsf{R}, l)$ suggests considering the following variation of Zorn's Lemma:

Every non-empty directed set x, in which every antichain has a least upper bound, has a maximal element.

A simple example shows that this statement is false in **NBG°**.

Example 2.4: Consider any well ordered set which does not have a maximal element, for instance, ω in its natural order. An antichain is a singleton $\{n\}$ with least upper bound n. But ω has no maximal element.

We conclude this section by considering combinations of the properties TR, AS, and C. Since there is no loss in generality in assuming relations are reflexive, TR & AS is P and TR & AS & C is L. So we are left with relations which are TR & C or AS & C. It follows from Example 2.2 that $\mathbf{Z}(\mathsf{AS}\ \&\ \mathsf{C}, \mathsf{TR})$ is false in **NBG°**, so we need not consider maximal principles of the form $\mathbf{Z}(\mathsf{AS}\ \&\ \mathsf{C}, U)$. Moreover, since L → AS & C → C and L → TR & C → C,

$$\mathbf{Z}(Q, \mathsf{AS}\ \&\ \mathsf{C}) \equiv \mathbf{Z}(Q, \mathsf{TR}\ \&\ \mathsf{C}) \equiv \mathbf{Z}(Q, \mathsf{L}) \equiv \mathbf{Z}(Q, \mathsf{C}),$$

where $Q =$ TR, P, D, R, F, or T.

The only cases left are maximal principles of the form **Z**(TR & C, U) where U is P, AS, AS & C, L, R, D, W, T, or F. Let

G = {**Z**(TR & C, U): U = W, T, or F}.
H = {**Z**(TR & C, U): U = AS, AS & C, P, D, R, or L}.

In a connected set, W, T, and F are equivalent, so each pair of statements in G is equivalent in **NBG°**. Also, in a transitive, connected set, AS, AS & C, P, D, R, and L are equivalent. Therefore, each pair of statements in H is equivalent in **NBG°**. Moreover, E → H because **Z**(TR, P) → **Z**(TR & C, P). (See Figure 2.2.) Also, G → B because **Z**(TR & C, W) → **Z**(L, W), and G → H because **Z**(TR & C, W) → **Z**(TR & C, P). Finally, we shall show that H → $\mathbf{AC}^{\mathbf{WO}}$. (This is similar to the proof given in [14] where it is shown **Z**(TR, P) → $\mathbf{AC}^{\mathbf{WO}}$.)

Theorem 2.7 **Z**(TR & C, P) → $\mathbf{AC}^{\mathbf{WO}}$.

Proof: Let x be a non-empty, well ordered set of non-empty sets and let y be the set of all choice functions on initial segments of x. Define a relation R on y by fRg iff $\mathfrak{D}(f) \subseteq \mathfrak{D}(g)$. R is a transitive connected relation on y. Also, since x is well ordered, any R-partially ordered subset of y is actually well ordered by R. Suppose y' is an R-partially ordered subset of y. We choose an upper bound f for y' as follows:

$$\mathfrak{D}(f) = \bigcup_{g \epsilon y'} \mathfrak{D}(g), \text{ and for each } u \epsilon \mathfrak{D}(f), f(u) = g(u)$$

where g is the R-first function in y' which has u in its domain. **Z**(TR & C, P) implies that y has an R-maximal element g, and g is clearly a choice function on x. Q.E.D.

It is shown in [14] that $\mathbf{AC}^{\mathbf{WO}}$ does not imply H in **NBG°**.

Theorem 2.8 **Z**(P, R) → **Z**(TR & C, L).

Proof: Suppose x is a non-empty set and S is a transitive, connected relation on x with the property that each linearly ordered subset of x has an S-upper bound. Define a relation S^* on x such that for all $u, v \epsilon x$, uS^*v iff $(uSv \ \& \ \neg vSu)$ or $u = v$. Then S^* is a partial ordering on x and x has an S-maximal element if and only if x has an S^*-maximal element. Suppose y is an S^*-ramified subset of x. Now, if $u, v \epsilon y$, $u \neq v$, uSv, and vSu then we claim u (or v) is an S-upper bound of y. For suppose $w \epsilon y$. Then, since S is connected, wSu or uSw, and wSv or vSw. If wSu holds we are all right. If wSv, then since S is transitive, we again obtain wSu. Therefore, the final case to consider is that uS^*w and vS^*w. Since y is ramified by S^*, this implies uS^*v or vS^*u. But each of these alternatives contradicts the assumption uSv and vSu. Now we claim that either u is an S-maximal element of x or that y has an S^*-upper bound. For suppose u is not an S-maximal element of x. Then there exists a $w \epsilon x$ such that uSw but not wSu. Then, it is easy to see that w is an S^*-upper bound of y.

Next, suppose uSv and vSu is false for all $u, v \epsilon y$, $u \neq v$. Then it follows that y is linearly ordered by S. So by hypothesis, y has an S-upper bound. It follows from the preceding argument that either x has an S-maximal element or y has an S^*-upper bound. Now, we have shown that either x has an S-maximal element or every S^*-ramified subset of x has an S^*-upper bound. If the latter alternative holds then it follows from $\mathbf{Z}(\mathsf{P}, \mathsf{R})$ that x has an S^*-maximal element. Consequently, we have shown that $\mathbf{Z}(\mathsf{P}, \mathsf{R}) \rightarrow \mathbf{Z}(\mathsf{TR} \,\&\, \mathsf{C}, \mathsf{L})$. Q.E.D.

Using a similar argument we can show that $\mathbf{Z}(\mathsf{P}, \mathsf{F}) \rightarrow \mathbf{Z}(\mathsf{TR} \,\&\, \mathsf{C}, \mathsf{W})$.

Our final result of this section is a rather unusual transitivity property for principles of cofinality and, therefore, for the corresponding forms of Zorn's Lemma.

Lemma 2.9 *If* Q_1 *is at least transitive and* Q_2 *is at least transitive and connected then* $\mathbf{C}(Q_1, Q_2) \,\&\, \mathbf{C}(Q_2, Q_3) \rightarrow \mathbf{C}(Q_1, Q_3)$.

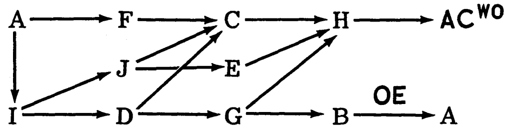
Proof: Suppose x is a set Q_1-ordered by R. Then, by $\mathbf{C}(Q_1, Q_2)$, x contains a quasi-cofinal subset y which is Q_2-ordered by R. Also, by $\mathbf{C}(Q_2, Q_3)$, y contains a quasi-cofinal subset z which is Q_3-ordered by R. It remains to be shown that z is quasi-cofinal in x. Suppose not. Then there is a $b \in x$ such that for all $u \in z$, $u \neq b$ and uRb. But y is quasi-cofinal in x, so there is a $v \in y$ such that $\neg vRb$. If there is a $u \in z$ such that vRu, then, since R is transitive on x, we would have vRb which contradicts the definition of v. Consequently, since R is connected on y, we must have uRv for all $u \in z$. But this contradicts the fact that z is quasi-cofinal in y. Thus, it follows that z is quasi-cofinal in x.

Corollary 2.10

(a) $\mathbf{C}(\mathsf{TR} \,\&\, \mathsf{C}, \mathsf{L}) \,\&\, \mathbf{C}(\mathsf{L}, \mathsf{W}) \equiv \mathbf{C}(\mathsf{TR} \,\&\, \mathsf{C}, \mathsf{W})$.
(b) $\mathbf{C}(\mathsf{D}, \mathsf{L}) \,\&\, \mathbf{C}(\mathsf{L}, \mathsf{W}) \equiv \mathbf{AC}$.

Similar results hold for the corresponding forms of Zorn's Lemma.

We summarize the results of this section in Figure 2.3. (See also Figure 4.1.)



A = $\{\mathbf{Z}(Q, U)$: $Q = \mathsf{TR}, \mathsf{P}, \mathsf{R}, \mathsf{F}$, or T, and $U = \mathsf{W}, \mathsf{L}$,
$\mathsf{TR} \,\&\, \mathsf{C}$, $\mathsf{AS} \,\&\, \mathsf{C}$, C, or D; or $Q = \mathsf{D}$ and $U = \mathsf{W}\}$,
B = $\{\mathbf{Z}(Q, U)$: $Q = \mathsf{L}$ or R, and $U = \mathsf{F}$ or T; or $Q = \mathsf{L}$ and $U = \mathsf{W}\}$,
C = $\{\mathbf{Z}(Q, \mathsf{R})$: $Q = \mathsf{P}$ or $\mathsf{D}\}$,
D = $\{\mathbf{Z}(Q, U)$: $Q = \mathsf{P}$ or D, and $U = \mathsf{F}$ or $\mathsf{T}\}$,
E = $\{\mathbf{Z}(\mathsf{TR}, U)$: $U = \mathsf{P}$ or $\mathsf{AS}\}$,
F = $\{\mathbf{Z}(\mathsf{D}, U)$: $U = \mathsf{C}$, $\mathsf{AS} \,\&\, \mathsf{C}$, $\mathsf{TR} \,\&\, \mathsf{C}$, or $\mathsf{L}\}$,
G = $\{\mathbf{Z}(\mathsf{TR} \,\&\, \mathsf{C}, U)$: $U = \mathsf{W}$, F, or $\mathsf{T}\}$,

H = {**Z**(TR & C, U): U = AS, AS & C, P, D, R, or L},
I = {**Z**(TR, U): U = F or T},
J = {**Z**(TR, R)},

Not ($\mathbf{NBG}^\circ \vdash \mathbf{AC}^{\mathbf{WO}} \rightarrow \mathrm{H}$)
Not ($\mathbf{NBG}^\circ \vdash \mathrm{B} \rightarrow \mathrm{A}$)
$\mathbf{NBG}^\circ \vdash \mathrm{A} \equiv \mathbf{AC}$
$\mathbf{NBG}^\circ \vdash (\mathrm{H}\ \&\ \mathrm{B}) \equiv \mathrm{G}$
$\mathbf{NBG}^\circ \vdash (\mathrm{F}\ \&\ \mathrm{B}) \equiv \mathbf{AC}$

Figure 2.3

3 *Variations of Hausdorff's Maximal Principle* In this section we consider variations of Hausdorff's Maximal Principle [13], which may be stated as:

Every partially ordered set contains a ⊆-maximal chain.

Again we vary two parts of the statement: the type of order on the set, and the type of ⊆-maximal subset. We denote such a variation by **H**(Q, U) to stand for the statement:

Every Q-ordered set contains a ⊆-maximal U-ordered subset.

As possibilities for Q and U we consider A, AS, TR, C, AS & C, TR & C, P, L, D, R, W, F, and T.

Using the fact that if $Q \rightarrow U$ then **H**(Q, U) is provable in $\mathbf{NBG}^\circ$ and that **H**(F, T) is provable by Lemma 2.1, we see that the following 66 statements are all provable in $\mathbf{NBG}^\circ$:

H(Q, U) for $Q = U$,
H(Q, A) for all Q,
H(Q, AS) for Q = AS & C, P, L, D, R, W, F, and T,
H(Q, TR) for Q = TR & C, P, L, D, R, W, F, and T,
H(Q, C) for Q = AS & C, TR & C, L, and W,
H(Q, AS & C) for Q = L and W,
H(Q, TR & C) for Q = L and W,
H(Q, P) for Q = L, D, R, W, F, and T,
H(Q, L) for Q = W,
H(Q, D) for Q = L and W,
H(Q, R) for Q = L, W, F, and T,
H(Q, F) for Q = W and T,
H(Q, T) for Q = W and F.

By using the set of integers in their natural order as a counterexample, we see that the following 30 statements are all false in $\mathbf{NBG}^\circ$: **H**(Q, U) for Q = A, AS, TR, C, AS & C, TR & C, P, L, D, or R, and U = W, F, or T.

We shall show that each of the remaining 73 statements are implied by **AC**; 52 of these statements imply **AC**, of the remaining 21, 10 imply **AC** if **LO** is assumed and the strength of the other 11 statements has not been determined.

To show that each of the 73 statements is implied by **AC**, we give another maximal principle called the *principle of finite character*. But first we define what it means for a property to be of *finite character*.

Definition 3.1 A non-empty property $\mathcal{P}$ is of *finite character* if a class X has the property $\mathcal{P}$ iff every finite subset of X has the property $\mathcal{P}$.

Each of the properties AS, TR, C, AS & C, TR & C, P, L, D, and R are properties of finite character while W, F, and T are not. The principle of finite character,

FC: *For every set x and every property* $\mathcal{P}$ *of finite character, there exists a* $\subseteq$*-maximal subset of x with the property* $\mathcal{P}$,

was shown independently by Teichmuller [28] and Tukey [29], to be equivalent to **AC**.

If being U-ordered is a property of finite character then $\mathbf{FC} \rightarrow \mathbf{H}(Q, U)$ for any Q. Also $\mathbf{H}(Q, \mathsf{L}) \equiv \mathbf{H}(Q, \mathsf{W})$ for $Q = \mathsf{F}$ or T. Thus, **FC** implies each of the following 73 statements:

$\mathbf{H}(\mathsf{A}, U)$ for $U =$ AS, TR, C, AS & C, TR & C, P, L, D, and R,
$\mathbf{H}(\mathsf{AS}, U)$ for $U =$ TR, C, AS & C, TR & C, P, L, D, and R,
$\mathbf{H}(\mathsf{TR}, U)$ for $U =$ AS, C, AS & C, TR & C, P, L, D, and R,
$\mathbf{H}(\mathsf{C}, U)$ for $U =$ AS, TR, AS & C, TR & C, P, L, D, and R,
$\mathbf{H}(\mathsf{AS}\ \&\ \mathsf{C}, U)$ for $U =$ TR, TR & C, P, L, D, and R,
$\mathbf{H}(\mathsf{TR}\ \&\ \mathsf{C}, U)$ for $U =$ AS, AS & C, P, L, D, and R,
$\mathbf{H}(\mathsf{P}, U)$ for $U =$ C, AS & C, TR & C, L, D, and R,
$\mathbf{H}(\mathsf{D}, U)$ for $U =$ C, AS & C, TR & C, L, and R,
$\mathbf{H}(\mathsf{R}, U)$ for $U =$ C, AS & C, TR & C, L, and D,
$\mathbf{H}(\mathsf{F}, U)$ for $U =$ C, AS & C, TR & C, L, D, and W,
$\mathbf{H}(\mathsf{T}, U)$ for $U =$ C, AS & C, TR & C, L, D, and W.

We shall prove first that each of the 28 statements in the preceding list for which $Q =$ P, D, R, F, or T implies **AC**. Using the fact that if $Q_1 \rightarrow Q_2$ then $\mathbf{H}(Q_2, U) \rightarrow \mathbf{H}(Q_1, U)$ for all U, we obtain the following implications:

$$\begin{aligned}
&\mathbf{H}(\mathsf{P}, \mathsf{L}) \rightarrow \mathbf{H}(\mathsf{D}, \mathsf{L}),\\
&\mathbf{H}(\mathsf{P}, \mathsf{L}) \rightarrow \mathbf{H}(\mathsf{R}, \mathsf{L}) \rightarrow \mathbf{H}(\mathsf{F}, \mathsf{L}),\\
&\mathbf{H}(\mathsf{P}, \mathsf{D}) \rightarrow \mathbf{H}(\mathsf{R}, \mathsf{D}) \rightarrow \mathbf{H}(\mathsf{F}, \mathsf{D}),\\
&\mathbf{H}(\mathsf{P}, \mathsf{R}) \rightarrow \mathbf{H}(\mathsf{D}, \mathsf{R}).
\end{aligned}$$

We also have that $\mathbf{H}(\mathsf{D}, \mathsf{L}) \rightarrow \mathbf{H}(\mathsf{P}, \mathsf{L})$ because a partially ordered set can be directed by simply adding on a greatest element. If $Q =$ P, D, or R then in a Q-ordered set C, AS & C, TR & C, and L are equivalent, and if $Q =$ F or T then in a Q-ordered set C, AS & C, TR & C, L, D, and W are all equivalent. Thus, we have,

$$\mathbf{H}(Q, \mathsf{C}) \equiv \mathbf{H}(Q, \mathsf{AS}\ \&\ \mathsf{C}) \equiv \mathbf{H}(Q, \mathsf{TR}\ \&\ \mathsf{C}) \equiv \mathbf{H}(Q, \mathsf{L})$$

for $Q =$ P, D, or R, and

$$\mathbf{H}(Q, \mathsf{C}) \equiv \mathbf{H}(Q, \mathsf{AS}\ \&\ \mathsf{C}) \neq \mathbf{H}(Q, \mathsf{TR}\ \&\ \mathsf{C}) \equiv \mathbf{H}(Q, \mathsf{L}) \equiv \mathbf{H}(Q, \mathsf{D}) \equiv \mathbf{H}(Q, \mathsf{W})$$

for $Q = \mathsf{F}$ or T. Moreover, a directed subset of a ramified set is a chain so that

$$\mathbf{H}(\mathsf{R}, \mathsf{D}) \equiv \mathbf{H}(\mathsf{R}, \mathsf{L}),$$

and it follows from Lemma 2.1 that $\mathbf{H}(\mathsf{F}, \mathsf{W}) \equiv \mathbf{H}(\mathsf{T}, \mathsf{W})$. It is well-known and easy to prove that $\mathbf{H}(\mathsf{P}, \mathsf{L}) \rightarrow \mathbf{Z}(\mathsf{P}, \mathsf{L})$. $\mathbf{Z}(\mathsf{P}, \mathsf{L})$, as mentioned in section 2, is the original form of Zorn's Lemma and is equivalent to **AC**. To complete the proof that each of the 28 statements implies **AC** it is sufficient to show that $\mathbf{H}(\mathsf{T}, \mathsf{W}) \rightarrow \mathbf{WO}$ and $\mathbf{H}(\mathsf{D}, \mathsf{R}) \rightarrow \mathbf{WO}$. (Felgner, [2], has shown that $\mathbf{H}(\mathsf{P}, \mathsf{D}) \rightarrow \mathbf{AC}$ and $\mathbf{H}(\mathsf{P}, \mathsf{R}) \rightarrow \mathbf{AC}$.)

Lemma 3.2 $\mathbf{H}(\mathsf{T}, \mathsf{W}) \rightarrow \mathbf{WO}$.

Proof: Let x be a non-empty set and w_x the set of well ordered subsets of x. w_x is a tree under the initial segment relation, I. By $\mathbf{H}(\mathsf{T}, \mathsf{W})$, w_x has a maximal well ordered subset, w^*. Now $\bigcup(\mathfrak{D}(w^*))$ ordered by $\bigcup(\mathfrak{R}(w^*))$ is a maximal well ordered subset of x. This must be a well ordering of all of x. Therefore, $\mathbf{H}(\mathsf{T}, \mathsf{W}) \rightarrow \mathbf{WO}$. Q.E.D.

Theorem 3.3 $\mathbf{H}(\mathsf{D}, \mathsf{R}) \rightarrow \mathbf{WO}$.

Proof: Let x be a set and w_x ordered by I the tree of well ordered subsets of x under the initial segment relation. (We will often write uIv for $\langle u, R_u\rangle \, I \, \langle v, R_v\rangle$ in this proof.) Since w_x is directed downwards by I, w_x ordered by I^*, where uI^*v iff vIu, is directed upwards. By $\mathbf{H}(\mathsf{D}, \mathsf{R})$, w_x has a $\subseteq$-maximal I^*-ramified subset, r. $r \neq \emptyset$ so there exists $\langle y, R_y\rangle \in r$, and since r is ramified $\overleftarrow{\langle y, R_y\rangle}$ is linearly ordered. Moreover, this set contains elements $\langle u, R_u\rangle$ such that yIu. Denote $\overleftarrow{\langle y, R_y\rangle}$ by $\overleftarrow{y}$. Since $\overleftarrow{y}$ is linearly ordered we can form $\bigcup_{\langle u, R_u\rangle \in \overleftarrow{y}} u$ and order this by $\bigcup_{\langle u, R_u\rangle \in \overleftarrow{y}} R_u$. We claim that $\bigcup R_u$ well orders $\bigcup u$ and that $\bigcup u = x$. This is obvious if one recognizes that $\overleftarrow{y}$ is a quasi-cofinal I-chain in w_x. But here is a verification.

Denote $\bigcup R_u$ by R^* and $\bigcup u$ by u^*. Let $\emptyset \neq s \subseteq u^*$. Then there exists $\langle u, R_u\rangle \in \overleftarrow{y}$ such that $s \cap u \neq \emptyset$. Let $n \in s \cap u$. If mR^*n and $m \in s$, then $m \in u$ because u is an initial segment of u^*. Thus $J_n = \{m \in s : mR^*n\} \subseteq u$. Since R^* and R_u agree on u, the R_u-least element of J_n is the R^*-least element of J_n and also the R^*-least element of s. Therefore, R^* well orders u^*.

Suppose $x \sim u^* \neq \emptyset$. Let $n \in x \sim u^*$. Consider r', ordered by I^*, where $r' = r \cup \{\langle u_n, R_n\rangle\}$, with the order R_n on $u_n = u^* \cup \{n\}$ being $R^* \cup \{\langle v, n\rangle : v \in u^*\} \cup \{\langle n, n\rangle\}$. We claim r' is a larger ramified subset of $\langle w_x, I^*\rangle$. Clearly, $r \subset r'$. Suppose r' is not ramified. Then some initial segment containing $\langle u_n, R_n\rangle$ is not linearly ordered. So there are pairs $\langle v, R_v\rangle$ and $\langle w, R_w\rangle \in r$ such that vIu_n and vIw, but w and u_n are not I-related. Now yIu_n, so yIv or vIy. If yIv, then yIw. But then by construction of u_n, wIu_n. On the other hand, suppose vIy. We have that v, y, and w are in r which is ramified. As y and w are both elements of $\overleftarrow{v}$, both are extensions of $\langle v, R_v\rangle$. $\overleftarrow{v}$ is linearly ordered so wIy or yIw. If wIy then wIu_n because yIu_n. Whereas if yIw, then wIu_n by construction of u_n. Thus every initial segment of r' is linearly ordered and hence r' is ramified. This contradicts the maximality of r. Therefore, $u^* = x$, and x is well ordered by R^*. Q.E.D.

This completes the proof that the 28 statements in the list for which $Q = \mathsf{P}, \mathsf{D}, \mathsf{R}, \mathsf{F}$, or T are each equivalent to **AC**.

Now we consider the remaining 45 statements from our list. In a transitive and connected set AS, AS & C, P, D, and R are each equivalent to L. Thus

$$\mathbf{H}(\mathsf{TR}\ \&\ \mathsf{C}, U) \equiv \mathbf{H}(\mathsf{TR}\ \&\ \mathsf{C}, \mathsf{L})$$

for $U = \mathsf{AS}, \mathsf{AS}\ \&\ \mathsf{C}, \mathsf{P}, \mathsf{D}$, and R. In an antisymmetric and connected set each of TR, TR & C, P, D, and R is equivalent to L, so

$$\mathbf{H}(\mathsf{AS}\ \&\ \mathsf{C}, U) \equiv \mathbf{H}(\mathsf{AS}\ \&\ \mathsf{C}, \mathsf{L})$$

for $U = \mathsf{TR}, \mathsf{TR}\ \&\ \mathsf{C}, \mathsf{P}, \mathsf{D}$, and R. In a connected set AS and AS & C are equivalent, TR and TR & C are equivalent, and each of P, D, and R is equivalent to L. Thus

$$\begin{gathered}\mathbf{H}(\mathsf{C}, \mathsf{AS}) \equiv \mathbf{H}(\mathsf{C}, \mathsf{AS}\ \&\ \mathsf{C}),\\ \mathbf{H}(\mathsf{C}, \mathsf{TR}) \equiv \mathbf{H}(\mathsf{C}, \mathsf{TR}\ \&\ \mathsf{C}),\\ \mathbf{H}(\mathsf{C}, \mathsf{P}) \equiv \mathbf{H}(\mathsf{C}, \mathsf{D}) \equiv \mathbf{H}(\mathsf{C}, \mathsf{R}) \equiv \mathbf{H}(\mathsf{C}, \mathsf{L}).\end{gathered}$$

In a transitive set AS and P are equivalent, C and TR & C are equivalent, and AS & C and L are equivalent. Therefore,

$$\begin{gathered}\mathbf{H}(\mathsf{TR}, \mathsf{AS}) \equiv \mathbf{H}(\mathsf{TR}, \mathsf{P}),\\ \mathbf{H}(\mathsf{TR}, \mathsf{C}) \equiv \mathbf{H}(\mathsf{TR}, \mathsf{TR}\ \&\ \mathsf{C}),\\ \mathbf{H}(\mathsf{TR}, \mathsf{AS}\ \&\ \mathsf{C}) \equiv \mathbf{H}(\mathsf{TR}, \mathsf{L}).\end{gathered}$$

Similarly, we have

$$\begin{gathered}\mathbf{H}(\mathsf{AS}, \mathsf{TR}) \equiv \mathbf{H}(\mathsf{AS}, \mathsf{P}),\\ \mathbf{H}(\mathsf{AS}, \mathsf{C}) \equiv \mathbf{H}(\mathsf{AS}, \mathsf{AS}\ \&\ \mathsf{C}).\\ \mathbf{H}(\mathsf{AS}, \mathsf{TR}\ \&\ \mathsf{C}) \equiv \mathbf{H}(\mathsf{AS}, \mathsf{L}).\end{gathered}$$

It is sufficient, therefore, to consider the following 24 statements:

$\mathbf{H}(\mathsf{A}, U)$ for $U = \mathsf{AS}, \mathsf{TR}, \mathsf{C}, \mathsf{AS}\ \&\ \mathsf{C}, \mathsf{TR}\ \&\ \mathsf{C}, \mathsf{P}, \mathsf{L}, \mathsf{D}$, and R,
$\mathbf{H}(\mathsf{AS}, U)$ for $U = \mathsf{P}, \mathsf{C}, \mathsf{L}, \mathsf{D}$, and R,
$\mathbf{H}(\mathsf{TR}, U)$ for $U = \mathsf{P}, \mathsf{C}, \mathsf{L}, \mathsf{D}$, and R,
$\mathbf{H}(\mathsf{C}, U)$ for $U = \mathsf{AS}, \mathsf{TR}, \mathsf{L}$,
$\mathbf{H}(Q, \mathsf{L})$ for $Q = \mathsf{AS}\ \&\ \mathsf{C}$ and $\mathsf{TR}\ \&\ \mathsf{C}$.

Using the preceding results plus the fact that if $Q_1 \rightarrow Q_2$ then $\mathbf{H}(Q_2, U) \rightarrow \mathbf{H}(Q_1, U)$ we obtain the following:

3.4

$$\begin{tikzcd}
 & \mathbf{H}(\mathsf{TR}, \mathsf{P}) \arrow[dr] & \\
\mathbf{H}(\mathsf{A}, \mathsf{AS}) \arrow[ur] \arrow[dr] & & \mathbf{H}(\mathsf{TR}\ \&\ \mathsf{C}, \mathsf{L}) \\
 & \mathbf{H}(\mathsf{C}, \mathsf{AS}) \arrow[ur] &
\end{tikzcd}$$

3.5

$$\begin{tikzcd}
 & \mathbf{H}(\mathsf{AS}, \mathsf{P}) \arrow[dr] & \\
\mathbf{H}(\mathsf{A}, \mathsf{TR}) \arrow[ur] \arrow[dr] & & \mathbf{H}(\mathsf{AS}\ \&\ \mathsf{C}, \mathsf{L}) \\
 & \mathbf{H}(\mathsf{C}, \mathsf{TR}) \arrow[ur] &
\end{tikzcd}$$

3.6
$$\begin{tikzcd}
 & \mathsf{H}(\mathsf{AS}, \mathsf{C}) \arrow[dr] & \\
\mathsf{H}(\mathsf{A}, \mathsf{C}) \arrow[ur] \arrow[dr] & & \mathsf{H}(\mathsf{P}, \mathsf{C}) \equiv \mathbf{AC} \\
 & \mathsf{H}(\mathsf{TR}, \mathsf{C}) \arrow[ur] &
\end{tikzcd}$$

3.7 $\mathsf{H}(\mathsf{A}, \mathsf{AS}\ \&\ \mathsf{C}) \rightarrow \mathsf{H}(\mathsf{AS}, \mathsf{C}) \rightarrow \mathsf{H}(\mathsf{P}, \mathsf{C}) \equiv \mathbf{AC}$

3.8 $\mathsf{H}(\mathsf{A}, \mathsf{TR}\ \&\ \mathsf{C}) \rightarrow \mathsf{H}(\mathsf{TR}, \mathsf{C}) \rightarrow \mathsf{H}(\mathsf{P}, \mathsf{C}) \equiv \mathbf{AC}$

3.9
$$\begin{tikzcd}
 & \mathsf{H}(\mathsf{AS}, \mathsf{P}) \arrow[r] & \mathsf{H}(\mathsf{AS}\ \&\ \mathsf{C}, \mathsf{L}) \\
\mathsf{H}(\mathsf{A}, \mathsf{P}) \arrow[ur] \arrow[r] \arrow[dr] & \mathsf{H}(\mathsf{C}, \mathsf{L}) \arrow[ur] \arrow[r] & \mathsf{H}(\mathsf{TR}\ \&\ \mathsf{C}, \mathsf{L}) \\
 & \mathsf{H}(\mathsf{TR}., \mathsf{P}) \arrow[ur] &
\end{tikzcd}$$

3.10
$$\begin{tikzcd}
 & & \mathsf{H}(\mathsf{P}, U) \equiv \mathbf{AC} \\
 & \mathsf{H}(\mathsf{AS}, U) \arrow[ur] \arrow[r] & \mathsf{H}(\mathsf{AS}\ \&\ \mathsf{C}, \mathsf{L}) \\
\mathsf{H}(\mathsf{A}, U) \arrow[ur] \arrow[r] \arrow[dr] & \mathsf{H}(\mathsf{C}, U) \arrow[ur] \arrow[r] & \mathsf{H}(\mathsf{TR}\ \&\ \mathsf{C}, \mathsf{L}) \\
 & \mathsf{H}(\mathsf{TR}, U) \arrow[ur] \arrow[r] & \mathsf{H}(\mathsf{P}, U) \equiv \mathbf{AC}
\end{tikzcd}$$

for $U = \mathsf{L}$, D, and R.

Lemma 3.11 $\mathsf{H}(\mathsf{AS}, \mathsf{C}) \equiv \mathsf{H}(\mathsf{C}, \mathsf{AS})$.

Proof: A relation is antisymmetric if and only if its complement is connected.

Consequently, it follows from 3.11 and 3.7 that $\mathsf{H}(\mathsf{C}, \mathsf{AS}) \equiv \mathbf{AC}$.

Lemma 3.12 $\mathsf{H}(\mathsf{TR}, \mathsf{P}) \rightarrow \mathbf{AC}$.

Proof: Let x be a non-empty set of non-empty pairwise disjoint sets. Define a relation S on $\bigcup x$ such that for $a, b \in \bigcup x$ where $a \in u \in x$ and $b \in v \in x$, aSb iff $u = v$. S is transitive and a maximum S-antisymmetric subset of $\bigcup x$ is a choice set for x. Q.E.D.

Next, suppose $\mathbf{AC^{LO}}$ is the axiom of choice for linearly ordered sets. That is, $\mathbf{AC^{LO}}$ is the statement:

For each non-empty, linearly ordered set of non-empty pairwise disjoint sets, there is a choice set.

Lemma 3.13 $\mathsf{H}(\mathsf{TR}\ \&\ \mathsf{C}, \mathsf{L}) \equiv \mathbf{AC^{LO}}$.

Proof: Let $\langle x, \leqslant\rangle$ be a non-empty, linearly ordered set of pairwise disjoint non-empty sets. Define a relation S on $\bigcup x$ such that for $a, b \in \bigcup x$, where $a \in u \in x$ and $b \in v \in x$, aSb iff $u \leqslant v$. S is transitive and connected. A maximal S-antisymmetric subset of $\bigcup x$ is a choice set for x.

Conversely, suppose $\mathbf{AC^{LO}}$ holds. Let x be a set, and R a relation on x which is transitive and connected. Define a relation S on x such that for $u, v \in x$,

$$uSv \text{ iff } (uRv \ \&\ vRu) \text{ or } u = v.$$

S is an equivalence relation on x. For $u \in x$, let $[u] = \{v \in x\colon vSu\}$ and let $y = \{[u]\colon u \in x\}$. Since R is transitive, if $u, v \in x$, $u', u'' \in [u]$ and $v', v'' \in [v]$, then $u'Rv'$ if and only if $u''Rv''$. Thus, we can define a relation T on y so that if $u, v \in x$,

$$[u]\ T[v] \text{ iff } (\forall u' \in [u])(\forall v' \in [v])u'Rv'.$$

The relation T linearly orders y and a choice set for y is a $\subseteq$-maximal linearly ordered subset of x. Q.E.D.

Now, it follows that each of the following maximal principles is equivalent to **AC** in **NBG°**:

$\mathrm{H}(\mathrm{A}, U)$ for $U = \mathrm{AS}, \mathrm{C}, \mathrm{AS}\ \&\ \mathrm{C}, \mathrm{TR}\ \&\ \mathrm{C}, \mathrm{P}, \mathrm{L}, \mathrm{D}$, and R,
$\mathrm{H}(\mathrm{AS}, U)$ for $U = \mathrm{C}, \mathrm{AS}\ \&\ \mathrm{C}, \mathrm{TR}\ \&\ \mathrm{C}, \mathrm{L}, \mathrm{D}$, and R,
$\mathrm{H}(\mathrm{TR}, U)$ for $U = \mathrm{AS}, \mathrm{C}, \mathrm{AS}\ \&\ \mathrm{C}, \mathrm{TR}\ \&\ \mathrm{C}, \mathrm{P}, \mathrm{L}, \mathrm{D}$, and R,
$\mathrm{H}(\mathrm{C}, U)$ for $U = \mathrm{AS}$ and $\mathrm{AS}\ \&\ \mathrm{C}$,
$\mathrm{H}(\mathrm{P}, U)$ for $U = \mathrm{C}, \mathrm{AS}\ \&\ \mathrm{C}, \mathrm{TR}\ \&\ \mathrm{C}, \mathrm{L}, \mathrm{D}$, and R,
$\mathrm{H}(\mathrm{D}, U)$ for $U = \mathrm{C}, \mathrm{AS}\ \&\ \mathrm{C}, \mathrm{TR}\ \&\ \mathrm{C}, \mathrm{L}$, and R,
$\mathrm{H}(\mathrm{R}, U)$ for $U = \mathrm{C}, \mathrm{AS}\ \&\ \mathrm{C}, \mathrm{TR}\ \&\ \mathrm{C}, \mathrm{L}$, and D,
$\mathrm{H}(\mathrm{F}, U)$ for $U = \mathrm{C}, \mathrm{AS}\ \&\ \mathrm{C}, \mathrm{TR}\ \&\ \mathrm{C}, \mathrm{L}, \mathrm{D}$, and W,
$\mathrm{H}(\mathrm{T}, U)$ for $U = \mathrm{C}, \mathrm{AS}\ \&\ \mathrm{C}, \mathrm{TR}\ \&\ \mathrm{C}, \mathrm{L}, \mathrm{D}$, and W.

It follows from 3.5 and the preceding results that,

$$\begin{tikzcd}
 & & \mathrm{H}(\mathrm{AS}, \mathrm{TR}) \equiv \mathrm{H}(\mathrm{AS}, \mathrm{P}) \arrow[dr] & \\
\mathbf{AC} \arrow[r] & \mathrm{H}(\mathrm{A}, \mathrm{TR}) \arrow[ur] \arrow[dr] & & \mathrm{A} \\
 & & \mathrm{H}(\mathrm{C}, \mathrm{TR}) \equiv \mathrm{H}(\mathrm{C}, \mathrm{TR}\ \&\ \mathrm{C}) \arrow[ur] &
\end{tikzcd}$$

where

$$\mathrm{A} = \{\mathrm{H}(\mathrm{AS}\ \&\ \mathrm{C}, U) \colon U = \mathrm{TR}, \mathrm{TR}\ \&\ \mathrm{C}, \mathrm{P}, \mathrm{L}, \mathrm{D}, \text{ or } \mathrm{R}\}.$$

Each statement in A is equivalent to $\mathrm{H}(\mathrm{AS}\ \&\ \mathrm{C}, \mathrm{L})$.

Using 3.9 and Lemma 3.13 we obtain

$$\begin{tikzcd}
 & & \mathrm{A} \\
\mathbf{AC} \arrow[r] & \mathrm{B} \arrow[ur] \arrow[dr] & \\
 & & \mathrm{C} \equiv \mathbf{AC}^{\mathbf{LO}}
\end{tikzcd}$$

where

$$\mathrm{B} = \{\mathrm{H}(\mathrm{C}, U) \colon U = \mathrm{P}, \mathrm{L}, \mathrm{D}, \text{ or } \mathrm{R}\},$$
$$\mathrm{C} = \{\mathrm{H}(\mathrm{TR}\ \&\ \mathrm{C}, U) \colon U = \mathrm{AS}, \mathrm{AS}\ \&\ \mathrm{C}, \mathrm{P}, \mathrm{L}, \mathrm{D}, \text{ or } \mathrm{R}\}.$$

Each statement in B is equivalent to $\mathrm{H}(\mathrm{C}, \mathrm{L})$, and each statement in C is equivalent to $\mathrm{H}(\mathrm{TR}\ \&\ \mathrm{C}, \mathrm{L})$. (Also see Figure 4.2.)

It is also true that $\mathrm{H}(\mathrm{TR}\ \&\ \mathrm{C}, \mathrm{AS}) \rightarrow \mathbf{Z}(\mathrm{TR}\ \&\ \mathrm{C}, \mathrm{AS})$. For let x be a set and R a transitive and connected relation on x such that every antisymmetric subset of x has an R-upper bound. $\mathrm{H}(\mathrm{TR}\ \&\ \mathrm{C}, \mathrm{AS})$ implies that x has a $\subseteq$-maximal antisymmetric subset y. It is easy to see that an R-upper bound for y is an R-maximal element for x. Thus, it follows from Lemma 3.13 that $\mathbf{AC}^{\mathbf{LO}} \rightarrow \mathbf{Z}(\mathrm{TR}\ \&\ \mathrm{C}, \mathrm{AS})$.

(We can also show that $\mathrm{H}(\mathrm{AS}\ \&\ \mathrm{C}, \mathrm{TR})$ does not imply **AC** in **NBG°**, by using the standard Fraenkel-Mostowski model [19] with a countable set of urelements U. The group is the set of all permutations on U and the filter is generated by subgroups which leave a finite number of urelements fixed.

Suppose x is a set in the model and R a relation on x which is antisymmetric and connected. Define a function f on the two element subsets of x such that if $u, v \in x, u \neq v$,

$$f(\{u, v\}) = u \textit{ iff } uRv.$$

Suppose R is also reflexive. f is in the model and has finite support. Let S = (support of f) $\cup$ (support of x). S is a finite subset of U. We claim that for $u \in x$, (support of u) $\subseteq S$. For suppose not. Let $a \in$ (support of u) $\sim S$ and let $b \in U \sim$ ((support of u) $\cup S$). Let σ be a permutation which permutes a and b and leaves all the other urelements fixed. Let $v = \sigma(u)$, then since $\sigma(x) = x, v \in x$. Furthermore, $\sigma(\langle\{u, v\}, u\rangle) = \langle\{u, v\}, v\rangle$, but this is a contradiction because σ leaves f fixed.

Let $\mathfrak{H}$ be the group of all permutations which leave S pointwise fixed. Then $\mathfrak{H}$ leaves x pointwise fixed so it follows from properties of Fraenkel-Mostowski models that x can be well ordered. Thus, we may choose a $\subseteq$-maximal transitive subset of x. Therefore, **H**(AS & C, TR) holds in the model but it is well known that **AC** does not. In fact, **H**(C, P) does not hold in this model. For let x be the set of all finite sequences of urelements without repetition. Order x by the relation R so that if $u, v \in x$, uRv iff length of $u \leqslant$ length of v. R is connected. If there were a $\subseteq$-maximal partially ordered subset of x, then we could choose a countably infinite subset of U. But this is impossible.

Moreover, in this same Fraenkel-Mostowski model, **Z**(L, W) is true but $\mathbf{AC}^{\mathrm{WO}}$ is false. Thus we can strengthen the result Not($\mathbf{NBG}^{\circ} \vdash \mathrm{B} \rightarrow \mathbf{AC}$) in Figure 2.3 to Not($\mathbf{NBG}^{\circ} \vdash \mathrm{B} \rightarrow \mathbf{AC}^{\mathrm{WO}}$). See also Figure 4.1.)

Let **H**$'(Q, U)$ be the statement:

In every Q-ordered set, every U-ordered subset can be extended to a $\subseteq$-maximal U-ordered subset.

Then it is easy to see that **FC** $\rightarrow$ **H**$'(Q, U)$ for all properties Q and U, where U is a property of finite character. Also, **H**$'(Q, U) \rightarrow$ **H**(Q, U) for all Q and U.

Lemma 3.14 **H**$'$(A, TR) $\rightarrow$ **AC**.

Proof: Let x be a non-empty set of non-empty pairwise disjoint sets. Suppose x has at least two elements. Define a relation S on $\bigcup x$ such that if $a, b \in \bigcup x$, where $a \in u \in x$ and $b \in v \in x$, then

$$aSb \textit{ iff } a = b \textit{ or } u \neq v.$$

Let $u, v \in x, u \neq v$, and $a \in u, b \in v$. Then $\{a, b\}$ is an S-transitive subset of x. Any $\subseteq$-maximal S-transitive subset of x which contains $\{a, b\}$ is a choice set for x. Q.E.D.

Let $\mathbf{AC}_{\mathbf{LO}}$ be the statement:

There is a choice set for each non-empty set of non-empty pairwise disjoint linearly ordered sets.

Lemma 3.15 $\mathrm{H}'(\mathrm{C}, \mathrm{TR}) \to \mathrm{AC}_{\mathrm{LO}}$.

Proof: Let x be a non-empty set of non-empty pairwise disjoint linearly ordered sets. ($\leq$ is the linear ordering.) Define a relation S on $\bigcup x$ such that for $a, b \in \bigcup x$, $a \in u$ and $b \in v$,

$$aSb \text{ iff } (u = v \ \& \ a \leq b) \text{ or } u \neq v).$$

S is connected on $\bigcup x$. Suppose $u, v \in x$, $u \neq v$, $a \in u$, and $b \in v$. A $\subseteq$-maximal transitive subset of $\bigcup x$ containing $\{a, b\}$ is a choice set for x. Q.E.D.

We have no further information about the relative strength of the statements $\mathrm{H}(\mathrm{A}, \mathrm{TR})$, $\mathrm{H}(\mathrm{AS}, \mathrm{TR})$, $\mathrm{H}'(\mathrm{C}, \mathrm{TR})$ or $\mathrm{H}(\mathrm{AS} \ \& \ \mathrm{C}, \mathrm{TR})$.

4 *Appendix* The following is a list of statements used in this paper with the abbreviations used to denote them.

AC: *For each non-empty set x of non-empty sets there is a function f (called a choice function) such that for each $u \in x$, $f(u) \in u$. (Equivalently, for each non-empty set x of non-empty pairwise disjoint sets, there is a set c (called a choice set) such that for each $u \in x$, $c \cap u$ is a singleton.)*

$\mathrm{AC}^{\mathrm{LO}}$: *There is a choice function (set) for each non-empty, linearly ordered set of non-empty (pairwise disjoint) sets.*

$\mathrm{AC}_{\mathrm{LO}}$: *There is a choice function (set) for each non-empty set of non-empty, (pairwise disjoint) linearly ordered sets.*

$\mathrm{AC}^{\mathrm{WO}}$: *There is a choice function (set) for each non-empty, well ordered set of non-empty (pairwise disjoint) sets.*

AR: *For each non-empty set x there is a $y \in x$ such that $y \cap x = \emptyset$.*

BPI: *Every Boolean algebra contains a prime ideal.*

Cof: *Every linearly ordered set contains a cofinal well ordered subset.*

$\mathrm{C}(Q, U)$: *Every Q-ordered set contains a quasi-cofinal U-ordered subset.*

FC: *For every set x and every property of finite character $\mathcal{P}$, there is a $\subseteq$-maximal subset of x with the property $\mathcal{P}$.*

GKA: *For every non-empty set $x = \{\langle u, s_u\rangle : s_u$ is a partial ordering on $u\}$, there is a function f on x such that for each $\langle u, s_u\rangle \in x$, $f(u, s_u)$ is a $\subseteq$-maximal antichain in u.*

$\mathrm{H}(Q, U)$: *Every Q-ordered set contains a $\subseteq$-maximal, U-ordered subset.*

$\mathrm{H}'(Q, U)$: *Every Q-ordered set has the property that each U-ordered subset can be extended to a $\subseteq$-maximal U-ordered subset.*

KA: *Every partially ordered set contains a $\subseteq$-maximal antichain.*

LO: *Every set can be linearly ordered.*

LW: *Every linearly ordered set can be well ordered.*

OE: *Every partial ordering can be extended to a linear ordering.*

WO: *Every set can be well ordered.*

Z(Q, U): *Every non-empty Q-ordered set, in which each U-ordered subset has an upper bound, has a maximal element.*

Z(Q, U, l): *Every non-empty Q-ordered set, in which each U-ordered subset has a least upper bound, has a maximal element.*

In Figures 4.1 and 4.2, see pp. 591 and 592, we summarize our results. So, for example, from Figure 4.1 we see that ¬**Z**(AS, TR) is provable in **NBG°**, while **Z**(P, R) is in the set C and the diagram at the bottom describes the relative strength of statements in this set.

Z(Q, U) Q U	A	AS	TR	C	AS & C	TR & C	P	L	D	R	W	F	T
A	x	x	x	x	x	x	x	x	x	x	x	x	x
AS	o	x	E	o	x	H	x	x	x	x	x	x	x
TR	o	o	x	o	o	x	x	x	x	x	x	x	x
C	o	o	AC	x	x	x	AC	x	F	AC	x	AC	AC
AS & C	o	o	AC	o	x	H	AC	x	F	AC	x	AC	AC
TR & C	o	o	AC	o	o	x	AC	x	F	AC	x	AC	AC
P	o	o	E	o	o	H	x	x	x	x	x	x	x
L	o	o	AC	o	o	H	AC	x	F	AC	x	AC	AC
D	o	o	AC	o	o	H	AC	x	x	AC	x	AC	AC
R	o	o	J	o	o	H	C	x	C	x	x	x	x
W	o	o	AC	o	o	G	AC	B	AC	AC	x	AC	AC
F	o	o	I	o	o	G	D	B	D	B	x	x	x
T	o	o	I	o	o	G	D	B	D	B	x	x	x

o = The negation is provable in **NBG°**

x = Provable in **NBG°** AC = Equivalent to AC in **NBG°**

$$\begin{tikzcd}
 & & \mathsf{AC}^{\mathsf{LO}} \arrow[r] & H \arrow[r] & \mathsf{AC}^{\mathsf{WO}} \\
\mathsf{AC} \arrow[urr] \arrow[r] \arrow[dd] & F \arrow[r] & C \arrow[ur] & & \\
 & J \arrow[ur] \arrow[r] & E & & \\
I \arrow[ur] \arrow[r] & D \arrow[uur] \arrow[r] & G \arrow[uuur] \arrow[r] & B \arrow[r, "\mathsf{OE}"] & \mathsf{AC}
\end{tikzcd}$$

Not(**NBG°** ⊢ B → $\mathsf{AC}^{\mathsf{WO}}$) Not(**NBG°** ⊢ $\mathsf{AC}^{\mathsf{WO}}$ → H)

NBG° ⊢ (H & B) ≡ G **NBG°** ⊢ (F & B) ≡ AC

Figure 4.1

H(Q, U) Q / U	A	AS	TR	C	AS & C	TR & C	P	L	D	R	W	F	T
A	x	x	x	x	x	x	x	x	x	x	x	x	x
AS	AC	x	AC	AC	x	C	x	x	x	x	x	x	x
TR	D	E	x	F	A	x	x	x	x	x	x	x	x
C	AC	AC	AC	x	x	x	AC	x	AC	AC	x	AC	AC
AS & C	AC	AC	AC	AC	x	C	AC	x	AC	AC	x	AC	AC
TR & C	AC	AC	AC	F	A	x	AC	x	AC	AC	x	AC	AC
P	AC	E	AC	B	A	C	x	x	x	x	x	x	x
L	AC	AC	AC	B	A	C	AC	x	AC	AC	x	AC	AC
D	AC	AC	AC	B	A	C	AC	x	x	AC	x	AC	AC
R	AC	AC	AC	B	A	C	AC	x	AC	x	x	x	x
W	o	o	o	o	o	o	o	o	o	o	x	AC	AC
F	o	o	o	o	o	o	o	o	o	o	x	x	x
T	o	o	o	o	o	o	o	o	o	o	x	x	x

o = The negation is provable in **NBG**$^\circ$

x = Provable in **NBG**$^\circ$ AC = Equivalent to **AC** in **NBG**$^\circ$

$$\begin{tikzcd}
 & & A \\
\mathbf{AC} \arrow[r] & B \arrow[ur] \arrow[dr] & \\
 & & C \equiv \mathbf{AC}^{\mathbf{LO}}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & E \arrow[dr] & \\
\mathbf{AC} \arrow[r] & D \arrow[ur] \arrow[dr] & & A \\
 & & F \arrow[ur] &
\end{tikzcd}$$

$\text{Not}(\mathbf{NBG}^\circ \vdash A \to B)$

Figure 4.2

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To be continued

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