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METACOMPLETENESS

ROBERT K. MEYER

In [1], a logic was called *coherent* provided that it could plausibly be interpreted in its own metalogic. By deepening and making more intuitive the logical analysis implicit in [1], we develop here a kindred notion of *metacompleteness*; a logic is metacomplete provided that *exactly* the sentences true on a certain preferred interpretation of that logic in its metalogic are theorems. Acquaintance with [1] is not presupposed. We shall show in particular that a number of familiar logics, e.g., of the intuitionist, modal, and relevant families, are metacomplete, and that accordingly these logics share with intuitionist calculi two interesting properties:

(A) If $A \vee B$ is a theorem, so is at least one of A or B.
(B) If $\exists x A(x)$ is a theorem, so is some substitution instance $A(t)$, for some term t.

Harrop, Rasiowa, and Kleene have found simple truth-functional-style arguments for (A) and (B) in the case of intuitionist theories in particular, as Kripke has called to my attention. Here, by building on the techniques of [1], we present such results in a wider context, applicable in particular to the relevant logics whose theory is set out in [5].[1]

Negation was treated classically in [1], but our main interest here will be in logics that are either negation-free or which formalize a negation acceptable from a generally constructive point of view. This is less restrictive than it sounds, in particular for the relevant logics, for which some of our results are new, since the methods of [2] may be applied to show, at least in their sentential parts, their negation-free theorems captured by their negation-free axioms and rules; the result is that the relevant logics satisfy the motivating conditions (A) and (B) in their positive parts.[2] As we shall note for the system **R** of relevant implication in particular, if the classical negation axioms for **R** of Anderson and Belnap are replaced with intuitionistically acceptable ones, getting a system **RJ**, (A) and (B) hold throughout.

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1 Who among us was not at first a bit astonished to learn that

(1) $p \vee . p \rightarrow q$

is a theorem of sentential logic? (I use, here and afterwards, 'p', 'q', etc., to refer to sentential variables, 'A', 'B', to refer to formulas, and dots according to the Curry conventions in [3]; I also associate to the left, rank binary connectives &, v, → in order of increasing scope, and assume that unary connectives and quantifiers have minimal scope to resolve ambiguities.) For on any logically vertebrate construal of '→', the right disjunct is just patently *false*. For everyone from Quine to Anderson agrees that logical implication is a matter of genuine formal connection, and there is none such between 'p' and 'q' in the formula '$p \rightarrow q$'. So, it would seem, if we are *seriously* to understand '→' as *implies*, (1) is falsifiable on assigning the truth-value **F** to 'p', '$p \rightarrow q$' being false in any event. That, of course, is not the way it works, though there remains the lingering suspicion that that is the way it ought to work. Reserving henceforth *entails* for that relation which obtains between A and B just when B is a logical consequence of A, we may, on well-worn ground, distinguish between two ways in which a logical system may attempt to capture the notion of entailment; let us suppose that the capturing connective is '→'. First, '→' may merely *indicate* entailment, i.e., furnish a sort of *clue*, as it were; the traditional clue is that A *entails* B, according to a logic **L**, just in case $A \rightarrow B$ is *provable* in **L**, or is *logically true* according to some semantical analysis of **L**. But it is *not* requisite for '→' to *indicate* entailment but '→' should be understood in any vertebrate sense to *mean* 'entails'. In particular, as we all know there is no classical objection to (1), since on classical semantical analysis the falsehood of 'p' suffices for the *truth* of '$p \rightarrow q$'. Nevertheless, 'p' does not classically entail 'q', since the necessary clue, that '$p \rightarrow q$' be a theorem, is missing.

For Lewis and his followers, a connective that merely *indicates* entailment is too vapid. Their favored connective not only indicates but *expresses* entailment.[3] Various deeper issues, of the sort contemplated in the footnote, arise here, but let us zero in on a surface truth-functional analysis of '$p \rightarrow q$', where '→' is intended to *mean* 'entails' and 'p' and 'q' are particular, distinct sentential variables. Are we to contemplate the possibility of interpreting this particular formal sentence as *true*, or shall we merely dismiss it as *false*? The correct answer, I think, is that it depends on how seriously we want to take our formalism. If we are thinking of 'p', 'q', and so forth, simply as *placeholders* for sentences of some communicative language, English, for example, or even an *applied* first-order logic, we can certainly think of '$p \rightarrow q$' as sometimes true; e.g., let 'p' be 'entropy is increasing everywhere' and 'q' be 'entropy is increasing in Santa Monica', for example. If, on the other hand, we are dead serious about our formal language, thinking of 'p', 'q' as *irreducible* atomic sentences, distinct and without structural connections, then our original naive intuition that '$p \rightarrow q$' is *false* is precisely correct, whether 'p' and 'q' be thought of as true or false, under the assumption that '→' expresses entailment.

I do not say that the dead serious view of a formalism in which entailment is to be expressed is the only one that might be adopted. For certain purposes, the placeholder, or *dummy*, in Quine's unforgettable phrase, view of sentential variables is natural and useful. But if it is entailment that we are trying to formalize, I insist, then there is something wrong with a formalism that does not *admit* an interpretation in which $A \to B$ is true exactly when it is provable. Not everything classically true on such an interpretation need be provable—usually $\overline{p \to q}$ would not be, since we want to accommodate the placeholder view as well[4]—but at least everything provable should be true. For if we have really succeeded in formalizing the intuitive notion of entailment, in the strong, expressive sense, in a logic **L**, what would count as more natural and compelling evidence that it is *true* that A entails B than that $A \to B$ is provable in **L**? That it is *false* that A entails B, that $A \to B$ is unprovable in **L**?

It was considerations like the above that produced the notion of *coherence* in [1]. For the idea there, for the particular system **E** of entailment, was that sentential variables might be evaluated as true or false; the truth-functional connectives, including classical -, were to be respected in the usual truth-tabular way, and $A \to B$ was to be evaluated as true if provable in **E** and false otherwise. That is what interpreting **E** in its own metalogic came to in [1], and, since all theorems of **E** turned out true on all valuations meeting the stated conditions, called *metavaluations*, **E** was proclaimed *coherent*. Other modal logics, e.g., S4, turned out coherent on a parallel analysis of necessity (that in the end comes to the same thing). In the sense in which **E** is coherent, classical logic is of course incoherent, (1) being a false theorem on the metavaluation that makes 'p' false; incoherent also is S5, on the slightly jazzed-up version of (1),

(1′) $p \vee . \mathsf{N}p \to q$

where $\mathsf{N}p$ may be construed as $(p \to p) \to p$. For (1′) is an S5 theorem, again false on the metavaluation that makes 'p' false. To repeat, there is no killing objection to the claim that incoherent logics may indicate entailment; the objection is that such logics are inadequate to express entailment. Nor is coherence a property that univocally determines the *right* formalization of entailment; other motivating considerations, well-known from the writings of Lewis, Ackermann, and Anderson-Belnap, enter in. But an incoherent logic does not adequately formalize entailment on the gracious assumption that it does, that A entails B iff $A \to B$ is provable in the given logic **L**, making mincemeat of the view that entailment is a matter of formal connection. In the author's view, that is bad.

2 Coherence is a kind of consistency, i.e., for a given logic **L**, all theorems of **L** are true on such and such a kind of interpretation. The question naturally arises, "How may we meddle with coherence to get a corresponding kind of completeness?" Answer—by compounding the deadly seriousness that gave us coherence in the first place.

Consider (1) again. On our initial pretension that we only half-understood truth-tables, we were tempted to reject it because '$p \to q$' *is* false, being unprovable, and because 'p' can be *made* false on an assignment. But on a thorough-going meta-point-of-view, we might have rejected 'p' with the same haste as we rejected '$p \rightarrow q$'; 'p', after all, is not provable either. This leads us straightforwardly to the notion of the *canonical metavaluation* V, defined recursively as follows for a sentential logic $\mathbf{L}$ all of whose connectives and constants occur explicitly in the list below:

(i) $\mathsf{V}(p) = \mathbf{F}$, for each sentential variable p;
(ii) $\mathsf{V}(\mathbf{t}) = \mathbf{T}$;
(iii) $\mathsf{V}(\mathbf{F}) = \mathbf{F}$;[5]
(iv) $\mathsf{V}(A \,\&\, B) = \mathbf{T}$ iff $\mathsf{V}(A) = \mathbf{T}$ and $\mathsf{V}(B) = \mathbf{T}$;
(v) $\mathsf{V}(A \vee B) = \mathbf{T}$ iff $\mathsf{V}(A) = \mathbf{T}$ or $\mathsf{V}(B) = \mathbf{T}$;
(vi) $\mathsf{V}(A \rightarrow B) = \mathbf{T}$ iff $A \rightarrow B$ is a theorem of $\mathbf{L}$;
(vii) $\mathsf{V}(\neg A) = \mathbf{T}$ iff $\neg A$ is a theorem of $\mathbf{L}$.

On the understanding that $\mathsf{V}(A) = \mathbf{F}$ or $\mathsf{V}(A) = \mathbf{T}$ always, the above specifications determine V uniquely on all formulas of $\mathbf{L}$. On motivating remarks, only (vii) requires further explanation; the explanation is that in this section the sense of negation being analyzed is *formal refutability*, not simple falsehood; we use '$\neg$' for this intuitionistically acceptable sense of negation; for it, of course, excluded middle is expected generally to fail. Preference among metavaluations for the canonical one rests on the absence of a classical, symmetrical negation. For since metavaluations deal with the classical negation '-' truth-functionally (on pain of having excluded middle fail), we should have to require $\mathsf{V}(\bar{p}) = \mathbf{T}$ to go with (i) above, for each sentential variable 'p'. Since classical '$\bar{p}$' has no better claims than 'p', that is uninteresting; put otherwise, where '-' is present, there is no *preferred* metavaluation.

Restriction of our interest to the canonical metavaluation yields a restricted notion of coherence. (This is apparent since, in cases of practical interest, only the canonical metavaluation need be looked at to determine coherence once '-' is gone.) Let us call a logic $\mathbf{L}$ *weakly coherent* provided that all its theorems turn out true on the canonical metavaluation V. A corresponding notion of completeness, not wanted before for reasons given, turns out now so trivial in most cases as to be hardly worth mention; to mention it, let us call a logic *properly metacomplete* provided that everything true on the canonical metavaluation V is a theorem. Finally, $\mathbf{L}$ is *metacomplete* provided that it is both weakly coherent and properly metacomplete, i.e., provided that *exactly* the formulas true on V are theorems. From the way that V is defined on formulas $A \vee B$ by (v), we note in anticipation that it is immediate and trivial that all metacomplete logics have the disjunctive property (A) of the introductory remarks.

The formula (1), which refuted whatever claims that classical logic had to be coherent, was false on the canonical metavaluation; so classical logic is not even weakly coherent. *A fortiori*, it is not metacomplete. (Neither, on the same grounds, is the positive part of S5, formalized with strict implication primitive as in [4].) On a little reflection, it is a little puzzling that the positive part of classical logic is not metacomplete, since when formulated with a conscientious eye on the separation of connectives, as, e.g., in [3], all the axioms turn out true on the canonical metavaluation; the trouble, one concludes, must lie in *modus ponens*.

That is right. Let P be $p \rightarrow q. \rightarrow p: \rightarrow p$, and let Q be (1). Then $\mathsf{V}(P) = \mathbf{T}$ and $\mathsf{V}(P \rightarrow Q) = \mathbf{T}$, since both are classical tautologies, by clause (vi) specifying the canonical metavaluation. But, despite the theoremhood of Q, we know already that $\mathsf{V}(Q) = \mathbf{F}$, by clause (v). So V does not respect *modus ponens*.

If there is anyone left who still confuses material implication with logical implication, he has no longer even a truth-functional ground left for his confusion. For, as we have just seen, the trouble with taking material implication to express entailment is that, so taken, it is not truth-functional *enough*; putative entailments with true antecedents and false consequents turn out true. For the logics that really respect truth-functionality, like the system **E** of entailment, intuitionist logic, and some modal logics, this never happens; true antecedents and false consequents falsify a putative entailment. (After all these years of attack on classical logic, it is difficult to find a novel complaint against it; but the discovery that, as a theory of entailment, it is not truth-functional, I think, qualifies; I add, lest the reader think otherwise, that in my view the fact that most arguments against classical logic are valid does not diminish its value; what they amount to, usually, is that if one wants to do such-and-such one cannot do it classically in a natural way, and that is usually true; but when such-and-such, e.g., expressing entailment, is not relevant to the concerns of the moment, the simplicity and straightforwardness of classical logic generally make it the more useful instrument; to pick a homely analogy, if logicians were deep-sea divers, the classicist would complain about the use of oxygen tanks, since one can, after all, hold his breath; but for most of the things that we want to do on land, carrying an oxygen tank about always would be a bit cumbersome; when it is recognized that logic is, as Aristotle said it was, an instrument, not a repository of eternal truths, the question that confronts any formalism becomes, "What is it good for?" not, "Is it true?" Against the claims of its defenders, there are interesting and important purposes which classical logic simply does not serve, or serves very poorly; it is not to be discarded thereby, any more than a hammer is to be discarded because it is frustrating and unsatisfying to use it to tighten screws.) If, however, the conditions that we have put on the canonical metavaluation V for a logic **L** are insufficient to insure that it respect entailment, that is not hard to correct. Let the *canonical quasi-valuation* V' be defined like V, keeping *mutatis mutandis* the recursive specifications (i)-(v) above, and replacing (vi) and (vii) as follows to insure that truth-functionality be respected:

(vi)′ $\mathsf{V}'(A \rightarrow B) = \mathbf{T}$ iff both $A \rightarrow B$ is a theorem of **L** and either $\mathsf{V}'(A) = \mathbf{F}$ or $\mathsf{V}'(B) = \mathbf{T}$;
(vii)′ $\mathsf{V}'(\neg A) = \mathbf{T}$ iff both $\neg A$ is a theorem of **L** and $\mathsf{V}'(A) = \mathbf{F}$.

Clearly the canonical metavaluation V would not always coincide with the quasi-valuation V′; e.g., in the classical case. But the following always holds:

Lemma 1 *Let* V *and* V′ *be as above, for a fixed logic* **L**. *Then for all formulas* A *of* **L**, *if* V′(A) = **T** *then* V(A) = **T**.

Proof by induction on the length of A. We may assume that the principal connective of A is $\rightarrow$ or $\neg$, other cases being trivial. But inspection of (vi) and (vi)′ shows that part of the truth-condition for $B \rightarrow C$ on V′ is that it be true on V, and similarly for $\neg$ by (vii) and (vii)′, ending the proof of the lemma.

Corollary *If all theorems of* **L** *are true on* V′, **L** *is weakly coherent.*

Proof by Lemma 1 and definition of weak coherence.

Lemma 1 and its corollary set up the hard part of proving metacompletness, establishing weak coherence for our logic **L**. We pause for the easy part. A logic is *reasonable* provided that it meets each of the following conditions (if a connective is not in the vocabulary of the logic, or a constant, we drop the corresponding condition):

(i) No sentential variables are theorems.
(ii) **t** is a theorem.
(iii) **F** is not a theorem.
(iv) If A and B are theorems, so is A & B.
(v) If one of A or B is a theorem, so is $A \vee B$.

Obviously the conditions are not arduous. Nevertheless, we have

Lemma 2 *Let* **L** *be a reasonable logic, as just defined. Then* **L** *is properly metacomplete,* i.e., *everything true on the canonical metavaluation is a theorem of* **L**.

(We note in passing that this means, by Lemma 1, also that everything true on V′ is a theorem.)

Proof: Let V be the canonical metavaluation, and show by induction on the length of A that if V(A) = **T**, A is a theorem of **L**. Since **L** is reasonable, if A is a sentential variable or constant, V(A) = **T** iff A is a theorem of **L**. If A is of the form B & C, then if V(A) = **T** both V(B) = **T** and V(C) = **T**, whence on inductive hypothesis both B and C are theorems, whence adjoining by (iv) above A is a theorem. The argument is similar, using (v), if A is of the form $B \vee C$. If the main connective of A is $\rightarrow$ or $\neg$, theoremhood of A in **L** is precisely the condition for V(A) = **T**, ending the proof of the lemma.

Proper metacompleteness being a trivial property for any reasonable logic, we return to weak coherence. A logic is *rational* provided that A1 below is a theorem and that R1-R3 are primitive or derivable rules, and that it may be axiomatized on some selection of axiom and rule schemata from the list below.

A1. $\mathbf{t}$
A2. $A \rightarrow A$
A3. $A \;\&\; B \rightarrow A$
A4. $A \;\&\; B \rightarrow B$
A5. $(A \rightarrow B) \;\&\; (A \rightarrow C) \rightarrow (A \rightarrow B \;\&\; C)$
A6. $A \rightarrow A \vee B$
A7. $B \rightarrow A \vee B$
A8. $(A \rightarrow C) \;\&\; (B \rightarrow C) \rightarrow (A \vee B \rightarrow C)$
A9. $A \;\&\; (B \vee C) \rightarrow A \;\&\; B \vee A \;\&\; C$
A10. $A \rightarrow B. \rightarrow : B \rightarrow C. \rightarrow .A \rightarrow C$
A11. $B \rightarrow C. \rightarrow : A \rightarrow B. \rightarrow .A \rightarrow C$
A12. $A \rightarrow. A \rightarrow B : \rightarrow. A \rightarrow B$
A13. $A \rightarrow A. \rightarrow B : \rightarrow B$, *provided that* A2 *is a theorem scheme*
A14. $A \rightarrow : A \rightarrow B. \rightarrow B$
A15. $\mathbf{t} \rightarrow. A \rightarrow A$
A16. $A \rightarrow. \mathbf{t} \rightarrow A$
A17. $\mathbf{t} \rightarrow A. \rightarrow A$, *provided that* A2 *is a theorem scheme*
A18. $A \rightarrow. B \rightarrow C : \rightarrow :A \rightarrow B. \rightarrow .A \rightarrow C$
A19. $A \rightarrow B. \rightarrow : C \rightarrow D. \rightarrow .A \rightarrow B$
A20. $A \rightarrow B. \rightarrow : C \rightarrow. A \rightarrow B$
A21. $A \rightarrow B. \rightarrow : A \rightarrow. B \rightarrow C \therefore \rightarrow : A \rightarrow B. \rightarrow. A \rightarrow C$
A22. $A \rightarrow. B \rightarrow C : \rightarrow. A \rightarrow B \therefore \rightarrow \therefore A \rightarrow. B \rightarrow C : \rightarrow. A \rightarrow C$
A23. $A \rightarrow B. \rightarrow. B \rightarrow C : \rightarrow : A \rightarrow B. \rightarrow. A \rightarrow C$
A24. $B \rightarrow C. \rightarrow. A \rightarrow B : \rightarrow : B \rightarrow C \;.\rightarrow. A \rightarrow C$
A25. $A \rightarrow C. \rightarrow. A \rightarrow B : \rightarrow : A \rightarrow C \;.\rightarrow. A \rightarrow B \;\&\; C$
A26. $C \rightarrow B. \rightarrow. A \rightarrow B : \rightarrow : C \rightarrow B \;.\rightarrow. C \vee A \rightarrow B$
A27. $A \rightarrow B. \rightarrow : A \rightarrow C \;.\rightarrow. A \rightarrow B \;\&\; C$
A28. $A \rightarrow C. \rightarrow : B \rightarrow C \;.\rightarrow. A \vee B \rightarrow C$
A29. $\mathbf{F} \rightarrow A$
A30. $A \;\&\; (A \rightarrow B) \rightarrow B$
A31. $(A \rightarrow B) \;\&\; (B \rightarrow C) \rightarrow . A \rightarrow C$
A32. $A \rightarrow \mathbf{F}. \rightarrow \neg A$
A33. $\neg A \rightarrow. A \rightarrow \mathbf{F}$

R1. *From* A *and* $A \rightarrow B$, *infer* B.
R2. *From* A *and* B, *infer* $A \;\&\; B$.
R3. *From either* A *or* B, *infer* $A \vee B$.
R4. *From* $A \;\&\; B$, *infer both* A *and* B.
R5. *From* $A \rightarrow B$ *and* $B \rightarrow C$, *infer* $A \rightarrow C$.
R6. *From* $A \rightarrow B$, *infer* $B \rightarrow C. \rightarrow. A \rightarrow C$.
R7. *From* $B \rightarrow C$, *infer* $A \rightarrow B. \rightarrow. A \rightarrow C$.
R8. *From* $C \rightarrow D$, *infer* $A \rightarrow B. \rightarrow. C \rightarrow D$.
R9. *From* A, *infer* $B \rightarrow A$.
R10. *From* A, *infer* $\mathbf{t} \rightarrow A$.

Obviously we could continue in this vein for quite a while. On checking the above list with [2], [3], [4], and [5], however, it is clear that we have provided axioms and rules sufficient for the positive parts $\mathbf{E}^+$, $\mathbf{R}^+$, and $\mathbf{T}^+$[6]

of the relevant logics, $S2^+$, $S3^+$, $S4^+$, and M^+ of the Lewis-style modal logics, and (adopting Curry's terminology) for the absolute logic **HA**, the intuitionist logic **HJ**, and the Johansson minimal calculus **HM**, together with sundry other logics got by selecting axioms and rules from the above list at will. That A1 and R2-R3 are required to hold (when the appropriate constant or connectives are present) is simply to insure that all rational logics be reasonable; these conditions could be weakened somewhat in most cases. Note also that *modus ponens* (R1) must hold. We have, following Hacking, taken strict implication as primitive for the Lewis-style modal logics, which suits our heuristic purposes, though an explicit theory of modality might be had if we define $\mathsf{N}A$ as $\mathbf{t} \rightarrow A$. (Alternatively, we might deal with 'N' directly, as in [1], setting $\mathsf{V}(\mathsf{N}A) = \mathbf{T}$ iff $\mathsf{N}A$ is a theorem, and $\mathsf{V}'(\mathsf{N}A) = \mathbf{T}$ iff both $\mathsf{N}A$ is a theorem and $\mathsf{V}'(A) = \mathbf{T}$; adding appropriate modal axioms, arguments in the style of Lemma 2 above and Lemma 3 below would work, considerably simplifying the similar arguments of [1].) We get the version **RJ** of relevant implication with intuitionist negation by taking as axioms A1-A17, A29, A32-A33, and as rules R1-R2; similar Lewy-style modal and relevant logics with strict negation are got likewise by adding A29, A32-A33 to positive axioms, in the style of [4]; if a minimal negation is wanted, drop A29.

Lemma 3 *Let* **L** *be a rational logic. Then* **L** *is weakly coherent.*

Proof: It suffices by Lemma 1 to show all theorems of **L** true on the canonical quasi-valuation V'. We prove this by induction on the length of proof of a given theorem A of **L**, showing that for each axiom of **L** that it is true on V' and that truth on V' is preserved under the rules of inference of **L**. We verify a few specimen axioms and rules, leaving the rest to the reader.

Ad R1. Suppose $\mathsf{V}'(A) = \mathbf{T}$ and $\mathsf{V}'(A \rightarrow B) = \mathbf{T}$. By (vi)$'$, $\mathsf{V}'(B) = \mathbf{T}$. (The reader will recall that it was exactly to verify *modus ponens* that we passed from V to V' in the first place.)

Ad A3. $A \& B \rightarrow A$ is on assumption a theorem of **L**. Hence it suffices by (vi)$'$ that it be true on V' if whenever $\mathsf{V}'(A \& B) = \mathbf{T}$ then $\mathsf{V}'(A) = \mathbf{T}$, which is trivial by the recursive specification of V'.

Ad A8. $(A \rightarrow C) \& (B \rightarrow C) \rightarrow. A \vee B \rightarrow C$ is on assumption a theorem of **L**. Suppose the antecedent is true on V'. Then, by definition of V', so are each of $A \rightarrow C$ and $B \rightarrow C$. By (vi)$'$, each is a theorem, whence adjoining and applying *modus ponens*, $A \vee B \rightarrow C$ is a theorem of **L**. Moreover $\mathsf{V}'(A \vee B \rightarrow C) = \mathbf{T}$, by (vi)$'$ since if $A \vee B$ is true on V', so by definition is one of A or B, whence so is C by $\mathsf{V}'(A \rightarrow C) = \mathbf{T}$ or $\mathsf{V}'(B \rightarrow C) = \mathbf{T}$ respectively. This suffices for the truth of the axiom.

Ad A16. $A \rightarrow. \mathbf{t} \rightarrow A$ is on assumption a theorem of **L**. Suppose $\mathsf{V}'(A) = \mathbf{T}$. By Lemma 2, since **L** is rational and hence reasonable, A is a theorem of **L**. So by *modus ponens* $\mathbf{t} \rightarrow A$ is a theorem of **L**, whence by (ii) and (vi)$'$, $\mathsf{V}'(\mathbf{t} \rightarrow A) = \mathbf{T}$, which suffices for the truth of the axiom.

Ad A18. $A \rightarrow. B \rightarrow C: \rightarrow :A \rightarrow B. \rightarrow .A \rightarrow C$ is on assumption a theorem of **L**. Suppose the antecedent true but, for *reductio*, the consequent false on V', in the hope of falsifying the axiom on V'. At any rate the consequent is a theorem, by *modus ponens* and truth of antecedent (and hence theoremhood), so that $V'(A \to B) = \mathbf{T}$ and $V'(A \to C) = \mathbf{F}$ is forced. Repeating the move, $V'(A) = \mathbf{T}$ and $V'(C) = \mathbf{F}$ is forced. Since $V'(A \to B) = \mathbf{T}$, $V'(B)$ is then $\mathbf{T}$. This contradicts $V'(A \to. B \to C) = \mathbf{T}$, ending the verification of the axiom.

Ad R5. Suppose both $A \to B$ and $B \to C$ true on V'. Since R5 is a rule of **L**, clearly $A \to C$ is a theorem. Furthermore, if $V'(A) = \mathbf{T}$ then $V'(B) = V'(C) = \mathbf{T}$ by (vi)$'$ and assumptions, whence $V'(A \to C) = \mathbf{T}$.

Enough has been done to leave verification of all other cases in the reader's hands, ending the proof of the lemma.

All this verification by cases is, frankly, annoying. There must, I am sure, be some principle at work independent of axioms chosen, but no apparent one has shown itself. Obviously the various axioms and rules are all intuitionistically valid; obviously, also, they do not interact much; we noted, e.g., that adjunction was needed explicitly to verify A8, but in general the verification of each axiom does not require the other axioms and rules; indeed, though we have stated the axioms as schemes, except for *modus ponens* and what is needed for reasonableness, A1 and R1-R3, nothing need be taken schematically; *any* set of substitution instances of a scheme, not necessarily even closed under substitution, may be taken as axioms and verified one by one, without upsetting the proof of weak coherence.

Nevertheless, certain temptations to generalize must be resisted. Obviously not every intuitionistically valid formula may be taken as an axiom, independent of other choices; $p \vee. p \to \mathbf{t}$ would not do on its own, for example, for the disjuncts may be falsified separately on V' unless $p \to \mathbf{t}$ holds too, which in the absence of other axioms it would not. Similarly, in the absence of R4, no axioms of the form A & B will do, in general. So a characterization of the general conditions that breed weak coherence is a problem of some complexity, accordingly left open in the light of present, specific purposes.

We got weak coherence here by weakening the intuitionist logic **HJ**; obviously the question of strengthening it comes up also. If we do not admit a rule of substitution for sentential variables, we may add some classical axioms; e.g., $\neg\neg p \to p$ might be added singly, since as a theorem of the appropriate system its truth on V' requires in addition only $V'(\neg\neg p) = \mathbf{F}$, which holds. On the other hand the substitution instance $\neg\neg(p \vee \neg p) \to p \vee \neg p$ would not do, its consequent being false on V' for the theory got by adding it as a single new axiom to **HJ**, but its antecedent being true, by (vii)$'$. Similar remarks hold around Peirce's law; $((p \to q) \to p) \to p$ poses no problem, but substitution instances do. On the other hand, for weaker systems $\neg\neg A \to A$ is not ruled out categorically, even as a scheme; for example, if *all* negation axioms were of that form, clearly nothing of the form $\neg B$ would be provable in an otherwise rational logic, whence falsity of antecedent always insures the truth of $\neg\neg A \to A$ on V' in the appropriate system. So there is no maximal weakly coherent system closed under substitution for sentential variables.

We note finally that weak coherence implies coherence for rational logics. For, in the present context, an arbitrary metavaluation V^* is defined like the canonical metavaluation V, except that $\mathsf{V}^*(A) = \mathbf{T}$ is permitted when A is a sentential variable. (I.e., in the absence of classical negation a metavaluation is just anything that satisfies (ii)-(vii) above, dropping (i).) Prove by induction on the length of a given formula A of a rational logic L that if $\mathsf{V}(A) = \mathbf{T}$ then $\mathsf{V}^*(A) = \mathbf{T}$, where V is the canonical metavaluation for L and V^* is any metavaluation. It follows by Lemma 3 that all rational logics are coherent. Metacompleteness is our next topic.

Theorem 1 *Let* L *be any rational sentential logic. Then* L *is metacomplete;* i.e., *exactly the theorems of* L *are true on the weak canonical metavaluation* V.

Proof by Lemmas 2 and 3.

Corollary *All rational sentential logics have the intuitionist disjunctive property*, i.e., $A \vee B$ *is a theorem iff at least one of* A *or* B *is a theorem.*

Proof immediate by definition of metacompleteness.

For the relevant logics $\mathbf{R}^+$, $\mathbf{E}^+$, and $\mathbf{T}^+$, the corollary offers an alternative proof of the principal result of [6].

3 Only minor modifications are required to extend the results of the previous section to first-order logics. The language of these logics may be assumed to contain, besides connectives from among $\to$, &, $\vee$, $\neg$ and constants from among $\mathbf{t}$, $\mathbf{F}$, quantifiers $\forall$ and $\exists$, individual variables x, etc., n-place predicate letters G, etc., for all non-negative n (0-ary predicates counting as sentential variables), operation letters g, etc., for all non-negative n (0-ary ones counting as individual constants), and the identity =.[7] Terms and formulas are characterized as usual; where $A(x)$ is a formula and t is a term such that no confusion of bound variables results from substituting t for free x in $A(x)$, $A(t)$ shall denote the result of that substitution; if there is confusion, free variables of t bound in $A(x)$ are rewritten by definite plan first.

Let L be a first-order logic based on a language as indicated. The canonical metavaluation V and the canonical quasi-valuation V' are defined as before, taking (i) now in the form below, adding (viii) and (ix), and characterizing V by (i)-(ix) and V' by the analogues of (i)-(v), (viii)-(ix) and (vi)$'$-(vii)$'$ as before.

(i) $\mathsf{V}(A) = \mathbf{F}$, whenever A is atomic, and without =;
(viii) $\mathsf{V}(\forall x A(x)) = \mathbf{T}$ iff $A(t) = \mathbf{T}$ for all terms t;
(ix) $\mathsf{V}(\exists x A(x)) = \mathbf{T}$ iff $A(t) = \mathbf{T}$ for some term t.

Where identity is present, we add also

(x) $\vee(t = t) = \mathbf{T}$, for each term t;
(xi) $\vee(t = u) = \mathbf{F}$, if t and u are distinct terms.

A first-order logic **L** is *reasonable* if it meets the old criteria of reasonableness, and if moreover:

(f) *If $A(x)$ is a theorem of* **L**, *so is $(x)A(x)$.*
(g) *If $A(t)$ is a theorem of* **L**, *so is $\exists xA(x)$, where $A(t)$ results from $A(x)$ on the conventions above.*
(h) *Rewriting bound variables is an admissible rule of* **L**.

If identity is present, we assume also

(j) *$t = u$ is a theorem of* **L** *iff t and u are the same term.*

A first-order logic **L** is *rational* if it meets the old criteria of rationality, if it is reasonable, and if its first-order axiom and rule schemata are from among the following:

B1. $\forall xA(x) \rightarrow A(t)$
B2. $A(t) \rightarrow \exists xA(x)$
B3. $\forall x(A \rightarrow B) \rightarrow. \forall xA \rightarrow \forall xB$
B4. $\forall x(A \rightarrow B) \rightarrow. \exists xA \rightarrow \exists xB$
B5. $\forall xA \ \& \ \forall xB \rightarrow \forall x(A \ \& \ B)$
B6. $\exists x(A \vee B) \rightarrow \exists xA \vee \exists xB$
B7. $A \rightarrow \forall xA$, *where x is not free in A*
B8. $\exists xA \rightarrow A$, *where x is not free in A*
B9. $\forall x(A \vee B) \rightarrow \forall xA \vee B$, *where x is not free in B*
B10. $\exists xA \ \& \ B \rightarrow \exists x(A \ \& \ B)$, *where x is not free in B*
B11. $\forall x((A \rightarrow A) \rightarrow A) \rightarrow. \forall xA \rightarrow \forall xA. \rightarrow \forall xA$, *provided* A2 *holds*
B12. $t = t$
B13. $t = u \rightarrow. A(t) \rightarrow A(u)$, *provided* A2 *holds*

S1. *From A, infer $\forall xA$.*
S2. *From A, infer A', where A' results from A by rewriting bound variables.*
S3. *From $A \rightarrow B$, infer $A \rightarrow \forall xB$, where x is not free in A.*
S4. *From $A \rightarrow B$, infer $\exists xA \rightarrow B$, where x is not free in B.*

Analogues of old lemmas are proved as before.

Lemma 1* *Let $\vee$ and $\vee'$ be as above, for a fixed quantificational logic* **LQ**. *Then for all formulas A, if $\vee'(A) = \mathbf{T}$ then $\vee(A) = \mathbf{T}$.* [*Proof* as above.]

Lemma 2* *Let* **LQ** *be a reasonable first-order logic, as just defined. Then* **LQ** *is properly metacomplete.*

Proof as of Lemma 2, by induction on the length of A. Atomic cases are as above, though if identity is present in **LQ** conditions (x) and (xi) on the canonical metavaluation $\vee$ and condition (j) on reasonableness of **LQ** must be invoked. For the inductive part of the argument, we need newly look only at the cases where A is of the form $\exists xB$ or $\forall xB$. Suppose first that $\vee(\exists xB(x)) = \mathbf{T}$. By (ix), $B(t)$ is true on $\vee$ and hence a theorem for some term t, by inductive hypothesis. So then by (g) is $\exists xB(x)$. (Rewriting enters only by definition of $B(t)$.) Finally, suppose A is of the form $\forall xB(x)$ and $\mathsf{V}(A) = \mathbf{T}$. By (viii), $\mathsf{V}(B(x)) = \mathbf{T}$; on inductive hypothesis $B(x)$ is a theorem, whence by (f) so is its generalization A. This ends the proof of Lemma 2*.

Lemma 3* *Let* **LQ** *be a rational first-order logic. Then* **LQ** *is weakly coherent.*

Proof: Like Lemma 3. It suffices, by Lemma 1*, to show for each theorem A that $\mathsf{V}'(A) = \mathbf{T}$, where V' is the canonical quasi-valuation. The method is induction on length of proof, assuming $\mathsf{V}'(B) = \mathbf{T}$ if B has a shorter proof than A and showing that $\mathsf{V}'(A) = \mathbf{T}$ when A is a theorem. Sentential axioms and rules are as before; if identity is present, B12 is nevertheless trivial. Other quantificational axioms and rules are verified case by case. Specimens follow, the rest being left to the reader.

Each axiom, we note, is in the form of an implication, so that it suffices to show, for each such, that the antecedent cannot be true on V' and the consequent false, since theoremhood is assured.

Ad B4. Suppose $\mathsf{V}'(\forall x(A \rightarrow B)) = \mathbf{T}$, but that $\mathsf{V}'(\exists xA \rightarrow \exists xB) = \mathbf{F}$. Since the latter is a theorem, we must have $\mathsf{V}'(\exists xA) = \mathbf{T}$ but $\mathsf{V}'(\exists xB) = \mathbf{F}$. But for some t, $A(t)$ is true on V', whence so is $A(t) \rightarrow B(t)$ by (viii), whence $\mathsf{V}'(B(t)) = \mathbf{T}$ (since V' respects truth-functionality), whence $\mathsf{V}'(\exists xB) = \mathbf{T}$ by (ix), a contradiction.

Ad B7. Suppose $\mathsf{V}'(A) = \mathbf{T}$. Since x is on assumption not free in A, A itself is the only instance of A, whence by (viii) $\mathsf{V}'(\forall xA) = \mathbf{T}$.

Ad B11. Suppose the antecedent of B11 true on V' and its consequent false. We must have (analogously to B4) $\mathsf{V}'(\forall xA) = \mathbf{F}$. So there is a term t such that $\mathsf{V}'(A(t)) = \mathbf{F}$. But $\mathsf{V}'(A(t) \rightarrow A(t). \rightarrow A(t)) = \mathbf{T}$, by (viii). But $A(t) \rightarrow A(t)$ is a theorem, as required for the axiomhood of B11, whence its truth on V' is trivial. This forces $\mathsf{V}'(A(t)) = \mathbf{T}$, a contradiction, ending the verification.

Verification of the rewriting rule S2 has already entered tacitly into the verification of some axioms; B4, for example. (For the unspecified definite plan of definitions relevant to proper substitution of t for x in $A(x)$ may yield different results in different contexts, e.g., if we choose the first variable in a list of variables that is foreign to a given context for rewriting purposes, to prevent confusion of bound variables, $[A \rightarrow B](t)$ is not necessarily $A(t) \rightarrow B(t)$, though the latter does result from the former by an explicit application of the rewriting rule S2.) The argument that rewriting A as A' makes no difference to its truth-value on V' proceeds by induction on length of formula. As interesting cases we select those in which A is of the form $B \rightarrow C$ and of the form $\forall xB$. Let $B' \rightarrow C'$ be A' in the former case. Then A is provable iff A' is provable by the rewriting rule, and $\mathsf{V}'(B) = \mathsf{V}'(B')$ and $\mathsf{V}'(C) = \mathsf{V}'(C')$ on inductive hypothesis, which suffices by (vi)$'$ that $\mathsf{V}'(A) = \mathsf{V}'(A')$. In the latter case, let A' be $\forall yB'$, where x and y are not necessarily distinct. $\forall xB$ is true if all its instances $B(t)$ are true, and each of these is clearly (at most) a rewriting of an instance of $\forall yB'$, whence on inductive hypothesis A and A' stand or fall together on V'.

The usual syntactical nonsense attending confusion of bound variables being disposed of, we may now look at the honest rules. The inductive hypothesis on length of proof having been irrelevant in verifying S2, we assume henceforth that rewriting is accomplished in zero steps. This enables us to assert, for each rational logic **LQ**, that if $A(x)$ is provable in **LQ**, $A(t)$ is provable in no more steps; for replace each step $B(x)$ in a proof of $A(x)$ with $B(t)$; up to rewriting, axioms go into axioms and premises of rules into premises of corresponding rules, in view of the schematic formulation of **LQ**, which suffices for the assertion. This verifies S1, on the inductive hypothesis that its premise A is true on V'; A is then by Lemma 2* a theorem, whence by the assertion just made each instance $A(t)$ is a theorem too, and, being provable in no more steps than A, for all terms t, $A(t)$ is true on V'; by (viii) so is $\forall xA$, ending the verification of S2. To verify S3 on the inductive hypothesis that $\mathsf{V}'(A \rightarrow B) = \mathbf{T}$, note by Lemma 2* that $A \rightarrow B$ is a theorem of **LQ** and hence, assuming x not free in A, $A \rightarrow \forall xB$ is a theorem. We must show also that if $\mathsf{V}'(A) = \mathbf{T}$ then $\mathsf{V}'(\forall xB) = \mathbf{T}$. But if $\mathsf{V}'(A) = \mathbf{T}$, B is then by definition of V' true on V'; furthermore, by the argument just gone through we can prove $[A \rightarrow B](t)$ for each t in no more steps, which amounts (perhaps rewriting) to $A \rightarrow B(t)$, whence each $B(t)$ is similarly true on V'; again, so is $\forall xB$. By parallel reasoning S4 is verified, ending the proof of Lemma 3*.

Extension of our principal results to quantification theory is now immediate.

Theorem 2 *Let* **LQ** *be any rational first-order logic. Then* **LQ** *is metacomplete*; i.e., *exactly the theorems of* **LQ** *are true on the weak canonical metavaluation* V.

Proof by Lemmas 2* and 3*.

Corollary *All rational first-order logics have the intuitionist disjunction property and the intuitionist existential property*, i.e., $A \vee B$ *is a theorem iff at least one of* A *or* B *is a theorem; and* $\exists xA(x)$ *is a theorem iff* $A(t)$ *is a theorem, for some term* t *of* **LQ**.

Since it has been proved in [7] that the relevant logics **RQ**, **EQ**, and **TQ** in particular are conservative extensions of the natural positive logics $\mathbf{RQ}^+$, $\mathbf{EQ}^+$, and $\mathbf{TQ}^+$, the corollary holds in particular for the negation-free theorems of these systems; indeed, $\mathbf{RQ}^+$ may be formulated by adding B1-B11, S1-S4 to $\mathbf{R}^+$; $\mathbf{EQ}^+$ results from $\mathbf{E}^+$ by the same additions; $\mathbf{T}^+$ by adding B1-B10, S1-S4. (These formulations are highly redundant, e.g., B1-B10, S1 suffice for $\mathbf{RQ}^+$.) In addition to the relevant logics, metacompleteness is of course proved for suitable formulations of first-order intuitionistic logic, Lewis modal logics $S2^+$, $S3^+$, $S4^+$, etc., Lewy logics, relevant implication with intuitionist negation, and so forth; of course the corollary to Theorem 2 holds for all these systems too. (I note that the corollary is a familiar fact about intuitionism, and that it was proved by other methods in [7] for a version of $\mathbf{RQ}^+$ with an additional intensional conjunction o.) In [7], much ado was made about the fact that relevant logics, though admitting classical negation, were constructively acceptable in their negation-free parts; Theorem 2 and its corollary repeat and deepen the sentiment, for together with the conservative extension results they show that one can use classical negation, *reductio* proofs, excluded middle, and so forth, to shorten proofs of negation-free theorems, while still being confident that one could have proved them in a constructive manner, e.g., existential quantifications from their instances.

We conclude by reflecting briefly on *what else* metacompleteness is good for. There is, after all, nothing in the notion of metacompleteness that requires that we confine it to logics; a similar analysis will work for applied logics, i.e., for theories; this has already been done, of course, for many intuitionist theories; *cf.* [3]. But the points about the harmlessness of classical negation make it a worth-while project to carry out the arguments for formal theories based on first-order relevant logics; **RQ** seems the natural candidate. This was accomplished in a rudimentary way by including identity in, though we subsequently were able to ignore it almost completely in carrying out arguments. Most formal theories have built into them sufficient machinery to assure that & means *and*, and sometimes $\forall$ is not far from *all*. The insistence that $\rightarrow$ really mean *entails* carries with it, at the level of rational logic, an assurance also that $\vee$ will really mean *or* and that $\exists$ will really mean *some*. Progress is really being made; if we can only get - to mean *not*, we are home, at the rational logical level; at the level of theories, not yet really plumbed, the task is rather to eliminate -; metacompleteness is one path worth looking down.[8]

Added in proof: The techniques of this paper are of considerably wider applicability than was known when it was submitted for publication. Thus, for example, the arguments here may be adapted to provide simpler proofs of the main results of its predecessor [1]; they also yield new proofs of the admissibility of Ackermann's rule γ for relevant logics, and of cut (as Dunn has noted) for certain Gentzen systems; the latter may be extended to higher-order cases, producing results not previously known for the relevant logics and new versions of the Takahashi-Prawitz verifications of Takeuti's conjecture in the Gentzen case. Other applications to relevant logics have been made by Dwyer, Routley and Wolf. Since the ideas here are very simple, it is not surprising, as Professor Kripke has pointed out to me, that they have occurred in a number of related forms to other authors, e.g., to Harrop, Rasiowa, Dwyer, and Fine, the first two of whom, with Kleene and others, have been interested in them in particular with intuitionist logics and mathematics in mind, while the latter two (and I) have been more interested in modal and relevant applications. But the ideas themselves, I think, should not be tied to any particular systems or concerns (though they are useful, as I shall argue elsewhere, in providing a rather comprehensive formal explication of a *coherence theory* of logical and mathematical truth). The key ideas, however, have that generality which goes with extreme simplicity (which is verified, in my own experience, by the fact that everyone with whom I have discussed them has come up with yet another kind of application). [February, 1976]

NOTES

1. For further remarks on the significance of (A) and (B), *cf.* below, and [7].

2. *Cf.* also [7].

3. Routley and I in [8]-[10] and Urquhart in [11] have sought deep structural semantical analyses of entailment in the style of Kripke; in a sense the present theory is but a fragment of that theory, which involves, as is the custom, a truth-functional *reduction* of the intensional connective $\rightarrow$. In fact, the present results will be used in [10], while no debt is owed here to [8]-[11].

4. My colleague, Nino Cocchiarella, has voiced dissatisfaction with this multiple accommodation, preferring precisely that $\overline{p \rightarrow q}$ be a theorem, when $\rightarrow$ is interpreted as strict implication. As he has also pointed out, one rapidly loses control of such systems, which turn out unaxiomatizable even at the level of first-order, on pain of being able to enumerate recursively the non-theorems of standard predicate logic. (The point is more easily seen with necessity; on the dead serious view, $-\Box A$ is a good guy iff A, or maybe $\Box A$, is a non-theorem; given a system that contains the first-order functional calculus as a well-defined subsystem, as most modal and relevant logics do, of course axiomatizing the dead serious good guys runs afoul of Church's theorem.) Reflected in Cocchiarella's view, I think, is the old Tractarian search for an ideal language that will mirror the world as it is; for my part, though, the dead serious interpretation is simply one among others, neither to be insisted upon or ruled out; the thought that if insisted upon only God, at best, can ascertain which are the logical truths causes me to opt for ambiguity against what appears to me theology.

5. I distinguish in general (*cf.* [12]) **F**, **T**, **f**, **t**. In **R**, my principal intuitive model, these are respectively the conjunction of all sentences, the disjunction thereof, the disjunction of all falsehoods, and the conjunction of all truths; intuitively, that is; appropriate infinitary machinery is not in **R** or **RQ**. Since we deal below with many systems, these meanings are not necessarily to be thought of below, though in the interesting systems they are either right or not far off, thus explaining the otherwise puzzling contrast here between '**F**' and '**t**'. '**f**', which does not appear, I tie to classical negation; '**F**', to intuitionistic negation, which is why it does appear. In systems of strict implication, '**t**' gets tied to '$\Box$' via the (in principle, Ackermann) definition '$\mathbf{t} \rightarrow A$'.

6. The system now called '**T**' by Anderson-Belnap was originally called '**P**'; '**T**' is mnemonic, for *ticket* entailment.

7. We do not, of course, *require* that '=' be present; an appropriate restriction, incidentally, is requisite for relevant identity; $a = a \rightarrow. \ Fb \rightarrow Fb$, licensed by B13 as stated here, is an evident fallacy of relevance.

8. My thanks are due to Professors Dunn, Woodruff, Anderson, and Belnap for helpful discussions, and for checking the arguments of [1] developed and simplified here. I am also grateful to the National Science Foundation for partial support of this research through grant GS-2648.

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Australian National University
Canberra, A.C.T., Australia