

DIRECT ANALOGUES OF THE SHEFFER STROKE
 IN m -VALUED LOGIC

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It has been shown by Peirce¹ and, independently, by Sheffer² that all functions of two-valued propositional logic can be defined in terms of a single two-place primitive. This result has been extended to the m -valued case, by the explicit definition of families of such functions, which we will call Sheffer functions, by Webb,³ Götlind,⁴ and the present author.⁵ Explicit characterization of the totality of such functions has been carried out for $m = 2$ by Post⁶ and for $m = 3$ by the present author.⁷ It is known that for $m > 2$, there are always a considerable number of such functions, not always describable in any simple fashion as a generalization of the two functions of the case $m = 2$.

If we limit our considerations to functions which seem analogous to those of the case $m = 2$ we notice that the Sheffers for $m = 2$ can be partially characterized by either of two descriptions equivalent for $m = 2$ but not otherwise:

- a) There exists a $K(0 \leq K \leq 1)$ such that if $i \neq j$, $f(i, j) = K$.
- b) There exist permutations a_0, a_1 and b_0, b_1 of the values $0, 1$ such that, if $i \leq j$, $f(a_i, a_j) = b_i$.

Generalizing these we obtain:

- c) There exists a $K(0 \leq K < m)$ such that if $i \neq j$, $f(i, j) = K$.
- d) There exist permutations a_0, \dots, a_{m-1} and b_0, \dots, b_{m-1} of the values $0, \dots, m - 1$ such that if $i \leq j$, $f(a_i, a_j) = b_i$.

Webb has produced Sheffer functions of both types.

In this paper we will prove a necessary and sufficient condition for a function of such types to be a Sheffer function.

Definitions:

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|------------------------|--------------------------|
| 1. Webb function | A function satisfying c) |
| 2. Preference function | A function satisfying d) |

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|---|---|
| <p>3. $f(p, q)$ is properly closed</p> | <p>There exists a proper subset of the values $0, \dots, m - 1$ closed under $f(p, q)$.</p> |
| <p>4. $f(p, q)$ has the proper substitution property</p> | <p>There exists a decomposition α of the values into at least 2 and at most $m - 1$ equivalence classes such that $f(p, q)$ induces a single-valued function $f'(\{a\}, \{b\})$ from $\alpha \times \alpha$ to α.</p> |
| <p>5. $T_i(p)$</p> | <p>$T_i(j) = i$, for any j.</p> |
| <p>6. $D_{x,y,z}(p)$</p> | <p>$D_{x,y,z}(z) = x$ and $D_{x,y,z}(r) = y$, if $z \neq r$.</p> |
| <p>7. $f^m(p)$</p> | <p>$f^1(p) = f(p)$ and $f^{K+1}(p) = ff^K(p)$.</p> |
| <p>8. $f_{i=1}^m(g_i(p))$</p> | <p>$f_{i=1}^1 g_i(p) = g, (p)$ and $f_{i=1}^{K+1}(g_i(p)) = f(f_{i=1}^K(g_i(p)), g_{K+1}(p))$.</p> |
| <p>9. $N_{x,y,z,w}(p, q)$</p> | <p>$N_{x,y,z,w}(z, w) = x$ and $N_{x,y,z,w}(p, q) = y$ if $p \neq z$ or $q \neq w$.</p> |
| <p>10. M_b-function</p> | <p>A function $M, (p, q)$ such that $M_b(b, i) = M_b(i, b) = i$, for any i (where b is a particular value).</p> |

The present author has elsewhere shown the following theorems:

Theorem A Any function of m values, which can define a permutation consisting of a single cycle, a permutation consisting of $m - 1$ cycles such that the cycle which contains two elements is such that the elements are adjacent in the cycle of the first permutation, and a function which takes $m - 1$ values, can define all one-place functions.⁸

Theorem B The set consisting of an M_b -function and the family $D_{i,b,z}(p)$, $i = 0, \dots, m - 2, z = 0, \dots, m - 1$, can define all one-place functions.⁹

Theorem C Any two-place function which can define all one-place functions is a Sheffer function.¹⁰

Theorem I If $f(p, q)$ is a Webb function, it is Sheffer function if and only if it is not properly closing.

That no properly closing function is a Sheffer function has been shown elsewhere.¹¹ Let $f(p, q)$ be a Webb function and let $r(p) = f(p, q)$.

Case 1: Let $r(p)$ take j values ($j < m - 1$). Then $f(p, q)$ takes no more than $j + 1$ values ($j + 1 \leq m - 1$) and hence is properly closing.

Case 2: Let $r(p)$ take $m - 1$ values. If there is an i such that $r(i) = K$ (where K is defined by c), $f(p, q)$ takes $m - 1$ values and hence is properly closing. If not and the set $\{r^i(K)\} i = 1, \dots, n - 1$ does not contain all elements other than K , the set consisting of K and $r^i(K) i = 1, \dots, n - 1$ is closed and $f(p, q)$ is properly closing. Hence the only subcase which is not properly closing satisfies the following conditions: a_0, \dots, a_{m-1} is a permutation of $0, \dots, m - 1$. $f(a_i, a_i) = a_{i+1}$ for $1 \leq i \leq m - 2$, $f(a_{m-1}, a_{m-1}) = a_b$ for some $b < m - 1$ and $f(i, j) = a_0$ for $i \neq j$.

Then let:

$$A_1(p) = f(p, r(p)) = Ta_0(p)$$

$$A_j(p) = f(p, B_{j-1}(p))$$

where:

$$B_1(p) = Ta_0(p)$$

$$B_j(p) = A_j^j(p).$$

By induction $A_{i+1}(a_{j-1}) = a_j$ for $j \leq i$ and $A_{i+1}(p) = a_0$ and $B_{i+1}(a_{j-1}) = a_{j-1}$ for $j \leq i$ and $B_{i+1}(p) = a_i$ otherwise. $A_m A_m^{m-1} f(B_m(p), A_m^{m-2} A_m(p))$ is a_j for $p = a_j$ and $j \leq m - 3$, a_{m-1} for $p = a_m$ and a_m for $p = a_{m-1}$. Consequently $A_m(p)$, $A_m^{m-1} f(B_{m-1}(p), A_m^{m-2} A_m(p))$, and $B_{m-1}(p)$ satisfy the conditions on Theorem A and hence $f(p, q)$ is a Sheffer function by Theorem B.

Case 3: Let $r(p)$ take m values. If $r(p)$ is of order less than $m - 1$ it is factorable into cycles. Then any cycle containing K determines a proper subset closed under $f(p, q)$ and $f(p, q)$ is properly closing.

If this is not the case, there exists a permutation a_0, \dots, a_{m-1} of the values $0, \dots, m - 1$ such that $f(a_i, a_i) = a_{i+1}$ for $i < m - 1$, $f(a_{m-1}, a_{m-1}) = a_0$ and $f(i, j) = a_{m-1}$ for $i \neq j$. Then $f(p, r(p)) = T_{a_m}(p)$ and $r^i T_{a_m}(p) = T_{a_i} D_{x,y,z}(p)$ is the family $\{r^i(r^j(p), T_k(p))\}$. Then

$$f(r^{m-1} f(p, q), T_{a_{m-1}}(p)) = N_{a_0}, a_{m-1}, a_0, a_0(p, q).$$

Then $N_{x,y,z}(p, q) = D_{x,y,a_0}, N_{a_0}, a_{m-1}, a_0, a_0(r^i(p), r^j(q))$ where i and j satisfy the equations $r^i(z) = r^j(w) = a_0$. We define the following functions:

$$A_0(p, q) = f(p, r^{m-1}(q))$$

$$A_1(p, q) = f(N_{a_0}, a_{m-1}, a_{m-1}, a_{m-1}(p, q), T_{a_{m-1}}(p))$$

$$A_i(p, q) = f(L_{i-1}(p, q), M_{i-1}(p, q)), \text{ for } i > 1$$

where: $M_0(p, q) = T_{a_m}(p, q)$, and

$$M_h(p, q) = A_h(M_{h-1}(p, q), A_h(N_{a_h}, a_{m-1}, a_{m-1}, a_{h+1}(p, q)), N_{a_h}, a_{m-1}, a_{h+1}, a_{m-1}(p, q))$$

$$L_h(p, q) = A_h(M_h(p, q), N_{a_0}, a_{m-1}, a_{m-1}, a_{m-1}(p, q)) \text{ for } h > 0.$$

By induction: for every $i(0 \leq i \leq m - 1)$, $A_i(a_{m-1}, a_{m-1}) = a_{m-1}$ and for every $j(0 \leq j \leq i)$, $A_i(a_{m-1}, a_j) = A_i(a_j, a_{m-1}) = a_j$. Hence $A_{m-1}(p, q)$ is an $M_{a_{m-1}}$ -function and $f(p, q)$ is a Sheffer function by Theorems B and C.

Theorem II *A preference function is a Sheffer function if and only if it neither is properly closing nor has the proper substitution property.*

The present author has shown necessity of the condition elsewhere.¹² Let $s(p) = f(p, p)$. $s(p)$ is a permutation. If it has more than one cycle, it is properly closing since the smallest cycle containing a_0 defines a set closed under $f(p, q)$. Let $f * (p, q) = s^{n-1} f(p, q)$. Then

$$D_{a_1}, a_{0_1} a_0(p) = f *_{i-1}^{m-1} (s^i(p))$$

$$T_{a_0}(p) = f *_{i-1}^m (s^i(p))$$

$$T_{a_i}(p) = s^i T_{a_0}(p), \text{ for } i > 0$$

$$D_{a_1}, a_1, a_0(p) = D_{a_1}, a_{0_1} a_0 D_{a_1}, a_0, a_0(p).$$

Let us call a set A compact relative to an ordered set B which contains it, provided $x, y, \epsilon A$ and $x < z < y$ implies $z \in A$.

Lemma 1 *Given an ordered set $0, \dots, m - 1$, the cyclic subgroup generated by a permutation of order m containing one cycle and an initial sequence I of the m elements, of length $n + 1$, either some element of the subgroup transforms I into a non-compact set or given a compact set A of $n + 1$ elements such that some element of the subgroup transforms A into I , every element of the subgroup transforms A into a set identical with or disjoint from I .*

Assume P_1 and P_2 are elements of the subgroup, $P_1(I) = A$, $P_2(A) \neq I$ and $P_2(A) \cap I \neq \emptyset$. Let $a \subseteq P_2(A) \cap I$. Since there are $n + 1$ elements of A and for any two elements x and y of the ordered set there exists exactly one element of the subgroup which maps x into y , there are exactly $n + 1$ elements of the subgroup which map an element of A into the initial element of I and exactly $n + 1$ elements of the subgroup which map an element of A into a . Since $a \in I$ and the only compact set of $n + 1$ elements containing the initial element of I is I , some element of the subgroup maps A into a non-compact set since $P_2(A)$ contains a but not the initial element of I .

Lemma 2 *If $f(p, q)$ is not properly closing, either $D_{a_0}, a_i, a_0(p), i = 0, \dots, m - 1$ is definable or $f(p, q)$ satisfies the proper substitution property.*

By induction on i : Assume true for $i \leq n - 1$. Then it is true without restriction or $A_{a_0}, a_i, a_0(P), i = 0, \dots, d$ are definable (and hence true for $d = n$).

Case 1: For every $j, s^j(a_k), k = 0, \dots, n$ is compact. Assume $s^x(a_i) = s^y(a_j), i, j \leq n$. Then $s^x(a_k), k = 0, \dots, m$ is a permutation of $s^y(a_k), k = 0, \dots, n$. For by Lemma 1, since $s^z(a_k), k = 0, \dots, n$ is always compact and $s^{n-y}(s^y(a_k)) = a_k$, every power of S which transforms an $\{s^x(a_k)\} k = 0, \dots, n$ into a set containing an element of $\{a_k\} k = 0, \dots, n$ transforms $\{s^x(a_k)\}$ into a_k . But $s^{n-y}(p)$ transforms $\{s^x(a_i)\}$ into a_j . Hence $s^{n-y}(p)$ transforms $\{s^x(a_k)\}$ into $\{a_k\}$. Hence $s^y(p)$ transforms $\{a_k\}$ into $\{s^x(a_k)\}$ and $\{s^x(a_k)\} = \{s^y(a_k)\}$. Hence a_0, \dots, a_{m-1} consists of a sequence of blocks of length $n + 1$ such that if a_i and a_j are in the same block, so are $s(a_i)$ and $s(a_j)$. Then since, if $i \leq j, f(a_i, a_j) = f(a_i, a_i) = s(a_i)$, the block to which $f(p, q)$ belongs is dependent only on the block to which p and q belong. Hence, given the decomposition of $0, \dots, m - 1$ into the blocks mentioned, if \bar{p} is the block to which p belongs, $f(p, q)$ induces the function $f^1(\bar{p}, \bar{q})$ and hence has the proper substitution property.

Case 2: There is a j such that $\{s^j(a_k)\} k = 0, \dots, n$ is not compact. Let $i \leq n$ be chosen such that if $a_x = s^j(a_0)$ and $a_y = s^j(a_i)$ then there exists an r such that $x < r < y$ or $y < r < x$ and $s^{n-j}(a_r)$ is not in $\{a_k\}, k = 0, \dots, n$. Suppose $x < r < y$. Then $s^{m-j}f^*(s^j D_{a_0}, a_1, a_0(p), T_{a_r}(p)) = s^{m-j}f^*(D_{a_x}, a_y, a_0(p), T_{a_r}(p)) = s^{m-j}D_{a_x}, a_r, a_0, s^{m-j}(a_r), a_0(p)$. If on the other hand, $y < r < x$, there exists an e such $s^e(a_y) = a_{m-1}$. Let $a_x = s^e(a_x)$ and $a_r = s^e(a_r)$. If $x^1 < r^1$,

$$\begin{aligned} & s^{m-e} f * (s^{j+e} D_{a_0}, a_i, a_0(p), T_{a_r'}(p)) \\ &= s^{m-e} f * (D_{a_{x'}}^j, a_{m-1}, a_0(p), T_{a_r'}(p)) \\ &= s^{m-e} D_{a_{x'}}, a_{r'}, a_0(p) = D_{a_x}, a_r, a_0. \end{aligned}$$

If $r' < x'$,

$$\begin{aligned} & s^{m-e} f * (s^e f * (s^e f)(s^j D_{a_0}, a_i, a_0(p), T_{a_r}(p), T_{a_{x'}}(p))) \\ &= s^{m-e} f * (s^e D_{a_r}, a_y, a_0(p), T_{a_x}(p)) \\ &= s^{m-e} f * (D_{a_r'}, a_{m-1}, a_0(p), T_{a_r'}(p)) \\ &= s^{m-e} D_{a_r'}, a_{x'}, a_0(p) = D_{a_r}, a_x, a_0(p). \end{aligned}$$

Then $D_{a_r}, a_x, a_x(p) = D_{a_r}, a_x, a_0 s^{n-j}(p)$ and $D_{a_x}, a_r, a_0(p) = D_{a_r}, a_x, a_x, D_{a_r}, a_x, a_x s^j(p)$. Hence $D_{a_x}, a_r, a_0(p)$ can be defined and $s^{n-j} D_{a_x}, a_r, a_0(p) = D_{a_0}, s^{n-j}(a_r), a_0(p)$.

Let $A_h = s^{n-j}(a_r)$. Then $h > n$. Let $h' < h$. Then $D_{a_0}, a_{h'}, a_0(p) = f * (D_{a_0}, a_h, a_0(p), T_{a_{h'}}(p))$. Hence the lemma follows.

Assume $f(p, q)$ is a preference function which is neither properly closing nor has the proper substitution property. Then by Lemma 2, $D_{a_0}, a_i, a_0(p)$ can be defined for $i = 1, \dots, m - 1$. Then $\{s^j D_{a_0}, a_i, a_0 s^k(p)\}$ contains $\{D_x, a_{m-1}, z(p)\}$. Since $f * (p, q)$ is an $M_{a_{n-1}}$ -function, $f(p, q)$ is a Sheffer function by Theorems B and C.

It should be noted that the conditions proved above are equivalent to the conditions proved in the case $m = 3$.¹³

Remark (added in 1974): This article was originally written in March, 1955. Since that time, a not unconsiderable literature on related questions has developed. The interested reader is referred to the index numbers of *The Journal of Symbolic Logic* or to the bibliography in Peter Rutz, *Zweiwertige und Mehrwertige Logik*, Ehrenwirth, München, 1973, for details. The reader should also note that many of the Sheffer function classes which are isolated in this literature, especially in the work of J. C. Muzio, overlap those of this paper.

NOTES

1. Charles S. Peirce, *Collected Papers*, vol. 4, paragraphs 12-20, 264-265.
2. Henry M. Sheffer, "A set of five independent postulates for Boolean algebras, with applications to logical constants," *Transactions of the American Mathematical Society*, vol. 14 (1913), pp. 487-488.
3. Donald L. Webb, "Definition of Post's generalized negative and maximum in terms of one binary operator," *American Journal of Mathematics*, vol. 58 (1936), pp. 193-194. "Generation of any n -valued logic by one binary operator," *Proceedings of the National Academy of Sciences*, vol. 21 (1935), pp. 252-254.
4. Erik Götlind, "Some Sheffer function in n -valued logic," *Portugaliae Mathematicae*, vol. 12 (1952), pp. 141-149.

5. Norman M. Martin, "Some analogues of the Sheffer stroke function in n -valued logic," *Indagationes Mathematicae*, vol. 12 (1950), pp. 373-400. "A note on Sheffer function in n -valued logic," *Methodos* (Milan), vol. III (1951), pp. 240-242.
6. Emil L. Post, "Introduction to the general theory of elementary propositions," *American Journal of Mathematics*, vol. 43 (1921), pp. 163-165.
7. Norman M. Martin, "The Sheffer functions of 3-valued logic," *The Journal of Symbolic Logic*, vol. 19 (1954), pp. 45-51.
8. Norman M. Martin, *Sheffer Function and Axiom Sets in m -valued Propositional Logic*, Unpublished Ph.D. Dissertation, University of California, Los Angeles, p. 55.
9. "A note on Sheffer functions in n -valued logic," p. 241.
10. "The Sheffer functions of 3-valued logic."
11. *Ibid.*
12. *Ibid.*
13. Theorem I is contained in the author's unpublished doctoral dissertation. Theorem II is a generalization of theorems contained in this dissertation. I am indebted to Mr. James E. Barry for aid in preparing this article.

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