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COMPLETENESS OF THE ALGEBRA OF SPECIES

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In [4] it was announced that the algebra of species is complete, in the sense that a formula of the algebra of species is provable in the algebra of species if and only if it is valid in every algebra of species of all subspecies of any infinite species.

1 We shall use the results of [3] to show that the algebra of species is complete in the above sense.

Theorem 1 *Let φ be a mapping of the formulae of the intuitionistic propositional calculus onto the formulae of the algebra of species such that, for every formula H of the propositional calculus, $\varphi(H) = (T = 1)$, is a formula of the algebra of species, and T is a term obtained from H by replacing every propositional variable, p, by a species variable, A, and every propositional functor, $\rightarrow$, $\vee$, $\wedge$, $\sim$, in H by the corresponding species-algebraic operation, $\Rightarrow$, $\cup$, $\cap$, $-$. Then, φ is a one-to-one mapping of the formulae of the propositional calculus onto the formulae of the algebra of species such that a formula H of the propositional calculus is provable in the propositional calculus if and only if $\varphi(H)$ is valid in every algebra of species of all subspecies of any infinite species* (*cf.* Theorem 4.11 in [3]).

Corollary 1 *The above mapping φ maps axioms* (1)-(11) *of the propositional calculus onto axioms* (1)-(11) *of the algebra of species.*

2 The axioms of the intuitionistic propositional calculus are of the form:

1. $H_1 \rightarrow (H_2 \rightarrow H_1)$
2. $H_1 \rightarrow (H_1 \rightarrow H_2) .\rightarrow. H_1 \rightarrow H_2$
3. $H_1 \rightarrow H_2. \rightarrow .(H_2 \rightarrow H_3) \rightarrow (H_1 \rightarrow H_3)$
4. $H_1 \wedge H_2 \rightarrow H_1$
5. $H_1 \wedge H_2 \rightarrow H_2$
6. $H_1 \rightarrow H_2. \rightarrow .(H_1 \rightarrow H_3) \rightarrow (H_1 \rightarrow H_2 \wedge H_3)$
7. $H_1 \rightarrow H_1 \vee H_2$
8. $H_2 \rightarrow H_1 \vee H_2$
9. $H_1 \rightarrow H_3. \rightarrow .(H_2 \rightarrow H_3) \rightarrow (H_1 \vee H_2 \rightarrow H_3)$

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10. $(H_1 \rightarrow \sim H_2) \rightarrow (H_2 \rightarrow \sim H_1)$
11. $H_1 \rightarrow (\sim H_1 \rightarrow H_2)$

The rules of proof in the intuitionistic propositional calculus are:

1. The rule of substitution: if the formula $H(p)$ is provable in the propositional calculus then the formula $H(p/H_*)$, which is obtained from $H(p)$ by substituting H_* for p throughout H, is also provable.
2. The rule of detachment: if the formula $H_1 \rightarrow H_2$ is provable and the formula H_1 is provable, then the formula H_2 is also provable.

3 *Provability in the intuitionistic propositional calculus* A formula H of the propositional calculus is said to be provable from a set of formulae X if H can be obtained from X by applying the rule of substitution and/or the rule of detachment a finite number of times. The following relations hold:

1. If H is an element of X then H is provable from X.
2. (a) If $(H_1 \rightarrow H_2)$ and H_1 are provable from X, then H_2 is also provable from X.
(b) If H is provable from X, then any formula which is obtained from H by applying the rule of substitution is also provable from X.

A formula is said to be provable in the intuitionistic propositional calculus if it is provable from the axioms of the propositional calculus.

4 The axioms of the algebra of species are formulae of the form:

1. $A \Rightarrow (B \Rightarrow A) = 1$
2. $[A \Rightarrow (A \Rightarrow B)] \Rightarrow (A \Rightarrow B) = 1$
3. $(A \Rightarrow B) \Rightarrow [(B \Rightarrow C) \Rightarrow (A \Rightarrow C)] = 1$
4. $(A \cap B) \Rightarrow A = 1$
5. $(A \cap B) \Rightarrow B = 1$
6. $(A \Rightarrow B) \Rightarrow [(A \Rightarrow C) \Rightarrow (A \Rightarrow (A \cap B))] = 1$
7. $A \Rightarrow (A \cup B) = 1$
8. $B \Rightarrow (A \cup B) = 1$
9. $(A \Rightarrow C) \Rightarrow [(B \Rightarrow C) \Rightarrow (A \cup B) \Rightarrow C] = 1$
10. $(A \Rightarrow \overline{B}) \Rightarrow (B \Rightarrow \overline{A}) = 1$
11. $A \Rightarrow (\overline{A} \Rightarrow B) = 1$

5 The rules of proof in the algebra of species are:

1. The rule of term-substitution: if the formula $H(A)$ of the algebra of species is provable, then the formula $H(A/T)$ which is obtained from $H(A)$ by substituting the term T for A throughout H, is also provable.
2. The rule of species-detachment: if the formula $T_1 \rightarrow T_2 = 1$, is provable and the formula $T_1 = 1$ is provable, then the formula $T_2 = 1$ is also provable.

6 *Provability in the algebra of species* A formula H of the algebra of species is said to be provable from a set of formulae X, if H can be obtained from X by applying the rule of term-substitution and/or species-detachment a finite number of times.

The following relations hold:

1. If H is an element of X, then H is provable from X.
2. (a) If $T_1 \Rightarrow T_2 = 1$ and $T_1 = 1$ are provable from X, then $T_2 = 1$ is also provable from X.
(b) If H is provable from X, then any formula which is obtained from H by applying the rule of term-substitution is also provable from X.

By the definition of φ, together with sections **2-5**, it follows that a formula H of the propositional calculus is provable from X if and only if H is provable from X in the algebra of species, and, by Theorem 1 and Corollary 1, we conclude that a formula H of the algebra of species is provable in the algebra of species if and only if it is valid in every algebra of species of all subspecies of any infinite species.

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