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# EPISTEMIC LOGIC WITH IDENTIFIERS

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**1** *Introduction* In this paper* we develop a system of quantificational epistemic logic, which we designate as **QKL**. The sentential part, **KL**, is interpreted in the usual Kripke-type structures. For the quantificational part, we elaborate these structures by bringing in formal machinery for the individuation (or cross-identification) of objects appearing in the domains of the structure. This machinery consists of a family of partial functions which map (parts of) domains into domains, and the system has been designed to bring out the logical properties that follow from imposing conditions on this family of identifiers.

An essential consideration for a system of epistemic logic is to preserve the kind of logical distinction that exists intuitively between statements of the forms:

1) It is known that some $x$ is $P$

and

2) Some $x$ is known to be $P$.

This distinction is related to somewhat controversial issues concerning opaque modal contexts, logical identity, and individuation. The system **QKL** reflects, in general, Hintikka's views (as expounded in [1], for example) as to how these matters should be handled. Briefly, this means that we treat the logic of opaque epistemic constructions, on the principle that quantification into such a construction refers to those objects which can be individuated throughout the set of possible states (worlds) that are relevant to it. **QKL** is a technical elaboration of this principle. Moreover, although 1) and 2) are closely related in **QKL**—in fact, 2) is logically stronger than 1)—the distinction between them is preserved.

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A defect of Hintikka's use of cross-identification as a means of elucidating the logic of opaquely construed modal contexts has seemed to be that the only methods for making such identifications are contingent and context dependent—so much so that no uniform description can be given for them.[1] This makes it look as if we should not expect to get any systematic account of logic based on cross-identification, and may even be viewed as a fulfillment of the prediction of Quine in [3] that we would not be able to articulate the logic of quantification into opaque constructions (without making highly questionable philosophical assumptions). We do not see that these conclusions are warranted, and our work with **QKL** bears against them. By positing a very minimal set of formal properties for identifiers, we are able to treat the problems we are interested in—mainly, that of the logical relationship between type 1) and type 2) statements—as specifically logical (as opposed to epistemological or ontological) ones.[2]

After setting up the semantics of **QKL**, we proceed to axiomatize it and to show that our axiomatization is semantically complete.[3] In order to obtain a suitable modeling—one that does not collapse the distinction between statements of types 1) and 2)—we modify the usual construction (as found, for example, in [4]), by employing a certain auxiliary sentential epistemic theory (*cf.* section **7**). Out of this we get a canonical type of model which is built around the system of natural numbers (as state indices) with successor and has an intransitive alternative relation. We find this interesting, because it suggests that we might well make use of such canonical modeling when we make the intuitive distinction between statements 1) and 2).

Another noteworthy feature of the modeling technique we use is that our auxiliary theory violates Hintikka's axiom: $KA \rightarrow KKA$. This does not mean that we could not model the axiom (the auxiliary language is kept separate from the language being modeled), but we would be unable to do our modeling if the axiom were accepted as a universal logical truth, governing every epistemic language. (Hintikka would not now defend it as such, anyway, as he makes clear in [5]).

**2** *Epistemic languages* A (first order) *epistemic predicate language* has symbols of the following types:

A. Logical Symbols

1) *Parentheses:* (, ).
2) *Connective Symbols:* $\rightarrow$, $\sim$.
3) The *epistemic operator* symbol: $K$.
4) A countably infinite set of *variables*.
5) The *equality symbol:* $=$.
6) For each $n = 0, 1, 2, \ldots$, the $n$'th *order* individuating symbol: $Q^n$.

B. Parameters

1) For each $n = 0, 1, 2, \ldots$, a countable set of $n$-place *predicate symbols.*
2) A countable set of *constant symbols.*

We shall suppose that the logical symbols are fixed for all such languages, so that different languages are obtained by choosing different sets of parameters. Further, we shall classify the equality symbol as a (special) 2-place predicate symbol and the $Q$-symbols as 1-place predicate symbols. An epistemic predicate language will also be referred to as a *QK language*.

An *epistemic sentential language* is a language whose logical symbols are just the parentheses, connective symbols, and the epistemic operator symbol; and whose parameters are just a set of 0-place predicate symbols—called *sentence symbols*. We will also refer to such a language as a *K language.*

Our notational conventions for handling the syntax of $K$ and $QK$ languages—which is developed in the usual way—are given in the following table:

| Notation | Used For |
|---|---|
| $x, y, z, \ldots$ | variables |
| $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$ | constant symbols |
| $r, s, t, \ldots$ | terms |
| $P$ | predicate symbols |
| $A, B, C, \ldots$ | formulas |
| $A \vee B$ | $\sim A \rightarrow B$ |
| $A \& B$ | $\sim(A \rightarrow \sim B)$ |
| $A \leftrightarrow B$ | $(A \rightarrow B) \& (B \rightarrow A)$ |
| $A \rightarrow B \rightarrow C$ | $A \rightarrow (B \rightarrow C)$ |
| $K^nA$; $n = 0, 1, 2, \ldots$ | $KK \ldots KA$ (with $n$ occurrences of $K$) |
| $J^nA$; $n = 0, 1, 2, \ldots$ | $\sim K^n \sim A$ |
| $\exists xA$ | $\sim(x) \sim A$ |

A term $r$ occurs *bound* in a formula $A$ iff it occurs in $A$ within the scope of a quantifier on $r$. Occurrences of $r$ other than bound ones are *free* occurrences. (Note that any occurrence of a constant term is free, by this definition.) The formula that results from substitution of $t$ for all free occurrences of $r$ in $A$ is designated $A^r_t$. If $t$ is *free for* $r$ in $A$; i.e., if the substituted occurrences of $t$ are free in $A^r_t$, we use $A_r(t)$—and sometimes just $A(t)$—for $A^r_t$.

Atomic formulas that are not $Q$-formulas (i.e., not of the form $Q^n r$) will be called *basic formulas.* The *K-profile,* $\mathrm{Pr}_x(A)$, *of A with respect to x* is defined recursively as follows:

1) $\mathrm{Pr}_x(A) = 0$, if $A$ is basic.

2) $$\mathrm{Pr}_x(Q^n r) = \begin{cases} n, \text{ if } x = r \\ 0, \text{ otherwise.} \end{cases}$$

3) $\mathrm{Pr}_x(\sim A) = \mathrm{Pr}_x(A)$.

4) $\mathrm{Pr}_x(A \rightarrow B) = \max(\mathrm{Pr}_x(A), \mathrm{Pr}_x(B))$.

5) $$\mathrm{Pr}_x(KA) = \begin{cases} \mathrm{Pr}_x(A) + 1, \text{ if } x \text{ occurs free in } A \\ \mathrm{Pr}_x(A), \text{ otherwise.} \end{cases}$$

6) $\mathrm{Pr}_x((y)A) = \begin{cases} \mathrm{Pr}_x(A), \text{ if } y \neq x \\ 0, \text{ otherwise.} \end{cases}$

$\mathrm{Pr}_x(A)$ gives the maximum "depth" of modal contexts for free occurrences of $x$ in $A$.

**3** *K-structures* A *K-frame* is a pair $\langle I, \geqslant\rangle$, where $I$ is a non-empty set and $\geqslant$ is a reflexive binary relation on $I$. $I$ is the set of *state indices* and $\geqslant$ is the *alternative relation* of the frame. A *K-structure* for a $QK$ language $\mathcal{L}$ over $\langle I, \geqslant\rangle$ is a mapping $\mathfrak{M}$ which assigns

1) a non-empty set $M_i$ to each $i \in I$;
2) to each $n$-place predicate symbol $P$ of $\mathcal{L}$ a function $P^{\mathfrak{M}}$ on $I$ such that, for each $i \in I$, $P_i^{\mathfrak{M}} (= P^{\mathfrak{M}}(i))$ is an $n$-ary relation on $M_i$;
3) to each constant symbol $\mathfrak{a}$ of $\mathcal{L}$ a partial function $\mathfrak{a}^{\mathfrak{M}}$ on $I$ such that $\mathfrak{a}_i^{\mathfrak{M}} \in M_i$, for each $i \in$ domain $\mathfrak{a}^{\mathfrak{M}}$.

We take a 0-ary relation to be just one of a pair—say 0 and 1—of designated truth-values. Thus, if $\mathcal{L}$ is a $K$ language, a $K$-structure for $\mathcal{L}$ over $\langle I, \geqslant\rangle$ simply assigns a truth-value $P_i^{\mathfrak{M}}$, for each $i$, to each sentence symbol $P$ of $\mathcal{L}$ (and no sets $M_i$ need be specified). $M_i$ is called the *domain* of $\mathfrak{M}$ at $i$, $P_i^{\mathfrak{M}}$ is called the *extension* of $P$ at $i$, and $\mathfrak{a}_i^{\mathfrak{M}}$ (when it exists) is called the *denotation* of $\mathfrak{a}$ at $i$. Note that, since $\mathfrak{a}$ may fail to denote in any domain $M_i$, we have here a modeling of so-called *free logic.*

We shall now develop machinery for the interpretation of epistemic languages in their $K$-structures. For a given frame $\langle I, \geqslant\rangle$, we define relations $\geqslant^n$ on $I$, for $n \geqslant 0$, by recursion as follows:

1) $j \geqslant^0 i$ iff $j = i$;
2) $j \geqslant^{n+1} i$ iff for some $k \in I$, $j \geqslant^n k$ and $k \geqslant i$.

We call $j$ an *n'th order alternative* to $i$ iff $j \geqslant^n i$. Note that, by the reflexivity of $\geqslant$, $j \geqslant^n i$ implies $j \geqslant^{n+1} i$. ($j \geqslant^n i$ holds whenever one can get from $i$ to $j$ in $n$ steps through the relation $\geqslant$.) Further, we define the *accessibility relation*, $\geqslant^\#$, on $I$ by taking

$$j \geqslant^\# i \text{ iff for some } n, j \geqslant^n i,$$

and we say that $j$ is *accessible from* $i$ whenever $j \geqslant^\# i$.

Now suppose that $M$ is a $K$-structure over a frame $\langle I, \geqslant\rangle$. We consider mappings $F$ defined on the relation $\geqslant^\#$ such that, whenever $j \geqslant^\# i$, $F_{j,i} (= F(j, i))$ is a partial function mapping $M_i$ into $M_j$. Given such an $F$, we define the *n'th order domain*, $M_i^n$, at $i$, by setting

$$M_i^n = \bigcap_{j \geqslant^n i} \text{domain } F_{j,i}.$$

$F$ will be called an *individuating map* for the structure $M$ iff it satisfies the following conditions:

1) $F_{i,i}(a) = a$, all $a \in M_i$ (i.e., $F_{i,i}$ is the identity function on $M_i$).

2) If $j \geqslant^{\#} k \geqslant i$, then $F_{j,i} \subseteq F_{j,k} \circ F_{k,i}$ (i.e., $F_{j,i}(a) = F_{j,k}(F_{k,i}(a))$, all $a \in$ domain $F_{j,i}$).
3) $M_i^n \neq \emptyset$, for $n = 0, 1, 2, \ldots$.

We shall call $F_{j,i}$ the *identifier function from $i$ to $j$*. Intuitively, $F_{j,i}(a) = b$ means that object $b$ in $M_j$ has been identified as "the same" as object $a$ in $M_i$. Of course, not every object in $M_i$ need be identified in $M_j$—$a$ may not "exist" in $M_j$—so $F_{j,i}$ is in general just a partial function. In order for $a$ to get into $M_i^n$, it must be identifiable under $F$ at all $n$'th order alternatives to $i$. We call such an $a$ an *$n$'th order individuated* element—briefly, an *$n$'th order individual*—of $M_i$.

Our condition 1) on $F$ says simply that each object of $M_i$ is identified with itself. Condition 2) is designed to prohibit a certain kind of splitting of individuals: it says that any identification from $i$ to $j$ will give the same result as a chain of identifications (of the same object) consisting of an identification from $i$ to $k \geqslant i$, followed by an identification from $k$ to $j$. Condition 3) is a kind of regularity condition on identifiers (and one we could dispense with without affecting our main results). It simply says that every domain we have to consider is non-empty—a usual convention in logic.

Say $a \in M_i$. We will call the object $a_j^i = F_{j,i}(a)$—whenever it exists—the *image* of *$a$ at $j$ from $i$*, and we call the set of images $a_j^i$ of $a$ from $i$ the *world line* of *$a$ from $i$*. Following Hintikka in [1], we may interpret the world line of an object from a given state as the set of realizations of some individual concept within the context of states accessible from the given state. Since $a \in M_i^n$ just in case its world line from $i$ runs through all $n$'th order alternatives to $i$, it is appropriate to say that such an object has been individuated in the context of this set of states.

An *assignment to variables* in a structure $\mathfrak{M}$ over a frame $\langle I, \geqslant\rangle$ is a function $u$ on $I$ which, for each $i \in I$, assigns a member of $M_i$ to each variable; i.e.,

$$u_i(x) \in M_i, \text{ for all variables } x.$$

If $y$ is a fixed variable and $a \in M_i$, we define a new assignment, $u(y\,|\,a)$, which agrees with $u$ except that it assigns objects to $y$ along the world line of $a$ from $i$. More explicitly,

$$u_j(y\,|\,a)(x) = \begin{cases} a_j^i, \text{ if } x = y \text{ and } a_j^i \text{ exists} \\ u_j(x), \text{ otherwise.} \end{cases}$$

It will be convenient to extend any assignment $u$ to terms of the language, whenever they denote in $\mathfrak{M}$. We do this by taking

$$u_i(\mathfrak{a}) = \mathfrak{a}_i^{\mathfrak{M}} \text{ whenever } \mathfrak{a}_i^{\mathfrak{M}} \text{ exists.}$$

The object $u_i(r)$—whenever it exists—will be called the *denotation* of term $r$ *at $i$ under $u$*. When we have an individuating map $F$ for $\mathfrak{M}$ we may use $r_j^i$ for $F_{j,i}(u_i(r))$—whenever it exists—and call $r_j^i$ the *image* of *$r$ at $j$ from $i$*

*under* $u$. The term $r$ will be said to be *n-rigid at i under u* iff $u_i(r) \in M_i^n$ and $u_j(r) = r_j^i$, for all $j \geqslant^n i$. Further, $r$ is *n-rigid at i* iff it is $n$-rigid at $i$ under all assignments $u$.

**4** *Satisfaction in K-structures* The basic mechanism for interpreting a language $\mathcal{L}$ in its structures is a *satisfaction predicate*, which will say, for any basic formula $A$, whether $A$ is *satisfied at* state $i$ *in* $\mathfrak{M}$ *under* an assignment $u$ in $\mathfrak{M}$. Our notation for this will be: $\models_i^{\mathfrak{M}} A[u]$. The notion of satisfaction in a structure is complicated by the fact that we allow non-denoting terms in our languages, and there are various ways of interpreting statements containing such terms.[4] We shall remain neutral with respect to most of these possibilities by defining a generic notion of satisfaction. In other words, we do not give a unique interpretation of formulas in a given structure. Rather, we lay down general conditions which allow a large number of particular interpretations. Roughly speaking, what we require is that a satisfaction predicate in $\mathfrak{M}$ respect the laws of equality, respect $\mathfrak{M}$ itself, and respect assignments to variables in $\mathfrak{M}$. We formulate these requirements as follows:

1a) $\models_i^{\mathfrak{M}} r = r[u]$.
1b) If $\models_i^{\mathfrak{M}} r = t[u]$, then either both $u_i(r)$ and $u_i(t)$ exist or neither do.
1c) If $u_i(r)$ and $u_i(t)$ exist, then $\models_i^{\mathfrak{M}} r = t[u]$ iff $u_i(r) = u_i(t)$.
2a) If $\models_i^{\mathfrak{M}} r_p = t_p[u]$; $p = 1, 2, \ldots, n$, then $\models_i^{\mathfrak{M}} Pr_1 \ldots r_n[u]$ iff $\models_i^{\mathfrak{M}} Pt_1 \ldots t_n[u]$.
2b) If $u_i(r_p)$ exists; $p = 1, 2, \ldots, n$, then $\models_i^{\mathfrak{M}} Pr_1 \ldots r_n$ iff $(u_i(r_1), \ldots, u_i(r_n)) \in P_i^{\mathfrak{M}}$.
3) If $u$ and $v$ are assignments in $\mathfrak{M}$ which agree at $i$ on all variables of $A$; i.e., $u_i(x) = v_i(x)$, for all variables $x$ which occur in $A$, then

$$\models_i^{\mathfrak{M}} A[u] \text{ iff } \models_i^{\mathfrak{M}} A[v].$$

Conditions 1b), 1c), and 2b) make $\models^{\mathfrak{M}}$ behave well with respect to $\mathfrak{M}$; 1a) and 2a) give the usual logical requirements for equality; and 3) makes satisfaction in $\mathfrak{M}$ depend upon the values assigned to the relevant variables. Note that 1b) stipulates that we cannot satisfy an equality statement between a denoting and a non-denoting term. Although it may eliminate some otherwise quite desirable interpretations, nothing more than a few technicalities hang on this condition.

If $\mathfrak{M}$ is a $K$-structure and $F$ is an individuating map for $\mathfrak{M}$, we set $\mathfrak{M}[F] = (\mathfrak{M}, F)$, and call $\mathfrak{M}[F]$ an *individuated structure*. Any satisfaction predicate $\models^{\mathfrak{M}}$ in $\mathfrak{M}$ can be extended to a *satisfaction predicate*, $\models^{\mathfrak{M}[F]}$, *in* $\mathfrak{M}[F]$ by the following clauses:

4) $\models_i^{\mathfrak{M}[F]} Q^n r[u]$ iff $r$ is $n$-rigid at $i$ under $u$.
5) $\models_i^{\mathfrak{M}[F]} \sim A[u]$ iff not-$\models_i^{\mathfrak{M}[F]} A[u]$.
6) $\models_i^{\mathfrak{M}[F]} (A \rightarrow B)[u]$ iff not-$\models_i^{\mathfrak{M}[F]} A[u]$ or $\models_i^{\mathfrak{M}[F]} B[u]$.
7) $\models_i^{\mathfrak{M}[F]} KA[u]$ iff for all $j \geqslant i$, $\models_j^{\mathfrak{M}[F]} A[u]$.
8) $\models_i^{\mathfrak{M}[F]} (x)A[u]$ iff for all $a \in M_i^n$, $\models_i^{\mathfrak{M}[F]} A[u(x|a)]$.

Note that we immediately have the following:

9) $\models_i^{\mathfrak{M}[F]} K^n A[u]$ iff for all $j \geqslant^n i$, $\models_j^{\mathfrak{M}[F]} A[u]$.
10) $\models_i^{\mathfrak{M}[F]} J^n A[u]$ iff for some $j \geqslant^n i$, $\models_j^{\mathfrak{M}[F]} A[u]$.
11) $\models_i^{\mathfrak{M}[F]} \exists x A[u]$ iff for some $a \in M_i^n$, $\models_i^{\mathfrak{M}[F]} A[u(x|a)]$.

Suppose $S$ is a set of formulas of $\mathcal{L}$ and $\mathfrak{M}[F]$ is an individuated structure for $\mathcal{L}$. Then $S$ *implies* $A$ *at* $i$ *in* $\mathfrak{M}[F]$—notation: $S \models_i^{\mathfrak{M}[F]} A$—iff for all satisfaction predicates $\models^{\mathfrak{M}}$ and all $u$ in $\mathfrak{M}$,

$$\models_i^{\mathfrak{M}[F]} A[u] \text{ whenever } \models_i^{\mathfrak{M}[F]} B[u], \text{ all } B \in S.$$

Also, $S$ *implies* $A$ *in* $\mathfrak{M}[F]$—notation: $S \models^{\mathfrak{M}[F]} A$—iff $S$ implies $A$ at all $i \in I$ in $\mathfrak{M}[F]$. Further, $S$ *logically implies* $A$—notation: $S \models A$—iff for every $\mathfrak{M}$ and $F$, $S$ implies $A$ in $\mathfrak{M}[F]$. Finally, $A$ is *valid at* $i$ in $\mathfrak{M}[F]$, *valid in* $\mathfrak{M}[F]$, or *logically valid*, according as $\models_i^{\mathfrak{M}[F]} A$, $\models^{\mathfrak{M}[F]} A$, or $\models A$. If $A$ is a sentence (a closed formula) we may use '$A$ is true' or '$A$ holds' in place of '$A$ is valid'. When no ambiguity results, we may write $\models$ for $\models^{\mathfrak{M}[F]}$ or $\models^{\mathfrak{M}}$.

## 5 *Semantical results*

Lemma 5.1 *Suppose $u$ and $v$ are assignments to variables in some $\mathfrak{M}$ such that, for any variable $x$ which occurs free in a given formula $A$, if $n = \mathrm{Pr}_x(A)$ then*

$$u_j(x) = v_j(x), \text{ all } j \geqslant^n i.$$

*Then we have*

$$\models_i A[u] \text{ if and only if } \models_i A[v].$$

*Proof:* This is just a generalization of condition 3) above, to cover all formulas. It is easily established by induction on $A$—using clauses 4)-8) for satisfaction predicates.

Lemma 5.2 *If $r$ is $\mathrm{Pr}_x(A)$-rigid at $i$ under $u$, then*

$$\models_i A_x(r)[u] \text{ if and only if } \models_i A[u(x|u_i(r))].$$

*Proof:* Let $a = u_i(r)$. Then, for $j \geqslant^n i$, where $n = \mathrm{Pr}_x(A)$, $u_j(x|a)(x) = a_j^i = r_j^i = u_j(r)$, since $r$ is $n$-rigid. Thus, $u(x|a)$ assigns the same object to $x$ at $j$ as $u$ does to $r$. The lemma depends on just this fact, and a routine induction on $A$ can be given.

Lemma 5.3 *If $r$ is $\mathrm{Pr}_x(A)$-rigid at $i$ under $u$, then*

$$\models_i \exists x A[u] \text{ whenever } \models_i A_x(r)[u].$$

*Proof:* If $\models_i A_x(r)[u]$, then $\models_i A[u(x|u_i(r))]$, by Lemma 5.2. Now $u_i(r) \in M_i^n$, where $n = \mathrm{Pr}_x(A)$, so we have $\models_i \exists x A[u]$.

Note that having $u_i(r) \in M_i^n$ is not, by itself, sufficient to allow us to generalize existentially on $r$ in $A_x(r)$. This is reasonable; we cannot infer

Some $x$ is known to be $P$

from

It is known that $r$ is $P$,

even though $r$ may name an object that can be individuated—because $r$ may fail to do the necessary individuating (i.e., may fail to be rigid).

Lemma 5.4 (Characterization of rigid terms)

$$\models_i Q^n r[u] \textit{ if and only if } \models_i \exists x K^n(x = r)[u].$$

Note: We assume $x$ is distinct from $r$.

*Proof:* If $r$ is $n$-rigid at $i$ under $u$, we can use Lemma 5.3 to generalize on $r$ in $K^n(r = r)$—which is a valid formula—to get $\models_i \exists x K^n(x = r)[u]$. Conversely, if $\models_i \exists x K^n(x = r)[u]$, we have some $a \in M_i^n$ with $\models_i K^n(x = r)[u(x|a)]$. From this we get $\models_j x = r[u(x|a)]$, all $j \geqslant^n i$, so $u_j(r)$ exists (by condition 1b) on satisfaction predicates) and $a_j^i = u_j(r)$, all $j \geqslant^n i$. Now

$$r_j^i = F_{j,i}(u_i(r)) = F_{j,i}(a) = a_j^i = u_j(r),$$

so $r$ is $n$-rigid at $i$ under $u$.

Lemma 5.5 *The formulas:* $Q_r^{n+1} \to KQ^n r$, *for* $n = 0, 1, 2, \ldots$, *are logically valid.*

*Proof:* Suppose $\models_i Q_r^{n+1}[u]$. Then $r_j^i = u_j(r)$, for all $j \geqslant^{n+1} i$. Say $k \geqslant i$. If $j \geqslant^n k$, then $j \geqslant^{n+1} i$ and, by condition 2) for individuating maps $F$,

$$u_j(r) = r_j^i = F_{j,k}(r_k^i) = F_{j,k}(u_k(r)) = r_j^k.$$

Thus $r$ is $n$-rigid at $k$ under $u$, for all $k \geqslant i$. This gives us $\models_i KQ^n r[u]$.

As a corollary of Lemmas 5.4 and 5.5 we have that

$$\exists x K^{n+1}(x = r) \to K \exists x K^n(x = r)$$

is logically valid. The antecedent of this formulates a statement of the form

Some $x$ is known to be $P$,

while the consequent has the form

It is known that some $x$ is $P$.

We shall show that the converse is not valid. For this purpose we choose states $0, 1, \ldots, n + 1$ with: $n + 1 \geqslant n \geqslant \ldots \geqslant 1 \geqslant 0$. We read this chain intransitively—e.g., $2 \not\geqslant 0$. Then we select distinct objects $a^0, b^0, a^1, a^2, \ldots, a^{n+1}$, where the superscript indicates the domain to which each object belongs—e.g., $M_0 = \{a^0, b^0\}$ and $M_1 = \{a^1\}$. Now we define identifiers as follows:

1) $F_{j,i}(a^i) = a^j$, whenever $0 \leqslant i \leqslant j \leqslant n$ or $1 \leqslant i \leqslant j \leqslant n + 1$.

2) $F_{j,0}(b^0) = \begin{cases} b^0, \text{ if } j = 0 \\ a^j, \text{ otherwise.} \end{cases}$

Then $F$ satisfies the conditions to be an individuating map. Note that the world line of $a^0$ is $\{a^0, \ldots, a^n\}$ and the world line of $a^1$ is $\{a^1, \ldots, a^{n+1}\}$. Thus, if we take a constant symbol $\mathfrak{a}$ and set

$$\mathfrak{a}_i^{\mathfrak{M}} = a^i; \text{ for } i = 0, 1, \ldots, n+1$$

we will have $\models_0^{\mathfrak{M}} Q^n\mathfrak{a}$ and $\models_1^{\mathfrak{M}} Q^n\mathfrak{a}$, so $\models_0^{\mathfrak{M}} KQ^n\mathfrak{a}$. However, $Q^{n+1}\mathfrak{a}$ does not hold at 0, because this would require $\mathfrak{a}_{n+1}^{\mathfrak{M}} = F_{n+1,0}(a^0)$, and $a_{n+1}^0 = F_{n+1,0}(a^0)$ does not exist. Our model falsifies the formula

$$KQ^n r \to Q^{n+1} r$$

when $r$ is $\mathfrak{a}$. Notice that if we had the "converse" of condition 2) for $F$; i.e., if we had

2′) $F_{j,k} \circ F_{k,i} \subseteq F_{j,i}$, whenever $j \geqslant^{\#} k \geqslant i$

then, since $F_{n+1,1}(F_{1,0}(a^0)) = F_{n+1,1}(a^1) = a^{n+1}$, we would have $F_{n+1,0}(a^0) = a^{n+1}$. One can readily show, in fact, that 2′) suffices to make the above formula valid (for any term $r$).

Lemma 5.6 *The following are logically valid:*

1) $(x)A \;\&\; Q^n r \to A_x(r)$.
2) $(x)(Q^n x \to A) \to (x)A$.
3) $(x)(C \to A) \to C \to (x)A$, *where* $n = \mathrm{Pr}_x(A)$ *and* $x$ *is not free in* $C$.

*Proof:* Part 1) involves a straightforward application of Lemma 5.2. The proof of 2) uses the fact that, for each $a \in M_i^n$, $x$ is $n$-rigid at $i$ under $u(x|a)$—whence $\models_i Q^n x[u(x|a)]$, for all $a \in M_i^n$. Part 3) requires the use of Lemma 5.1, but is easy.

## 6 *An axiom system for* **QKL**:

*Axioms*

For $\to$: 1) $A \to B \to A$
2) $(A \to B) \to (A \to B \to C) \to (A \to C)$

For $\sim$: 1) $\sim\sim A \to A$
2) $A \to \sim\sim A$
3) $(A \to B) \to \sim B \to \sim A$

For $\vee$: 1) $(A \to C) \to (B \to C) \to A \vee B \to C$
2) $A \to A \vee B$
3) $B \to A \vee B$

For &: 1) $A \to B \to A \;\&\; B$
2) $A \;\&\; B \to A$
3) $A \;\&\; B \to B$

For $K$: 1) $KA \to A$
2) $K(A \to B) \to KA \to KB$

For =: 1) $r = r$
2) $r_1 = t_1 \;\&\; \ldots \;\&\; r_n = t_n \to Pr_1 \ldots r_n \to Pt_1 \ldots t_n$

For $Q$: 1) $Q^0 x$
2) $\exists x Q^n x$; $n = 0, 1, 2, \ldots$
3) $r = t \to Q^0 r \to Q^0 t$

4) $Q^n r \;\&\; Q^n t \;\&\; r = t \rightarrow K^n(r = t)$; $n = 0, 1, 2, \ldots$
5) $Q^{n+1} r \rightarrow KQ^n r$; $n = 0, 1, 2, \ldots$

For $(x)$: 1) $(x)A \;\&\; Q^n r \rightarrow A_x(r)$
2) $(x)(Q^n x \rightarrow A) \rightarrow (x)A$
3) $(x)(C \rightarrow A) \rightarrow C \rightarrow (x)A$ where $n = \mathrm{Pr}_x(A)$ and $x$ is not free in $C$

*Rules of Inference*

Modus Ponens (MP): From $A$ and $A \rightarrow B$ infer $B$
The $K$-rule: From $A$ infer $KA$
Generalization (Gen): From $A$ infer $(x)A$

As usual, a formula is a *formal theorem* of the system iff it follows from axioms by inference rules. A sequence of formulas leading from axioms to $A$ via inference rules is a *formal proof* of $A$. $A$ is *deducible from* a set $S$ of formulas iff there are formulas $A_1, \ldots, A_m \in S$ such that $A$ can be obtained from $A_1, \ldots, A_m$ and theorems of **QKL** by use of the rules MP and Gen, with the proviso that Gen may not be applied with respect to a variable which occurs free in $A_1, A_2, \ldots, A_m$. Note that $A$ is deducible from $S$ just in case there is a sequence of formulas leading from formulas $A_1, \ldots, A_m$ of $S$, and logical axioms, to $A$ via inference rules, and in which

1) the $K$-rule is applied only to theorems of **QKL**;
2) every application of Gen is either to a theorem of **QKL** or is on a variable which does not occur free in $A_1, \ldots, A_m$.

Such a sequence is a *deduction* of $A$ *from* $S$ and the formulas $A_1, \ldots, A_m$ are *assumption formulas* for the deduction. We write $S \vdash A$ to mean: $A$ is deducible from $S$. Clearly, $A$ is a theorem of **QKL** just in case $\vdash A$ (i.e., $A$ is deducible from the empty set).

We have already shown that certain of our axioms are valid (Lemmas 5.5 and 5.6) and it is elementary to check the remaining ones. It is also easy to see that our inference rules preserve logical validity. These facts are used in proving

Theorem 6.1 (The Soundness Theorem) *If* $S \vdash A$ *then* $S \vDash A$.

*Proof:* Say $B_1, \ldots, B_n$ is a deduction of $A$ from $S$ with assumption formulas $A_1, \ldots, A_m$. Then we can show $A_1, \ldots, A_m \vDash B_p$; $p = 1, 2, \ldots, n$ by induction on the given deduction. The only case of any consequence is when $B_p$ is $(x)B_q$, for some $q < p$. Suppose that $A_1, \ldots, A_m$ are satisfied at $i$ in some structure $\mathfrak{M}[F]$ under an assignment $u$. Since $x$ cannot occur free in an assumption formula, they will also be satisfied under $u(x|a)$, for all $a \in M_i^n$, where $n = \mathrm{Pr}_x(B_q)$. By induction hypothesis, so will $B_q$. This gives us the satisfaction of $(x)B_q$ at $i$ under $u$.

Lemma 6.2 (The Deduction Theorem) *If* $S, A \vdash B$ *then* $S \vdash A \rightarrow B$.

*Proof:* If $B_1, \ldots, B_n$ is a deduction of $B$ from $S + A$, then by a routine induction on this deduction we get

$$S \vdash A \rightarrow B_p, \textit{for } p = 1, 2, \ldots, n.$$

Corollary 6.3 *If* $S \vdash A_1, \ldots, S \vdash A_n$ *and* $A_1, \ldots, A_n \vdash A$, *then* $S \vdash A$.

*Proof:* By the deduction theorem

$$(*) \qquad A_1 \to A_2 \to \ldots \to A_n \to A$$

is a theorem of **QKL**. When we combine a proof of (*) with deductions of $A_1, \ldots, A_n$ from $S$, we get a deduction of $A$ from $S$.

Lemma 6.4 (Alphabetic Variant Lemma) *If* $y$ *does not occur free in* $A$, $(x)A \vdash (y)A_x(y)$.

*Proof:* From axiom $(x)$ 1) we get $(x)A, Q^n y \vdash A_x(y)$, where $n = \mathrm{Pr}_x(A)$. Hence $(x)A \vdash Q^n y \to A_x(y)$. Since $y$ is not free in $A$ we can introduce $(y)$ to get $(x)A \vdash (y)(Q^n y \to A_x(y))$. Now we use axiom $(y)$ 2) to get $(x)A \vdash (y)A_x(y)$.

Corollary 6.5 *If* $y$ *does not occur free in* $A$, $(y)A_x(y) \vdash (x)A$.

*Proof:* Since $x$ does not occur free in $A_x(y)$, we can apply Lemma 6.4 to get $(y)A_x(y) \vdash (x)(A_x(y))_y(x)$. Now $(A_x(y))_y(x) = A^{xy}_{yx} = A$, since $y$ has no free occurrence in $A$.

Lemma 6.6 (Generalization on Constants) *If* $S \vdash A_x(\mathfrak{a})$ *then* $S \vdash (x)A$, *provided* $\mathfrak{a}$ *does not occur in* $A$ *or in any formula of* $S$.

*Proof:* Let $B_1, \ldots, B_n$ be a deduction of $A_x(\mathfrak{a})$ from $S$, and let $y$ be a variable which does not occur in any $B_p$. For each $p$, take $B'_p$ to be $(B_p)^{\mathfrak{a}}_y$. Then $B'_1, B'_2, \ldots, B'_n$ is a deduction of $(A_x(\mathfrak{a}))^{\mathfrak{a}}_y = A^{x\,\mathfrak{a}}_{\mathfrak{a}\,y} = A_x(y)$—since $a$ does not occur in $A$—from $S$. Thus $S \vdash A_x(y)$, with assumption formulas among $B_1, \ldots, B_n$. By our choice of $y$ we can introduce $(y)$ to get $S \vdash (y)A_x(y)$. If $y = x$, we are done. Otherwise, $y$ does not occur in $A$ so we can apply Corollary 6.5 to obtain $S \vdash (x)A$.

Lemma 6.7 *If* $x$ *does not occur free in* $S$, *and if* $\mathrm{Pr}_x(A) \leqslant \mathrm{Pr}_x(B)$, *then*

$$S \vdash A \to B \textit{ implies } S \vdash (x)A \to (x)B.$$

*Proof:* Say $m = \mathrm{Pr}_x(A)$ and $n = \mathrm{Pr}_x(B)$. Since $m \leqslant n$, we have $Q^n x \vdash Q^m x$, by Axioms Q5) and K1). Now $(x)A, Q^m x \vdash A$, so $(x)A, Q^n x \vdash A$. By assumption, $S, A \vdash B$, so we have $S, (x)A, Q^n x \vdash B$; whence $S, (x)A \vdash Q^n x \to B$. Now $x$ is not free in $S$, so we may introduce $(x)$ here to get $S, (x)A \vdash (x)(Q^n x \to B)$. Then $S, (x)A \vdash (x)B$, by Axiom $(x)$ 2).

Corollary 6.8 *If* $x$ *does not occur free in* $S$, *and if* $\mathrm{Pr}_x(A) \geqslant \mathrm{Pr}_x(B)$, *then* $S \vdash A \to B$ *implies* $S \vdash \exists xA \to \exists xB$.

*Proof:* Apply Lemma 6.7 to $S \vdash \sim B \to \sim A$.

For our next result (the replacement theorem) we need to define the *K-profile*, $\mathrm{Pr}(A)$, of a formula $A$. $\mathrm{Pr}(A)$ is just the depth of the modal context for $A$, and is given by the following recursive clauses:

1) $\mathrm{Pr}(A) = 0$, if $A$ is basic.
2) $\mathrm{Pr}(Q^n r) = n$.
3) $\mathrm{Pr}(\sim A) = \mathrm{Pr}(A)$.

4) $\mathrm{Pr}(A \rightarrow B) = \max(\mathrm{Pr}(A), \mathrm{Pr}(B))$.
5) $\mathrm{Pr}(KA) = \mathrm{Pr}(A) + 1$.
6) $\mathrm{Pr}((x)A) = \mathrm{Pr}(A)$.

For a given formula $C$, we use $C_A$ for the formula which results by replacing a specified subformula of $C$ by $A$.

Lemma 6.9 (Formula Replacement) *Suppose $A$ and $B$ each have exactly $x_1, \ldots, x_m$ as free variables, and that*

$$\mathrm{Pr}_{x_p}(A) = \mathrm{Pr}_{x_p}(B), \textit{ for } p = 1, 2, \ldots, m.$$

*Then*

$$K^n(x_1)(x_2) \ldots (x_m)(A \leftrightarrow B) \vdash C_A \leftrightarrow C_B$$

*where* $n = \mathrm{Pr}(C)$.

*Proof:* By induction on $C$. For the quantifier case, apply Lemma 6.7, making use of the fact that

$$\mathrm{Pr}_{x_p}(C_A) = \mathrm{Pr}_{x_p}(C_B);\ p = 1, 2, \ldots, m.$$

Lemma 6.10 (Term Replacement) $K^n(r = t) \vdash A_x(r) \leftrightarrow A_x(t)$, *where* $n = \mathrm{Pr}_x(A)$.

*Proof:* By induction on $A$. For the quantifier case $A$ has the form $(y)B$. Now $r$ and $t$ must be distinct from $y$; otherwise they would not be free for $x$ in $A$. Thus we can apply Lemma 6.7 to the induction hypothesis $K^n(r = t) \vdash B_x(r) \leftrightarrow B_x(t)$, to get, $K^n(r = t) \vdash (y)B_x(r) \leftrightarrow (y)B_x(t)$.

Lemma 6.11 $\vdash (x)(y)(x = y \rightarrow A(x) \leftrightarrow A(y))$.

*Proof:* By Lemma 6.10 and Axiom Q 4), $Q^n x,\ Q^n y,\ x = y \vdash A(x) \leftrightarrow A(y)$, where $n = \mathrm{Pr}_x(A(x))$. From this we have

$$Q^n x \vdash Q^n y \rightarrow x = y \rightarrow A(x) \leftrightarrow A(y).$$

If we introduce $(y)$ and use $(y)$ 2) we get

$$Q^n x \vdash (y)(x = y \rightarrow A(x) \leftrightarrow A(y)).$$

By repeating this procedure on $x$ we get the lemma.

**7** *A sentential epistemic theory* If we take some set $X$ of sentences of a $K$ or $QK$ language as (non-logical) axioms for a theory in that language, we are doing more than simply treating $X$ as a fixed set of assumption formulas. The reason is that, as axioms, the sentences of $X$ are given the status of logical validities and are subject to the application of all inference rules; in particular, the $K$-rule. We can achieve this by taking, not $X$, but the closure $X^\#$ of $X$ under the $K$-rule as the set of assumed formulas for the theory. Then we define the *theory based on* axiom set $X$ to be the set

$$\mathrm{Th}(X) = \{A : X^\# \vdash A\}.$$

Further, we take

$$S \vdash_X A \text{ iff } S + \mathsf{Th}(X) \vdash A$$

and say that $A$ is *deducible from* $S$ *in* $\mathsf{Th}(X)$ whenever $S \vdash_X A$. Also, we say that $A$ is a *theorem* of $\mathsf{Th}(X)$ whenever $\vdash_X A$. Note that

1) $S \vdash_X A$ if and only if $S + X^{\#} \vdash A$;
2) $A \in \mathsf{Th}(X)$ if and only if $\vdash_X A$.

Also note that $\mathsf{Th}(X)$ is closed under Gen, MP, and the $K$-rule. On the basis of this we are able to "transfer" all of the results for $\vdash$ obtained in section **6**, to the deducibility relation $\vdash_X$ induced by $\mathsf{Th}(X)$.

Suppose $\mathsf{Th}(X)$ is a theory in language $\mathcal{L}$. A structure $\mathfrak{M}[F]$ for $\mathcal{L}$ is a *model of* $\mathsf{Th}(X)$ iff every axiom of $\mathsf{Th}(X)$—i.e., every sentence in $X$—is valid in $\mathfrak{M}[F]$. A set $S$ of formulas of $\mathcal{L}$ *implies* $A$ *in* $\mathsf{Th}(X)$—notation: $S \vDash_X A$—iff $S$ implies $A$ in every model of $\mathsf{Th}(X)$.

Lemma 7.1 *$S \vDash_X A$ if and only if $S + X^{\#} \vDash A$.*

*Proof:* Say $S + X^{\#} \vDash A$ and $\mathfrak{M}[F]$ is a model of $\mathsf{Th}(X)$. Suppose $S$ is satisfied at $i$ under $u$ in $\mathfrak{M}[F]$. Since $X^{\#}$ is valid in $\mathfrak{M}[F]$, $A$ will also be satisfied at $i$ under $u$. Thus $S \vDash_X A$.

Say $S \vDash_X A$ and $S + X^{\#}$ is satisfied at $i$ under $u$ in a structure $\mathfrak{M}[F]$. We form the substructure $\mathfrak{M}'[F']$ of $\mathfrak{M}[F]$ by choosing $I' = \{j \in I: j \geqslant^{\#} i\}$. Since $X^{\#}$ is closed under the $K$-rule, $\mathfrak{M}'[F']$ will be a model of $\mathsf{Th}(X)$. Clearly $S$ is satisfied at $i$ under $u'$ in $\mathfrak{M}'[F']$ and so $A$ is also. But then $A$ is likewise satisfied at $i$ under $u$ in $\mathfrak{M}[F]$, so $S + X^{\#} \vDash A$.

Corollary 7.2 *If $S \vdash_X A$ then $S \vDash_X A$.*

*Proof:* By the soundness theorem plus Lemma 7.1.

If we confine ourselves to $\mathcal{K}$ languages, we are not involved with the complications of individuation, and we can invoke

Theorem 7.3 (The Completeness Theorem for $\mathcal{K}\mathcal{L}$)[5] *If $S$ is a set of formulas of a $\mathcal{K}$ language,*

$$S \vDash A \textit{ implies } S \vdash A.$$

Corollary 7.4 *If $S$ and $X$ are sets of formulas of a $\mathcal{K}$ language,*

$$S \vDash_X A \textit{ implies } S \vdash_X A.$$

*Proof:* By the completeness theorem plus Lemma 7.1.

A set $S$ is *consistent* in a theory $\mathsf{Th}(X)$ iff for no formula $A$ do we have $S \vdash_X A$ & $\sim A$. $S$ is (*simply*) *consistent* iff it is consistent in $\mathsf{Th}(\emptyset)$, the empty theory. $S$ is *complete in* $\mathsf{Th}(X)$ iff for each formula $A$ of the language $\mathcal{L}$, either $S \vdash_X A$ or $S \vdash_X \sim A$.

We now specify a certain $\mathcal{K}$-language $\mathcal{L}_0$, whose sentence symbols are: $P_0, P_1, \ldots$. We shall study a theory $\mathsf{Th}(X_0)$ in $\mathcal{L}_0$, whose axiom set $X_0$ consists of the following sentences:

$X_0 1)$ $P_n \leftrightarrow KP_{n+1}$,

$X_0 2)$ $JKP_n \rightarrow KP_n$,
$X_0 3)$ $JP_n \rightarrow P_n$,

for $n = 0, 1, 2, \ldots$.

Also, we adopt the notation that $\vdash_0$ and $\models_0$ stand for $\vdash_{X_0}$ and $\models_{X_0}$, respectively.

Lemma 7.5 *$P_n$ & $J \sim P_n$ is consistent in* $\mathrm{Th}(X_0)$, *for* $n = 0, 1, 2, \ldots$.

*Proof:* We construct a model $\mathfrak{M}_0$ for $\mathrm{Th}(X_0)$—which we call the *canonical model*—and show that $P_n$ & $J \sim P_n$ is satisfiable in $\mathfrak{M}_0$, for all $n$. We start with a set $I_0 = \{i_0, i_1, \ldots\}$ of distinct state indices and we stipulate that

$$i_n \geqslant i_m \text{ iff } n = m \text{ or } n = m + 1.$$

This gives us a sentential frame $\langle I_0, \geqslant\rangle$. To obtain $\mathfrak{M}_0$ we specify that

$$P_n \text{ is ``true'' at } i_m \text{ iff } m \leqslant n.^6$$

Thus, for fixed $n$, $P_n$ holds at states $i_n, i_{n+1}, \ldots$, and $P_n$ fails at all previous states. This gives us, for each $n$, $\models_{i_n}^{\mathfrak{M}_0} P_n \,\&\, J \sim P_n$. Also, we have

$$\models_{i_n}^{\mathfrak{M}_0} KP_{n+m}, \text{ all } m \geqslant 1, \text{ and } \models_{i_n}^{\mathfrak{M}_0} K \sim P_m, \text{ all } m < n.$$

From these facts it can be readily seen that $\mathfrak{M}_0$ is a model of $\mathrm{Th}(X_0)$.

Lemma 7.6 1) $P_n$ & $J \sim P_n \vdash_0 K^m P_{n+m}$, *for all* $m$,
2) $P_n$ & $J \sim P_n \vdash_0 \sim K^{m+1} P_{n+m}$, *for all* $m$.

*Proof:* By induction on $m$. For $m = 0$ this is trivial. Assume both 1) and 2) for given $m$. Since

$$\vdash_0 K^m P_{n+m} \rightarrow K^{m+1} P_{n+m+1} \text{ and } \vdash_0 K^{m+2} P_{n+m+1} \rightarrow K^{m+1} P_{n+m}$$

both by $X_0 1)$, we readily get 1) and 2) for $m + 1$.

If $\mathfrak{M}$ is a structure for $\mathcal{L}_0$, the (*atomic*) *diagram* of $\mathfrak{M}$ at a state $i$ is the set of atomic sentences and negations of atomic sentences of $\mathcal{L}_0$ which hold at $i$. If $\mathfrak{M}$ is a model of $\mathrm{Th}(X_0)$ and $P_n$ & $J \sim P_n$ holds at $i$, then the diagram of $\mathfrak{M}$ at $i$ is just

$$\sim P_0, \sim P_1, \ldots, \sim P_{n-1}, P_n, P_{n+1}, \ldots.$$

For, by Lemma 7.6, $P_n, P_{n+1}, \ldots$ all hold at $i$ and, by $X_0 1)$, $J \sim P_n \rightarrow \sim P_{n-1}$, $J \sim P_{n-1} \rightarrow \sim P_{n-2}$, etc. are all valid in $\mathfrak{M}$.

Now, $i$ has some alternative state $i(+)$ at which $\sim P_n$ holds. By Lemma 7.6, $P_{n+1}$ will hold at $i(+)$ and, by $X_0 1)$, $J \sim P_{n+1}$ also holds at $i(+)$. By the same reasoning we used on $i$, the diagram of $i(+)$ must be

$$\sim P_0, \sim P_1, \ldots, \sim P_n, P_{n+1}, P_{n+2}, \ldots.$$

We call $i(+)$ a *plus alternative* to $i$. Clearly, we can generate a whole sequence

$$i, i(+), i(+,+), i(+,+,+), \ldots$$

of plus alternatives starting from $i$. An alternative to $i$ which is not a plus alternative will be called a *star alternative*, and we denote such a state by $i(*)$. Any star alternative to $i$ will have the same diagram as $i$. For, since $i(*)$ is not a plus alternative, $P_n$ must hold at $i(*)$. Also, $J \sim P_n \rightarrow KJ \sim P_n$ is valid in $\mathfrak{M}$, by $X_0 2)$, so $J \sim P_n$ holds at $i(*)$ as well. As we saw above, these two sentences determine the diagram.

If $P_n$ & $J \sim P_n$ holds at a state $i$ of a model $\mathfrak{M}$ of $\mathrm{Th}(X_0)$, we define the *index*, $n(i)$, *of* $i$ in $\mathfrak{M}$ to be just $n$. Our above work shows that $n(i)$ is uniquely determined and that, when $n(i)$ exists we have $n(i(+)) = n(i) + 1$ and $n(i(*)) = n(i)$, for all plus and star alternatives to $i$.

Lemma 7.7 *If* $n(i) = n(j)$, *then* $\models_i^{\mathfrak{M}} A$ *if and only if* $\models_j^{\mathfrak{M}} A$.

*Proof:* By induction on $A$. It is true for atomic $A$ since, if $n(i) = n(j)$, then $i$ and $j$ have the same diagram. Suppose it holds for $A$ and $B$. Then it obviously holds also for $\sim A$ and for $A \rightarrow B$. For the $K$-case, note that $KA$ holds at $i$ just in case $A$ holds at $i$, at all $i(+)$ and at all $i(*)$. By induction hypothesis, this is equivalent to $A$ holding at $j$, at all $j(+)$ and at all $j(*)$; i.e., to $KA$ holding at $j$.

Suppose $i_n$ is a state of some model $\mathfrak{M}$ of $\mathrm{Th}(X_0)$, having index $n$. We define states $i_{n+1}, i_{n+2}, \ldots$ of $\mathfrak{M}$ by taking $i_{m+1} = i_m(+)$, for $m = n, n+1, \ldots$. The submodel $\mathfrak{M}_n$ of $\mathfrak{M}$ obtained by setting $I_n = \{i_n, i_{n+1}, \ldots\}$ will be called a *prime submodel* of $\mathfrak{M}$ with *index* $n$. It is easy to see that there is, up to isomorphism, only one prime model of $\mathrm{Th}(X_0)$ with index $n$. This leads us to expect the following result:

Lemma 7.8 *If* $\mathfrak{M}$ *and* $\mathfrak{N}$ *are models of* $\mathrm{Th}(X_0)$ *containing, respectively, states* $i_n$ *and* $j_n$, *both with index* $n$, *then*

$$\models_{i_m}^{\mathfrak{M}_n} A \text{ if and only if } \models_{j_m}^{\mathfrak{N}_n} A, \text{ all } m \geq n.$$

*Proof:* An easy induction on $A$.

Lemma 7.9 *If* $\mathfrak{M}$ *has a prime submodel* $\mathfrak{M}_n$, *then, for all* $m \geq n$,

$$\models_{i_m}^{\mathfrak{M}} A \text{ if and only if } \models_{i_m}^{\mathfrak{M}_n} A.$$

*Proof:* By induction on $A$. We shall do the $K$-case. $KA$ holds at $i_m$ in $\mathfrak{M}$ just in case $A$ holds at $i_m$, at all $i_m(+)$ and at all $i_m(*)$. By Lemma 7.7, this is equivalent to $A$ holding at $i_m$ and at $i_{m+1}$ in $\mathfrak{M}$ which, by induction hypothesis, is equivalent to $A$ holding at these states in $\mathfrak{M}_n$; i.e., to $KA$ holding at $i_m$ in $\mathfrak{M}_n$.

Theorem 7.10 $P_n$ & $J \sim P_n \models_0 A$ *if and only if* $\models_{i_n}^{\mathfrak{M}_0} A$.

*Proof:* If $P_n$ & $J \sim P_n \models_0 A$, then $\models_{i_n}^{\mathfrak{M}_0} A$ because $P_n$ & $J \sim P_n$ holds at $i_n$ in the canonical model $\mathfrak{M}_0$. Suppose $\models_{i_n}^{\mathfrak{M}_0} A$ but that $P_n$ & $J \sim P_n$ does not imply $A$ in $\mathrm{Th}(X_0)$. Then there will be a model $\mathfrak{N}$ of $\mathrm{Th}(X_0)$ which has a state $j_n$ with index $n$, but in which $A$ fails at $j_n$. By Lemma 7.9, $\models_{j_n}^{\mathfrak{N}_n} \sim A$; whence, by Lemma 7.8, $\models_{i_n}^{\mathfrak{M}_n} \sim A$. But then Lemma 7.9 applies again to give us $\models_{i_n}^{\mathfrak{M}_0} \sim A$, contrary to assumption.

Corollary 7.11 *$P_n$ & $J \sim P_n \vdash_0 A$ if and only if* $\models_{i_n}^{\mathfrak{M}_0} A$.

*Proof:* By Corollaries 7.2 and 7.4—the soundness and completeness theorems for **QK**.

Corollary 7.12 *$P_n$ & $J \sim P_n$ is complete in* $\mathrm{Th}(X_0)$.

*Proof:* $\models_{i_n}^{\mathfrak{M}_0} A$ or $\models_{i_n}^{\mathfrak{M}_0} \sim A$, for all $A$.

Corollary 7.13 *If $P_n$ & $J \sim P_n \vdash_0 KA$, then $P_{n+1}$ & $J \sim P_{n+1} \vdash_0 A$.*

*Proof:* If $\models_{i_n}^{\mathfrak{M}_0} KA$ then $\models_{n+1}^{\mathfrak{M}_0} A$.

**8** *Completeness of the axiom system for* **QKL** Let $S$ be a set of formulas of some **QK** language $\mathcal{L}$. We say that $S$ is *maximal in* $\mathcal{L}$ iff for all formulas $A$ of $\mathcal{L}$, either $A \in S$ or $\sim A \in S$. We say that $S$ is *$Q$-saturated in* $\mathcal{L}$ iff for every formula $A$ of $\mathcal{L}$, $(x)A \in S$ whenever $A_x(r) \in S$ for all terms $r$ of $\mathcal{L}$ such that $Q^n r \in S$, where $n = \mathrm{Pr}_x(A)$.[7]

Lemma 8.1 *Suppose $S$ is maximal and consistent in $\mathcal{L}$. Then $S$ is $Q$-saturated in $\mathcal{L}$ if and only if for every formula $A$ of $\mathcal{L}$, if $\exists xA \in S$ there is some term $r$ of $\mathcal{L}$ such that $A_x(r) \in S$ and $Q^n r \in S$, where $n = \mathrm{Pr}_x(A)$.*

*Proof:* We prove the sufficiency of the given condition for $Q$-saturation. Suppose we have a formula $A$ and $A_x(r) \in S$, for all $r$ with $Q^n r \in S$. If $(x)A \notin S$, then $\exists x \sim A \in S$, by the maximal consistency of $S$. By the given condition there is some term $r$ with $\sim A_x(r) \in S$ and $Q^n r \in S$. Then we have both $A_x(r)$ and $\sim A_x(r)$ in $S$, in contradiction to the consistency of $S$. Thus, we must have $(x)A \in S$.

We shall now work with a fixed language $\mathcal{L}$ which has an infinite number of constant symbols. For any set $\Delta$ of constant symbols of $\mathcal{L}$, we shall use $\mathcal{L}(\Delta)$ for the sublanguage of $\mathcal{L}$ whose parameters are the predicate parameters of $\mathcal{L}$ plus the constant symbols in $\Delta$. We denote the set of constant symbols not in $\Delta$ (i.e., the complement of $\Delta$) by $\overline{\Delta}$. For any set $S$ of formulas, we use $\Delta(S)$ for the set of constant symbols of $S$, i.e., that occur in formulas of $S$. We call $S$ a *regular* set iff $\overline{\Delta(S)}$ is infinite, and we call $S$ a *model set* iff it is regular, maximal, consistent, and $Q$-saturated in $\mathcal{L}(\Delta(S))$.[8]

Lemma 8.2 (The Extension Lemma) *If $S$ is regular consistent in $\mathcal{L}$, there is a model set $S'$ extending $S$, i.e., such that $S \subset S'$.*

*Proof:*[9] Since $\overline{\Delta(S)}$ is infinite, we can partition it into two infinite sets, $\Delta_1$ and $\Delta_2$. Take $\Delta' = \Delta(S) \cup \Delta_1$. Now let

$$(A_1, x_1), (A_2, x_2), \ldots$$

be an enumeration of all pairs $(A, x)$, where $A$ is a formula of $\mathcal{L}(\Delta')$ and $x$ is a variable. We define a sequence $B_1, B_2, \ldots$ of formulas by taking $B_m$ to be

$$\exists x_m A_m \rightarrow (A_m)_{\mathfrak{a}}^{x} \mathbin{\&} Q^n \mathfrak{a},$$

where $\mathfrak{a}$ is the first constant symbol in $\Delta_1$—under some fixed well-ordering—which does not occur in $A_m$ or in any $B_k$, for $k < m$, and where $n = \mathrm{Pr}_{x_m}(A_m)$.

Now we take $S_0 = S + \{B_1, B_2, \ldots\}$. We claim that $S_0$ is consistent. If not, there will be a smallest $m \geqslant 0$ such that $S, B_1, \ldots, B_m \vdash \sim B_{m+1}$. Say $B_{m+1}$ is $\exists xA \rightarrow A_x(\mathfrak{a})\ \&\ Q^n\mathfrak{a}$. Then we will have both $S, B_1, \ldots, B_m \vdash \exists xA$ and $S, B_1, \ldots, B_m \vdash Q^n\mathfrak{a} \rightarrow \sim A_x(\mathfrak{a})$. By the choice of $\mathfrak{a}$, it does not occur in $S$, in $B_1, \ldots, B_m$ or in $Q^n x \rightarrow \sim A$. This allows us to apply Lemma 6.6 and generalize on $\mathfrak{a}$ to get $S, B_1, \ldots, B_m \vdash (x)(Q^n x \rightarrow \sim A)$. By Axiom $(x)2$) we then have $S, B_1, \ldots, B_m \vdash (x){\sim}A$ and so $S, B_1, \ldots, B_m \vdash \sim \exists xA$. But this makes $S + \{B_1, \ldots, B_m\}$ inconsistent, in violation of the condition by which $m$ was chosen. We conclude that $S_0$ is consistent.

Now let $C_1, C_2, \ldots$ be an enumeration of the formulas of $\mathcal{L}(\Delta')$. We define sets $S_1, S_2, \ldots$ by recursion as follows:

$$S_{n+1} = \begin{cases} S_n + C_{n+1}, \text{ if } S_n + C_{n+1} \text{ is consistent} \\ \\ S_n + \sim C_{n+1}, \text{ otherwise.} \end{cases}$$

These sets are all consistent. For, suppose that $S_n$ is consistent but $S_{n+1}$ is not, where $n \geqslant 0$. Then $S_{n+1} = S_n + \sim C_{n+1}$ and $S_n \vdash C_{n+1}$. But we also have $S_n \vdash \sim C_{n+1}$, since $S_n + C_{n+1}$ must be inconsistent. Thus, $S_n$ is inconsistent, in contradiction to our assumptions.

Finally, we take $S' = \bigcup_n S_n$. $S'$ is regular because $\overline{\Delta(S')} = \Delta_2$, and $S'$ is clearly consistent and maximal. Suppose $\exists xA \in S'$. Then, for some $m$, $B_m$ is the formula $\exists xA \rightarrow A_x(\mathfrak{a})\ \&\ Q^n\mathfrak{a}$, for some $\mathfrak{a} \in \Delta'$—with $n = \mathrm{Pr}_x(A)$. Since $B_m \in S'$, we have both $A_x(\mathfrak{a}) \in S'$ and $Q^n\mathfrak{a} \in S'$. By Lemma 8.1, $S'$ is $Q$-saturated in $\mathcal{L}(\Delta')$.

If we were to follow the usual completeness constructions, we would now build a $K$-structure using model sets as state indices (*cf.* [4], for example), and define an alternative relation on them by taking

$$T \geqslant S \text{ iff } A \in T \text{ whenever } KA \in S.$$

However, this relation is much too rich for our purposes, because we must also produce an individuating map $F$ for the $K$-structure we build, and this does not seem to be possible with the usual alternative relation. We shall employ the theory $\mathrm{Th}(X_0)$ of section **7** to "thin out" this relation and remove the obstacles to using the "natural" map $F$ (*cf.* our definition below).

We shall now use $i, j, k, \ldots$ to stand for natural numbers. We call the language $\mathcal{L}_0$ of section **7** the *auxiliary language* and we define, for $i = 0, 1, 2, \ldots$, the *auxiliary set* $H_i$ by taking

$$H_i = \{A : P_i\ \&\ J \sim P_i \vdash_0 A\},$$

i.e., $H_i$ is the deductive closure of $P_i\ \&\ J \sim P_i$ in $\mathrm{Th}(X_0)$. By our results in section **7**, we know that $H_i$ is maximal consistent in $\mathcal{L}_0$. Also we have, by Corollary 7.13, Lemma 8.3 *If* $KA \in H_i$ *then* $A \in H_{i+1}$.

For each model set $S$, we define the *indexed model sets*, $S(i)$; $i = 0, 1, 2, \ldots$ by setting $S(i) = S \cup H_i$. $S(i)$ has *index* $i$. Then we take our set $I$ of state indices to be the set of all indexed model sets. We specify a relation $\geqslant$ on $I$ as follows:

$$T(j) \geqslant S(i) \text{ iff } \begin{array}{l} 1)\ A \in T \text{ whenever } KA \in S, \\ 2)\ j = i \text{ or } j = i + 1, \\ 3)\ \text{if } T \neq S \text{ then } j = i + 1. \end{array}$$

Note that $\geqslant$ is reflexive. Also, we can use Lemma 8.3 to extend the condition 1) over $T(j)$ and $S(i)$. That is, we have

Lemma 8.4 *If* $T(j) \geqslant S(i)$, *then* $A \in T(j)$ *whenever* $KA \in S(i)$.

Note: We are assuming that $\mathcal{L}$ and $\mathcal{L}_0$ have no parameters in common.

Lemma 8.5 (The Counter-example Lemma) *If* $KA \notin S(i)$, *then there exists some alternative* $T(j) \geqslant S(i)$ *such that* $\sim A \in T(j)$.

*Proof:* If $\sim A \in S(i)$ we may take $T(j) = S(i)$. Otherwise, $A \in S(i)$.

Case 1 $A \in S$. We set $S^0 = \{B: KB \in S\}$ and observe that $S^0 + \sim A$ is consistent. If it were not, we would have $S^0 \vdash A$ and this would give us $S \vdash KA$, contrary to assumption. Since $S^0 + \sim A$ is also regular, we can apply the extension lemma to get a model set $T$ extending it. Then $T(i + 1)$ serves as the required $T(j)$.

Case 2 $A \in H_i$. We claim that $\sim A \in H_{i+1}$. If not, $P_{i+1}\ \&\ J \sim P_{i+1} \vdash_0 A$, so $P_{i+1}, A \vdash_0 J \sim P_{i+1}$. Now $P_{i+1} \in H_i$ and (by assumption) $A \in H_i$, so $H_i \vdash_0 J \sim P_{i+1}$. But $H_i \vdash KP_{i+1}$ (*cf.* Lemma 7.6), so we have a contradiction. We can now take $S(i + 1)$ as the required $T(j)$.

Corollary 8.6 $K^n A \in S(i)$ *if and only if for all* $T(j) \geqslant^n S(i)$, $A \in T(j)$.

*Proof:* By induction on $n$.

Suppose that $T(j) \geqslant^{\#} S(i)$. We define the *rank*, $\mathrm{rank}(T(j)/S(i))$, *of* $T(j)$ *over* $S(i)$ to be the least $n$ such that $T(j) \geqslant^n S(i)$.

Lemma 8.7 *If* $U(k) \neq S(i)$ *and* $T(j) \geqslant^{\#} U(k) \geqslant S(i)$, *then* $\mathrm{rank}(T(j)/S(i)) = \mathrm{rank}(T(j)/U(k)) + 1$.

*Proof:* Let $\mathrm{rank}(T(j)/U(k)) = m$. Clearly, $T(j) \geqslant^{m+1} S(i)$. Suppose $T(j) \geqslant^n S(i)$. Then we want to show that $n \geqslant m + 1$. By the choice of $m$, one has to use $m$ indexed model sets to get from $U(k)$ to $T(j)$. From the definition of $\geqslant$, we see that we therefore have $j = k + m$. Since $U(k) \neq S(i)$, we also have $k = i + 1$. Hence $j = i + m + 1$. From Lemma 7.6 we have $K^m P_{i+m} \in S(i)$. Now, if $n \leqslant m$ we would also have $K^n P_{i+m} \in S(i)$ and thus, by Corollary 8.6, $P_{i+m} \in T(j)$. Then $KP_j \in T(j)$, by Axiom $X_0 1)$. This contradicts $J \sim P_j \in T(j)$.

It follows from Lemma 8.7 that all paths (without repetitions) from $S(i)$ to $T(j)$, through the relation $\geqslant$, are of equal length; namely of length

$\mathrm{rank}(T(j)/S(i))$. In particular, $\geqslant$ never "loops back" on itself—it is strictly intransitive.

We now construct a $K$-structure $\mathfrak{M}$ over the frame $\langle I, \geqslant\rangle$. For any term $r$ and model set $S$ of , we define the *connotation of r at S* to be

$$[r]_S = \{t: r = t \in S\}.$$

We shall use $\Pi(S)$ for the set of terms appearing in $S$. It is easy to see that $[r]_S = \emptyset$ if and only if $r \notin \Pi(S)$ and that, when $r, t \in \Pi(S)$ we have $[r]_S = [t]_S$ if and only if $r = t \in S$. We define the *domain* $M_S$ *at* $S$ by taking

$$M_S = \{[r]_S: Q^0 r \in S\}.$$

Note that $M_S \neq \emptyset$, since $Q^0 x \in S$, for all variables $x$. We shall use $r^S$ for $[r]_S$ when $Q^0 r \in S$. For any constant symbol $\mathfrak{a}$ of $\mathcal{L}$ we define the *denotation of* $\mathfrak{a}$ *at* $S$ to be $\mathfrak{a}_S^{\mathfrak{M}} = \mathfrak{a}^S$, whenever $\mathfrak{a}^S$ exists. Also, if $P$ is an $n$-place predicate parameter of $\mathcal{L}$, we define the *extension of P at S* to be the $n$-ary relation $P_S^{\mathfrak{M}}$, defined on $M_S$ by

$$(r_1^S, \ldots, r_n^S) \in P_S^{\mathfrak{M}} \textit{ iff } Pr_1 \ldots r_n \in S.$$

To see that $P_S^{\mathfrak{M}}$ is well-defined, suppose that $r_p^S = t_p^S$, for $p = 1, 2, \ldots, n$. Since $r_p, t_p \in \Pi(S)$, we have $r_p = t_p \in S$, for $p = 1, 2, \ldots, n$. By an equality axiom we get $Pr_1 \ldots r_n \in S$ if and only if $Pt_1 \ldots t_n \in S$.

Now for each $S(i) \in I$, we set $M_{S(i)} = M_S$, $\mathfrak{a}_{S(i)}^{\mathfrak{M}} = \mathfrak{a}_S^{\mathfrak{M}}$ (whenever it exists), and $P_{S(i)}^{\mathfrak{M}} = P_S^{\mathfrak{M}}$. This defines a $K$-structure $\mathfrak{M}$ over $\langle I, \geqslant\rangle$.

In order to interpret $\mathcal{L}$ in $\mathfrak{M}$ we need to produce a satisfaction predicate. Suppose $u$ is an assignment to variables in $\mathfrak{M}$. For each $S$, we extend $u_S (= u_{S(i)})$ to a function $u_S^*$ defined on all terms of $\mathcal{L}$, by simply taking $u_S^*(\mathfrak{a}) = [\mathfrak{a}]_S$, for all $\mathfrak{a}$. Then, for each term $r$ there will be some term $r^*$ such that $u_S^*(r) = [r^*]_S$. Note that $u_S(r)$ exists just in case $Q^0 r^* \in S$.

We define satisfaction for equality formulas by taking

$$\models_{S(i)}^{\mathfrak{M}} r = t[u] \text{ iff } u_S^*(r) = u_S^*(t).$$

We need to check that this definition meets the requirements of condition 1), section **4**, on satisfaction predicates. We shall do so for 1b). Suppose $\models_{S(i)}^{\mathfrak{M}} r = t[u]$. Then $[r^*]_S = [t^*]_S$. If $Q^0 r^* \in S$, then $[r^*]_S \neq \emptyset$ so $r^*, t^* \in \Pi(S)$ and we have $r^* = t^* \in S$. By one of our $Q$-axioms, $Q^0 t^* \in S$. Thus, if $u_S(r)$ exists, so does $u_S(t)$. Parts 1a) and 1c) are immediate.

We define satisfaction for atomic parameter formulas of $\mathcal{L}$ by taking

$$\models_{S(i)}^{\mathfrak{M}} Pr_1 \ldots r_n[u] \text{ iff } Pr_1^* \ldots r_n^* \in S.$$

It is easy enough to see that this is a proper definition. In so doing, one also verifies condition 2a), section **4**, for $\models^{\mathfrak{M}}$. Part 2b) is immediate from the definition of $P_S^{\mathfrak{M}}$. Since our definition "extracts" from $u$ just the values assigned to the free variables of whatever (basic) formula is being interpreted, condition 3) is also met.

We next define the identifiers $F_{T(j), S(i)}$, when $T(j) \geqslant^{\#} S(i)$. We take

$$F_{T(j), S(i)}(r^S) = r^T,$$

whenever $Q^n r \in S$, where $n = \mathrm{rank}(T(j)/S(i))$. There are certain things to check in order to see that this definition is proper. For one thing, we want to know that $r^T$ exists when $T(j) \geqslant^n S(i)$ and $Q^n r \in S$. By a $Q$-axiom we get $K^n Q^0 r \in S$ from the latter; whence $Q^0 r \in T$, by the former. We must also check that the value $r^T$ is uniquely determined. Suppose $r^S = t^S$ with $Q^n r$, $Q^n t \in S$. Then $r = t \in S$ and we can apply a $Q$-axiom to get $K^n(r = t) \in S$. If $T(j) \geqslant^n S(i)$, we then have $r = t \in T$, as required.

Lemma 8.8 $M_S^n = \{r^S : Q^n r \in S\}$.

*Proof:* Suppose $Q^n r \in S$. Then, if $T(j) \geqslant^n S(i)$, $r^S \in \mathrm{domain}\ F_{T(j),S(i)}$, by definition. Thus $r^S \in M_S^n$.

Suppose $r^S \in M_S^n$. Then, for each $T(j) \geqslant^n S(i)$, we have $Q^m r \in S$, where $m = \mathrm{rank}(T(j)/S(i))$. We take

$$m_0 = \max\{\mathrm{rank}(T(j)/S(i)) : T(j) \geqslant^n S(i)\}.$$

We want to show that $m_0 = n$. From our choice of $m_0$ we have

$$T(j) \geqslant^{m_0} S(i), \text{ whenever } T(j) \geqslant^n S(i).$$

It follows by Corollary 8.6 that

$$(*) \qquad K^n A \in S(i), \text{ whenever } K^{m_0} A \in S(i),$$

for all formulas $A$ (of $\mathcal{L}$ or $\mathcal{L}_0$). By Lemma 7.6, $K^{m_0} P_{i+m_0} \in S(i)$ and $\sim K^{m_0+1} P_{i+m_0} \in S(i)$. From this we see that we cannot have $n > m_0$, because $K^n P_{i+m_0} \in S(i)$, by (*). Thus $m_0 = n$ and we have $Q^n r \in S$.

We are now in a position to show that the map $F$ satisfies the conditions 1)-3) of section **3**, to be an individuating map for $\mathfrak{M}$. $F_{S(i),S(i)}(r^S) = r^S$, since $Q^0 r \in S$, for all $r^S \in M_S$. Also, $M_S^n \neq \emptyset$, because $\exists x\, Q^n x \in S$, by a $Q$-axiom. Thus there is some $\mathfrak{a}$ with $Q^n \mathfrak{a} \in S$—since $S$ is $Q$-saturated. By Lemma 8.8, $\mathfrak{a}^S \in M_S^n$. For condition 2), suppose $T(j) \geqslant^{\#} U(k) \geqslant S(i)$. Let $n = \mathrm{rank}(T(j)/S(i))$ and suppose $Q^n r \in S$. If $m = \mathrm{rank}(T(j)/U(k))$ and $p = \mathrm{rank}(U(k)/S(i))$, we need to show that $Q^m r \in U$ and $Q^p r \in S$. Obviously, $p = 0$ or $p = 1$. If $p = 0$, then $m = n$ and we are done. If $p = 1$, then $U(k) \neq S(i)$ and, by Lemma 8.7, $n = m + 1$. Then $Q^{m+1} r \in S$, so $Q^m r \in U$ and $Q^1 r \in S$, as required.

For a given model set $S$, we can readily obtain a constant symbol $\mathfrak{a} \notin \Pi(S)$. Then $\mathfrak{a} = \mathfrak{a}$ holds at $S(i)$ in $\mathfrak{M}[F]$, but $\mathfrak{a} = \mathfrak{a} \notin S$. This shows that the converse of what we call the Basic Lemma—Lemma 8.11, below—fails for $\mathfrak{M}[F]$. Thus, we do not have an equivalence between membership in $S$ and satisfaction in $\mathfrak{M}[F]$. Because of this we shall have to use an unusual kind of induction—we call it *semantical induction*—in proving Lemma 8.11.[10] A set $W$ of formulas of $\mathcal{L}$ is *semantically inductive* iff

1) $A$ and $\sim A$ belong to $W$, for all atomic $A$.
2) If $\sim A$ and $B$ belong to $W$, then so does $A \to B$.
3) If $A$ and $\sim B$ belong to $W$, then so does $\sim(A \to B)$.
4) If $A$ belongs to $W$, then so do $KA$ and $JA$.
5) If $A_x(r)$ belongs to $W$, for all terms $r$ of $\mathcal{L}$, then so do $(x)A$ and $\exists x A$.
6) If $\models A \leftrightarrow B$, then $A$ belongs to $W$ if and only if $B$ does.

Let $W^*$ be the smallest inductive set, i.e., the intersection of all inductive sets. Note that if, for instance, $\sim(A \rightarrow B) \in W^*$, then we must have $A, \sim B \in W^*$, since the only way $\sim(A \rightarrow B)$ can get into $W^*$ is via condition 2). Analogous considerations apply to each of the other conditions.

Lemma 8.9 *$A \in W^*$ if and only if $\sim A \in W^*$.*

*Proof:* By induction on the length of $A$, where length is taken as the number of occurrences of logical symbols. For atomic $A$, we have the lemma by condition 1), above. Suppose it holds for $A$ and $B$. Since $\sim\sim A$ is logically equivalent to $A$, it will then hold for $\sim A$ by condition 6). Since $A \rightarrow B \in W^*$ is equivalent to $\sim A$, $B \in W^*$ and $\sim(A \rightarrow B) \in W^*$ is equivalent to $A$, $\sim B \in W^*$, it also holds for $A \rightarrow B$. Also, $KA \in W^*$ is equivalent to $A \in W^*$ and $\sim KA \in W^*$ is equivalent—by condition 6)—to $J \sim A \in W^*$. Since the latter is equivalent to $\sim A \in W^*$, we have the lemma for $KA$. Finally, suppose the lemma holds for all "instances" $A_x(r)$ of $A$. $(x)A \in W^*$ just in case all these instances belong to $W^*$. Also, $\sim(x)A \in W^*$ just in case $\exists x \sim A \in W^*$, and the latter holds just in case $\sim A_x(r) \in W^*$, for all $r$. From this we see that the lemma holds for $(x)A$.

Corollary 8.10 *$W^*$ contains all formulas of $\mathcal{L}$.*

*Proof:* By a routine induction on the length of formulas.

Let $u^0$ be the assignment to variables in $\mathfrak{M}$ defined by taking $u^0_S(x) = x^S$, for all variables $x$. We call $u^0$ the *canonical assignment* in $\mathfrak{M}$. Note that $u^0_S(r) = r^S$, whenever $r^S$ exists. Moreover, if $Q^n r \in S$ then $r$ is $n$-rigid under $u^0$ at $S(i)$. For we have $r^S \in M^n_S$, by Lemma 8.8, and for $T(j) \geqslant^n S(i)$,

$$r^S_T = F_{T(j),S(i)}(u^0_S(r)) = F_{T(j),S(i)}(r^S) = r^T = u^0_T(r).$$

Lemma 8.11 (The Basic Lemma) *If $A \in S$ then $\vDash^{\mathfrak{M}|F}_{S(i)} A[u_0]$.*

*Proof:* By semantical induction. We let $W$ be the set of formulas $A$ such that the lemma holds for $A$ and all $S$. Then we show that $W$ is semantically inductive. If $A$ is atomic then $A$ and $\sim A$ belong to $W$, by the definition of $\vDash^{\mathfrak{M}}$. Suppose $\sim A$, $B \in W$. If $A \rightarrow B \in S$ then $\sim A \in S$ or $B \in S$, by the maximal consistency of $S$. Then $\vDash_{S(i)} \sim A[u^0]$ or $\vDash_{S(i)} B[u^0]$, by induction assumption, so $\vDash_{S(i)}(A \rightarrow B)[u^0]$. Thus $A \rightarrow B \in W$ and we have condition 2) for $W$. The other sentential conditions are similar.

Suppose $A_x(r) \in W$, for all $r$. If $(x)A \in S$ then $A_x(r) \in S$ whenever $Q^n r \in S$, where $n = \mathrm{Pr}_x(A)$. By induction hypothesis, $\vDash_{S(i)} A_x(r)[u^0]$, for all $r$ with $Q^n r \in S$. Since every such $r$ is $n$-rigid at $S(i)$ under $u^0$, we can apply Lemma 5.2 to get

$$\vDash_{S(i)} A[u^0(x|r^S)], \text{ for all } r^S \in M^n_S$$

which gives us $\vDash_{S(i)}(x)A[u^0]$. In the case where $\exists x A \in S$, we have $A_x(r) \in S$ for some $r$ with $Q^n r \in S$. By an argument similar to the one just given, we get $\vDash_{S(i)} \exists x A[u^0]$.

Lemma 8.12 (The Satisfaction Lemma) *If $S$ is a consistent set of formulas in any $QK$-language $\mathcal{L}'$, then $S$ is satisfiable.*

*Proof:* If we form a language $\mathcal{L}$ from $\mathcal{L}'$ by adjoining infinitely many new constant symbols, then it is routine to check that $S$ will still be consistent in $\mathcal{L}$. Moreover, it is regular in $\mathcal{L}$ so we can apply the extension lemma to obtain a model set $S'$ extending it. By the basic lemma, $S'$ is satisfiable at any $S'(i)$ under $u^0$, so $S$ is also.

Theorem 8.13 (The Completeness Theorem) *If* $S \vDash A$ *then* $S \vdash A$.

*Proof:* If $S \vdash A$ fails, then $S + \sim A$ is consistent and, so, satisfiable. Consequently, $S \vDash A$ fails.

## NOTES

1. In [2] Hintikka describes two general types of cross-identification—perceptual and physical. He has not, however, paid much attention to the formal properties that might be extracted from either of these.
2. In this connection, we emphasize that our conditions on identifiers are formal, generic ones, and are not meant to be descriptive of any methods of individuation.
3. We employ set-theoretical semantics, not the descriptive semantics of model systems used by Hintikka. We do not see any important theoretical advantage in using descriptive semantics, and we find it technically unwieldy.
4. We could, for instance, stipulate that no (basic) formula is to be satisfiable if it contains a non-denoting term. This particular interpretation is used in [6].
5. The completeness of $K\mathcal{L}$ can be obtained by the usual modeling techniques—as in [4], for example—in modal logic. Since our modeling in section **8** will be an elaboration of these techniques, one can readily extract a completeness proof for $K\mathcal{L}$ from our work there.
6. To be correct, we should frame the definition as follows:

$$(P_n)_{i_m}^{\mathfrak{M}_0} = \begin{cases} 1, \text{ if } m \leqslant n \\ 0, \text{ otherwise.} \end{cases}$$

   Note that there is only one satisfaction predicate for $\mathcal{L}_0$ in $\mathfrak{M}_0$, and that we have

$$\vDash_{i_m}^{\mathfrak{M}_0} P_n \text{ iff } m \leqslant n.$$

7. In this section we shall take $A_x(r)$ to be the result of making *free substitutions* of $r$ for $x$ in $A$. That is, we use an alphabetic variant of $A$ in place of $A$, where necessary in order to ensure that $r$ is free for $x$. By our results (6.4 and 6.5) on alphabetic variants, we can do this modulo logical equivalence.
8. Our use of the term 'model set' differs from Hintikka's (in [7], for example) in that Hintikka does not require his model sets to be maximal.
9. Our proof of the extension lemma is a straightforward adaptation of the one for ordinary first-order logic in [8].
10. A similar kind of induction is used in [4], for the same reason.

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