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A SHORT EQUATIONAL AXIOMATIZATION OF MODULAR ORTHOLATTICES

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By definition, *cf.* e.g., [2], p. 52, a modular ortholattice is an ortholattice satisfying the following formula:[1]

F1 $[abc]:a,\ b,\ c \in A\ .\ a \leqslant c\ .\supset.\ a \cup (b \cap c) = (a \cup b) \cap c$

In this note it will be proved that:

(**A**) *Any algebraic system*

$$\mathfrak{A} = \langle A, \cup, \cap, {}^{\perp}\rangle$$

where $\cup$ *and* $\cap$ *are two binary operations and* $^{\perp}$ *is a unary operation defined on the carrier set A, is a modular ortholattice, if it satisfies the following three mutually independent postulates:*

A1 $[abcd]:a,\ b,\ c,\ d \in A\ .\supset.\ ((a \cap b) \cup (a \cap c)) \cup (d \cap d^{\perp}) = ((c \cap a) \cup b) \cap a$
A2 $[ab]:a,\ b \in A\ .\supset.\ a = (b \cup a) \cap a$
A3 $[abc]:a,\ b,\ c \in A\ .\supset.\ (a \cup b) \cup c = a \cup (b^{\perp} \cap c^{\perp})^{\perp}$[2]

Remark I: It is easy to prove that, in the field of any lattice formula, *F1* is inferentially equivalent to formula

B1 $[abc]:a,\ b,\ c \in A\ .\supset.\ (a \cap b) \cup (a \cap c) = ((c \cap a) \cup b) \cap a$
[*Cf.* *A9* in section **2.2** below]

Concerning this equivalence, *cf.* [3], [4], and [5]. The shortest postulate-system of modular lattices known up to now was obtained by J. Ričan who proved in [4] that any algebraic system which satisfies *B1* and

B2 $[abc]:a,\ b,\ c \in A\ .\supset.\ a = (c \cup (b \cup a)) \cap a$

1. Throughout this paper *A* indicates an arbitrary but fixed carrier set. The so-called closure axioms are assumed tacitly.

2. Of course, in this postulate-system, the operations $\cup$, $\cap$ and $\perp$ are not mutually independent.

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is a modular lattice. It remains an open problem whether, in Ričan's axiom-system, *B2* can be substituted by *A2* given above. In section **2.2** below it will be shown that the axiom system $\{A9\ (B1);\ A2\}$ generates rather a strong algebraic system although, probably, not a modular lattice.

Proof of (**A**):

1 Since in the field of any lattice $\{F1\} \rightleftarrows \{B1\}$, it is obvious that in the field of any ortholattice $\{F1\} \rightleftarrows \{A9\} \rightleftarrows \{A1\}$. Hence, clearly, the postulates *A1*, *A2*, and *A3* given in (**A**) are the theses of any modular ortholattice.

2 In this section it will be shown that axioms *A1*, *A2*, and *A3* imply the theses which are needed in order to prove that the system under investigation is a modular ortholattice. The deductions presented here will be divided into three parts. In the first it will be established that $\{A1;\ A2;\ A3\}$ imply

$$A8 \quad [ab] : a,\ b \in A\ .\supset.\ a = a \cup (b \cap b^{\perp})$$

and, therefore, also *A9*. In the second part it will be proved that the system $\{A9;\ A2\}$ is a latticoid in which, probably, the associative laws for $\cup$ and for $\cap$ do not hold. (At least, I am unable to obtain them.) Finally, in the third part, using the previously obtained results, it will be proved that in the field of $\{A1;\ A2;\ A3\}$ the needed formula *A23* holds.

2.1 Let us assume *A1*, *A2*, and *A3*. Then:

$A4 \quad [acd] : a,\ c,\ d \in A\ .\supset.\ a = ((a \cap a) \cup (a \cap c)) \cup (d \cap d^{\perp})$

PR $[acd] : \mathrm{Hp}\ (1)\ .\supset.$

$a = ((c \cap a) \cup a) \cap a = ((a \cap a) \cup (a \cap c)) \cup (d \cap d^{\perp})$ $\quad [1;\ A2,\ b/c \cap a;\ A1,\ b/a]$

$A5 \quad [ad] : a,\ d \in A\ .\supset.\ a \cap (d \cap d^{\perp}) = d \cap d^{\perp}$

PR $[ad] : \mathrm{Hp}\ (1)\ .\supset.$

$a \cap (d \cap d^{\perp}) = (((a \cap a) \cup (a \cap a)) \cup (d \cap d^{\perp})) \cap (d \cap d^{\perp})$ $\quad [1;\ A4,\ c/a]$

$= d \cap d^{\perp}$ $\quad [A2,\ a/d \cap d^{\perp},\ b/(a \cap a) \cup (a \cap a)]$

$A6 \quad [ad] : a,\ d \in A\ .\supset.\ a = (a \cap a) \cup ((d \cap d^{\perp})^{\perp} \cap (d \cap d^{\perp})^{\perp})^{\perp}$

PR $[ad] : \mathrm{Hp}\ (1)\ .\supset.$

$a = ((a \cap a) \cup (a \cap (d \cap d^{\perp}))) \cup (d \cap d^{\perp})$ $\quad [1;\ A4,\ c/d \cap d^{\perp}]$

$= ((a \cap a) \cup (d \cap d^{\perp})) \cup (d \cap d^{\perp})$ $\quad [A5]$

$= (a \cap a) \cup ((d \cap d^{\perp})^{\perp} \cap (d \cap d^{\perp})^{\perp})^{\perp}$ $\quad [A3,\ a/a \cap a,\ b/d \cap d^{\perp},\ c/d \cap d^{\perp}]$

$A7 \quad [ad] : a,\ d \in A\ .\supset.\ a \cap ((d \cap d^{\perp})^{\perp} \cap (d \cap d^{\perp})^{\perp})^{\perp} = ((d \cap d^{\perp})^{\perp} \cap (d \cap d^{\perp})^{\perp})^{\perp}$

PR $[ad] : \mathrm{Hp}\ (1)\ .\supset.$

$a \cap ((d \cap d^{\perp})^{\perp} \cap (d \cap d^{\perp})^{\perp})^{\perp}$

$= ((a \cap a) \cup ((d \cap d^{\perp})^{\perp} \cap (d \cap d^{\perp})^{\perp})^{\perp}) \cap ((d \cap d^{\perp})^{\perp} \cap (d \cap d^{\perp})^{\perp})^{\perp}$ $\quad [1;\ A6]$

$= ((d \cap d^{\perp})^{\perp} \cap (d \cap d^{\perp})^{\perp})^{\perp}$ $\quad [A2,\ a/((d \cap d^{\perp})^{\perp} \cap (d \cap d^{\perp})^{\perp})^{\perp},\ b/a \cap a]$

$A8 \quad [ab] : a,\ b \in A\ .\supset.\ a = a \cup (b \cap b^{\perp})$

PR $[ab] : \mathrm{Hp}\ (1)\ .\supset.$

$a = ((a \cap a) \cup (a \cap ((b \cap b^{\perp})^{\perp} \cap (b \cap b^{\perp})^{\perp})^{\perp})) \cup (b \cap b^{\perp})$

$[1;\ A4,\ c/((b \cap b^{\perp})^{\perp} \cap (b \cap b^{\perp})^{\perp})^{\perp},\ d/b]$

$= ((a \cap a) \cup ((b \cap b^{\perp})^{\perp} \cap (b \cap b^{\perp})^{\perp})^{\perp}) \cup (b \cap b^{\perp})$ $\quad [A7,\ d/b]$

$= a \cup (b \cap b^{\perp})$ $\quad [A6,\ d/b]$

A9 $[abc]: a, b, c \epsilon A .\supset. (a \cap b) \cup (a \cap c) = ((c \cap a) \cup b) \cap a$

PR $[ab]:$ Hp (1) $.\supset.$

$(a \cap b) \cup (a \cap c) = ((a \cap b) \cup (a \cap c)) \cup (b \cap b^{\perp})$ [1; *A8*, $a/(a \cap b) \cup (a \cap c)$]

$= ((c \cap a) \cup b) \cap a$ [*A1*, d/b]

Thus, $\{A1; A2; A3\} \rightarrow \{A9; A8\}$.

2.2 Now, let us assume only *A9* and *A2*. Then:

A10 $[ac]: a, c \epsilon A .\supset. a = (a \cap a) \cup (a \cap c)$

PR $[ac]:$ Hp (1) $.\supset.$

$a = ((c \cap a) \cup a) \cap a = (a \cap a) \cup (a \cap c)$ [1; *A2*, $b/c \cap a$; *A9*, b/a]

A11 $[ac]: a, c \epsilon A .\supset. a \cap a = (a \cap (a \cap c)) \cup (a \cap a)$

PR $[ac]:$ Hp (1) $.\supset.$

$a \cap a = ((a \cap a) \cup (a \cap c)) \cap a = (a \cap (a \cap c)) \cup (a \cap a)$ [1; *A10*; *A9*, $b/a \cap c$, c/a]

A12 $[a]: a \epsilon A .\supset. a \cap a = a \cap (a \cap a)$

PR $[a]:$ Hp (1) $.\supset.$

$a \cap a = ((a \cap a) \cup (a \cap a)) \cap (a \cap a) = a \cap (a \cap a)$ [1; *A2*, $a/a \cap a$, $b/a \cap a$; *A10*, c/a]

A13 $[a]: a \epsilon A .\supset. a = a \cap a$

PR $[a]:$ Hp (1) $.\supset.$

$a = (a \cap a) \cup (a \cap a) = (a \cap (a \cap a)) \cup (a \cap a) = a \cap a$ [1; *A10*, c/a; *A12*; *A11*, c/a]

A14 $[a]: a \epsilon A .\supset. a = a \cup a$

PR $[a]:$ Hp (1) $.\supset.$

$a = (a \cap a) \cup (a \cap a) = a \cup a$ [1; *A10*, c/a; *A13*; *A13*]

A15 $[ab]: a, b \epsilon A .\supset. a = a \cup (a \cap b)$

PR $[ab]:$ Hp (1) $.\supset.$

$a = (a \cap a) \cup (a \cap b) = a \cup (a \cap b)$ [1; *A10*, c/b; *A13*]

A16 $[ab]: a, b \epsilon A .\supset. a \cap b = a \cap (a \cap b)$

PR $[ab]:$ Hp (1) $.\supset.$

$a \cap b = (a \cup (a \cap b)) \cap (a \cap b) = a \cap (a \cap b)$ [1; *A2*, $a/a \cap b$, b/a; *A15*]

A17 $[ab]: a, b \epsilon A .\supset. (a \cap b) \cup a = (a \cup b) \cap a$

PR $[ab]:$ Hp (1) $.\supset.$

$(a \cap b) \cup a = (a \cap b) \cup (a \cap a) = ((a \cap a) \cup b) \cap a = (a \cup b) \cap a$ [1; *A13*; *A9*, c/a; *A13*]

A18 $[ab]: a, b \epsilon A .\supset. a = (a \cap b) \cup a$

PR $[ab]:$ Hp (1) $.\supset.$

$a = a \cap a = (a \cup (a \cap b)) \cap a = ((a \cap a) \cup (a \cap b)) \cap a$ [1; *A13*; *A15*; *A13*]

$= (a \cap (a \cap b)) \cup (a \cap a) = (a \cap b) \cup a$ [*A9*, $b/a \cap b$, c/a; *A16*; *A13*]

A19 $[ab]: a, b \epsilon A .\supset. a = (a \cup b) \cap a$ [*A18*; *A17*]

A20 $[ab]: a, b \epsilon A .\supset. a = a \cap (a \cup b)$

PR $[ab]:$ Hp (1) $.\supset.$

$a = (a \cup (a \cup b)) \cap a = (((a \cup b) \cap a) \cup (a \cup b)) \cap a$ [1; *A19*, $b/a \cup b$; *A19*]

$= (a \cap (a \cup b)) \cup (a \cap (a \cup b)) = a \cap (a \cup b)$ [*A9*, $b/a \cup b$, $c/a \cup b$; *A14*, $a/a \cap (a \cup b)$]

A21 $[ab]:a, b \epsilon A .\supset. a \cup b = b \cup a$
PR $[ab]:\mathrm{Hp}\ (1) .\supset.$
$a \cup b = ((b \cup a) \cap a) \cup ((b \cup a) \cap b) = ((b \cap (b \cup a)) \cup a) \cap (b \cup a)$
[1; *A2*; *A18*, a/b, b/a; *A9*, $a/b \cup a$, b/a, c/b]
$= (b \cup a) \cap (b \cup a) = b \cup a$ [*A20*, a/b, b/a; *A13*, $a/b \cup a$]
A22 $[ab]:a, b \epsilon A .\supset. a \cap b = b \cap a$
PR $[ab]:\mathrm{Hp}\ (1) .\supset.$
$a \cap b = (a \cap b) \cup (a \cap b) = ((b \cap a) \cup b) \cap a = b \cap a$
[1; *A14*, $a/a \cap b$; *A9*, c/b; *A18*, a/b, b/a]

Thus, since the theses *A9* and *A2* imply *A14*, *A13*, *A15*, *A20*, *A21*, and *A22*, it is proved that system {*A9*; *A2*} is a latticoid in which, probably, the associative laws for ∪ and ∩ do not hold.

2.3 Now, assume only *A3*, *A21*, and *A22*. Then:

A23 $[abc]:a, b, c \epsilon A .\supset. (a \cup b) \cup c = (c^{\perp} \cap b^{\perp})^{\perp} \cup a$
PR $[abc]:\mathrm{Hp}\ (1) .\supset.$
$(a \cup b) \cup c = a \cup (b^{\perp} \cap c^{\perp})^{\perp} = (b^{\perp} \cap c^{\perp})^{\perp} \cup a = (c^{\perp} \cap b^{\perp})^{\perp} \cup a$
[1; *A3*; *A21*, $b/(b^{\perp} \cap c^{\perp})^{\perp}$; *A22*, $a/b^{\perp}$, $b/c^{\perp}$]

Thus, it follows from sections **2.1**, **2.2**, and **2.3** that {*A1*; *A2*; *A3*} → {*A23*; *A20*; *A8*}.

3 Since, on the basis of deductions presented in [6], L. Beran has proved in [1] that any algebraic system which satisfies theses *A23*, *A20*, and *A8* is an ortholattice, it follows from Remark I and sections **1** and **2** that any algebraic system which satisfies postulates *A1*, *A2*, and *A3* is a modular ortholattice.

4 The mutual independence of axioms *A1*, *A2*, and *A3* is established by using the following algebraic tables:[3]

𝔐1

∪	α	β
α	α	α
β	α	α

∩	α	β
α	α	β
β	α	β

x	$x^{\perp}$
α	β
β	α

𝔐2

∪	α	β
α	α	α
β	α	α

∩	α	β
α	α	α
β	α	β

x	$x^{\perp}$
α	α
β	α

𝔐3

∪	α	β	γ
α	α	β	α
β	β	β	β
γ	α	β	γ

∩	α	β	γ
α	α	α	γ
β	α	β	γ
γ	γ	γ	γ

x	$x^{\perp}$
α	γ
β	γ
γ	γ

3. Concerning 𝔐1 and 𝔐2 *cf.* [3], pp. 385-386, and [5], p. 85. Table 𝔐3 is given in [6], p. 143, as table 𝔐4.

Namely:

(a) $\mathfrak{M}1$ verifies *A2* and *A3*, but falsifies *A1* for a/β, b/α, c/α, and d/α: (i) $((\beta \cap \alpha) \cup (\beta \cap \alpha)) \cup (\alpha \cap \alpha^{\perp}) = (\alpha \cup \alpha) \cup (\alpha \cap \beta) = \alpha \cup \beta = \alpha$, (ii) $((\alpha \cap \beta) \cup \alpha) \cap \beta = (\beta \cup \alpha) \cap \beta = \alpha \cap \beta = \beta$.
(b) $\mathfrak{M}2$ verifies *A1* and *A3*, but falsifies *A2* for a/β and b/α: (i) $\beta = \beta$, (ii) $(\alpha \cup \beta) \cap \beta = \alpha \cap \beta = \alpha$.
(c) $\mathfrak{M}3$ verifies *A1* and *A2*, but falsifies *A3* for a/γ, b/α, and c/α: (i) $(\gamma \cup \alpha) \cup \alpha = \alpha \cup \alpha = \alpha$, (ii) $\gamma \cup (\alpha^{\perp} \cap \alpha^{\perp})^{\perp} = \gamma \cup (\gamma \cup \gamma)^{\perp} = \gamma \cup \gamma^{\perp} = \gamma \cup \gamma = \gamma$.

5 It follows immediately from sections **3** and **4** that the proof of (**A**) is complete.

Remark II: It seems to me that the open problem mentioned in Remark I is rather difficult. We have to note that an addition of one of the associative laws, i.e., either the formula:

P1 $[abc]: a, b, c \in A .\supset. (a \cup b) \cup c = a \cup (b \cup c)$

or the formula:

R1 $[abc]: a, b, c \in A .\supset. (a \cap b) \cap c = a \cap (b \cap c)$

as a new axiom to the system $\{A9; A2\}$ generates a modular lattice. Namely:

(α) Assume *A9, A2,* and *P1*. Then:

B2 $[abc]: a, b, c \in A .\supset. a = (c \cup (b \cup a)) \cap a$
PR $[abc]$: Hp (1) $.\supset.$
$a = ((c \cup b) \cup a) \cap a = (c \cup (b \cup a)) \cap a$ $[1; A2, b/c \cup b; P1, a/c, c/a]$

Thus, the addition of *P1* as a new axiom to $\{A9; A2\}$ generates Ričan's postulate-system for modular lattices, *cf.* [4] and Remark I above.

(β) Assume *A9, A2,* and *R1*. Then, we have also *A13* and *A22, cf.* section **2.2** above. Hence:

K1 $[abcd]: a, b, c, d \in A .\supset. ((a \cap b) \cap c) \cup (a \cap d) = ((d \cap a) \cup (c \cap b)) \cap a$
PR $[abcd]$: Hp (1) $.\supset.$
$((a \cap b) \cap c) \cup (a \cap d) = (a \cap (b \cap c)) \cup (a \cap d)$ $[1; R1]$
$= ((d \cap a) \cup (b \cap c)) \cap a = ((d \cap a) \cup (c \cap b)) \cap a$
$[A9, b/b \cap c, c/d; A22, a/b, b/c]$
K2 $[ab]: a, b \in A .\supset. (a \cup (b \cap b)) \cap b = b$ $[A2, a/b, b/a; A13, a/b]$

Thus, the addition of *R1* as a new axiom to $\{A9; A2\}$ generates Kolibiar's postulate-system for modular lattices, *cf.* [3] and [5], p. 81.

But, I am able neither to obtain *P1* or *R1* in the field of system $\{A9; A2\}$ nor to prove that these formulas are not the consequences of this system.

Remark III: We have to note that, although, clearly, axiom *A1* is constructed in a rather mechanical way by combining formulas *A9* and *A8*, *A1* is an organic formula in the sense defined in [7], p. 60, point (c).

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