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EFFECTIVE DENSITY TYPES

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1 *Introduction* Recursive density types were introduced in [6] and studied in [4] and [5]. It was found that algebraic operations can be defined on the set of types so that the set becomes an interesting algebraic system roughly analogous to the algebra of recursive equivalence types. Further results of a more model-theoretic nature were announced by P. Aczel in [1] and [2]. (The details of [2] occur in a manuscript in our possession which is unpublished as far as we know.*) The main result in [2] is that certain ideals discussed in [4] and [5] satisfy the same universal sentences as the isols. (This is not quite precise as the language is richer, e.g., $\leqslant$ is a primitive relation, and a larger class of functions than the recursively combinatorial functions are allowed.) All this suggests that a further study of the model-theoretic properties of the recursive density types would be of interest. So far, it seems that the algebra of density types enjoy some saturation properties, and on the basis of work by Nerode and Barback the types look more like regressive isols than arbitrary isols. We hope to study all this in future papers.

In this paper, which is still at a pre-model-theoretic level we plan to study a subsystem of the algebra of density types which was motivated by [3]. This will add a new system to the systems Δ_s, Δ_u, and Δ_t studied in [4] and [5]. In [3] Arslanov introduced the concept of effectively hyperimmune set and studied the properties of such sets. (Actually he was interested primarily though not exclusively in effectively hypersimple sets.) In this paper we show that effectiveness is a property of the density type only so that one can introduce the concept of effective density type. Furthermore we show that the set of effective density types forms an ideal properly included in Δ_s though not containing Δ_u. In particular, effectiveness is sufficient to guarantee that such types satisfy the cancellation law as stated in Theorem 5 in [4]. Thus the concept of effectiveness leads to a purely algebraic consequence! This is what generated the interest in studying effective density types.

*Added in proof: P. Aczel, "Recursive density types and Nerode extensions of arithmetic," *Journal of the American Mathematical Society*, Series A, vol. 20 (1975), pp. 146-158.

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This paper is independent of [3]. Actually there is very little overlap since the emphasis in [3] is on effectively hypersimple sets. The only theorem in [3] which interests us here is the theorem that states that an effectively hyperimmune set cannot have a hyperimmune complement. We reprove this result here since we use the point of view of recursive boundedness whereas [3] emphasizes discrete arrays.

We use the terminology of [4] and [5]. In particular we identify a set and the function enumerating it in order of size.

2 *Definition and elementary properties of* Δ_e A set α is effectively hyperimmune if there exists a recursive function h such that for every index e of a recursive function g there exists an $n \leqslant h(e)$ such that $\alpha(n) > g(n)$. This is a heuristically reasonable way to effectivize the concept of recursive unboundedness, and it is shown in [3] that this is equivalent to an analogous effectivization using discrete arrays. This result will not be needed here.

For convenience let ϕ_e be an enumeration of the partially recursive functions. It is clear that such sets exist, e.g.,

$$\alpha(n) = \max_{\substack{m \leqslant n \\ \phi_m \text{ is recursive}}} [\phi_m(m) + n + 1] \text{ is effectively hyperimmune with}$$

$h(n) = n$. It is also clear that h may always be chosen so as to be strictly increasing. Furthermore it is enough to find an h which works for strictly increasing g. In fact, there exists a recursive function k such that $\phi_{k(e)}(n) = \max_{m \leqslant n} \phi_e(m) + n$. Then if h works for strictly increasing g then the composition $h \circ k$ works for arbitrary g.

Theorem 1 *If* $\beta \leqslant \alpha$ *and* α *is effectively hyperimmune then* β *is effectively hyperimmune.*

Proof: Choose g strictly increasing recursive such that $(\forall n)[\alpha(n) \leqslant g\,\beta(n)]$, and choose an h which works for α. Choose k so that $(\forall e)(\forall n)\{\phi_{k(e)}(n) = g[\phi_e(n)]\}$. Then for arbitrary ϕ_e there exists an $n \leqslant hk(e)$ such that $\alpha(n) > \phi_{k(e)}(n) = g\,\phi_e(n)$. Hence $g\,\beta(n) \geqslant \alpha(n) > g\,\phi_e(n)$, hence $\beta(n) > \phi_e(n)$. This shows that $h \circ k$ works for β.

Corollary 1 *If* $\alpha \sim \beta$ *and* α *is effectively hyperimmune then* β *is effectively hyperimmune.*

Definition The density A is effectively hyperimmune if it has an effectively hyperimmune representative. Let Δ_e be the set of all effectively hyperimmune densities.

Corollary 2 *If* $A \in \Delta_e$ *and* $B \leqslant A$ *then* $B \in \Delta_e$.

Theorem 2 *An effectively hyperimmune set cannot have a hyperimmune complement,* i.e., $\Delta_e \subset \Delta_t$.

Remark: This result will be strengthened in the next section.

Proof: There exists a function k such that $\phi_{k(a)}(n) = 2n + 2a$. Let α be effectively hyperimmune and h work for α. Then $(\forall a)[\exists n \leq hk(a)][\alpha(n) > \phi_{k(a)}(n)]$, i.e., letting $g = hk$ we obtain $(\forall a)[\exists n \leq g(a)][\alpha(n) > 2n + 2a]$. Fix a. If $n > a$ then, of course, $\alpha(n) > 2n$. If $n \leq a$ then $\alpha(a) \geq \alpha(n) > 2a$. Thus in either case we obtain $(\forall a)(\exists n)[a \leq n \leq g(a) \wedge \alpha(n) > 2n]$. If α' is the complement of α then $\alpha(n) > 2n \to \alpha'(n) \leq 2n$. Hence $\alpha'(a) \leq \alpha'(n) \leq 2n \leq 2g(a)$. Thus $2g$ is a recursive bound to α'.

Corollary 3 $C \in \Delta_e \wedge A + C = B + C \to A = B$.

Proof: This follows from the theorem and [5].

3 *Relations between* Δ_e, Δ_u, *and* Δ_s

Theorem 3 $\Delta_e \not\subset \Delta_u$.

Proof: Choose $A \in \Delta_e$. $(\exists B)(B \leq A)(B \notin \Delta_u)$ by [5]. By Corollary 2, $B \in \Delta_e$.

Theorem 4 $\Delta_u \not\subset \Delta_e$.

Proof: Enumerate the strictly increasing recursive functions $g_1, g_2, \ldots, g_n, \ldots$. (There is no necessary relation with the sequence ϕ_e.) We define α inductively in pieces. We use the notation g^i for the i'th iterate of g. Let e satisfy $\phi_e(i) = g_1^i(0)$ and let $\alpha(i) = g_1^i(0)$ for $i \leq g_1(e)$. Now suppose α is defined up to m and $g_1, g_2, \ldots, g_n$ have been considered. Let $h = \max(g_1, g_2, \ldots, g_{n+1})$ and define f: $f(x) = \alpha(x)$ if $x \leq m$ and $f(x) = h^{x-m}[\alpha(m)]$ if $x > m$. Let $f = \phi_e$. Choose any number p such that $p > m$ and $p \geq g_{n+1}(e)$. Let $\alpha(i) = h^{i-m}[\alpha(m)]$ for $m < i \leq p$. It is clear that by brute force we obtained a uniformly hyperimmune set which is not effectively hyperimmune.

Corollary 4 $\Delta_s \not\subset \Delta_e$.

This is clear from Theorem 4 since $\Delta_u \subset \Delta_s$.

Theorem 5 $\Delta_e \subset \Delta_s$.

Remark: This result is somewhat surprising. In fact, it seems reasonable at first to search for an effectively hyperimmune set which is not strongly hyperimmune. There is also the plausible fear that the result may depend on the Gödel numbering. Actually, the only property of the Gödel numbering which is used in the paper is that for every recursive functional F there exists a recursive function g such that $F[\phi_e] = \phi_{g(e)}$.

Proof: Roughly, the idea of the proof is to find an increasing sequence of Gödel numbers such that the functions go up much faster than the numbers. This loose statement will be made precise in the proof.

Let α be effectively hyperimmune and h strictly increasing work for α. Let f be arbitrary strictly increasing recursive. We must find an n_0 such that $(\forall n)[n \geq n_0 \to \alpha(n) > f(n)]$. There exists a strictly increasing recursive function k such that $(\forall e)(\forall n)\{\phi_{k(e)}(n) = fh\phi_e h\phi_e(n)\}$. We may choose k so that $k(0) > 0$. This guarantees that $(\forall n)[hk(n) > n]$. Let $k = \phi_a$. Then $\phi_{k(a)}(n) = fhkhk(n)$. It follows by induction that $(\forall n)[\phi_{k^i(a)}(n) \geq f(hk)^{2^i}(n)]$ for all $i \geq 1$. In fact, assume the latter is valid for i. Then *a fortiori* $\phi_{k^i(a)}(n) \geq (hk)^{2^i}(n)$.

$$\phi_{k^{i+1}(a)}(n) = fh\phi_{k^i(a)}h\phi_{k^i(a)}(n) \geqslant f\phi_{k^i(a)}\phi_{k^i(a)}(n) \geqslant f(hk)^{2^i}(hk)^{2^i}(n) \geqslant f(hk)^{2^i}(hk)^2(n) = f(hk)^{2i+2}(n).$$

Since h works for α, $[\exists n \leqslant hk^i(a)][\alpha(n) > \phi_{k^i}(n)]$. Hence *a fortiori* $[\exists n \leqslant (hk)^i(a)][\alpha(n) > f(hk)^{2^i}(n)]$. Since $hk(0) > 0$, $hk^a(0) \geqslant a$. Hence for $i \geqslant a + 1, f(hk)^{2^i}(0) \geqslant f(hk)^{i+a+1}(0) = f(hk)^{i+1}f^a(0) \geqslant f(hk)^{i+1}(a)$. We have

$$[\exists n \leqslant hk^i(a)][\alpha(n) > f(hk)^{2^i}(0)].$$

Hence for $i \geqslant a + 1$,

$$\alpha[hk^i(a)] > f(hk)^{2^i}(0) \geqslant f(hk)^{i+1}(a).$$

We claim that $hk^{a+1}(a)$ works as an n_0. In fact if $n \geqslant n_0$ then $(\exists i \geqslant a + 1)(hk)^i(a) \leqslant n < (hk)^{i+1}(a)$. Then

$$\alpha(n) \geqslant \alpha(hk)^i(a) > f(hk)^{i+1}(a) \geqslant f(n).$$

It follows from the proof that n_0 can be chosen effectively from f. Specifically, there exists a recursive function g such that if $n \geqslant g(e)$ then $\alpha(n) > \phi_e(n)$. Let us tentatively call such a set α an effectively strongly hyperimmune set. We then obtain the following.

Corollary 5 *A set is effectively hyperimmune if and only if it is effectively strongly hyperimmune.*

Proof: One direction follows from the above. The other direction is trivial.

The concept of an effectively strongly hyperimmune set is useful as a stepping stone towards the proof of the next theorem.

Theorem 6 *The union of two effectively hyperimmune sets is effectively hyperimmune.*

Proof: By the previous corollary we know that the sets are effectively strongly hyperimmune. We now simply note that the proof in [4] can trivially be "effectivized." Note that there is no loss in generality in assuming that the two sets are disjoint.

Remark: In [3] the analogous result for effectively hypersimple sets is proved. In that case the usual proof for hypersimple sets can be effectivized. For arbitrary hyperimmune sets the corresponding result is actually false. Hence no such technique is possible.

Theorem 7 $A \in \Delta_e$ *and* $B \in \Delta_e \rightarrow A + B \rightarrow A_e$.

Proof: This follows immediately from Theorem 6.

We have now shown Δ_e is an ideal in Δ properly included in Δ_S.

4 *Conclusion* It is easy to see that $\Delta_e \cap \Delta_u \neq \emptyset$. It follows from the unpublished work by P. Aczel referred to in the introduction that Δ_e satisfies the same universal sentences as the isols. Thus we have a further example to which model-theoretic studies can be applied.

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