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LAMBDA-CALCULUS TERMS THAT REDUCE TO THEMSELVES

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The usual example of a λ-calculus term which does not have a normal form is

(1) $D \equiv (\lambda x.xx)(\lambda x.xx)$.

The term D has the additional property of being a redex identical to its contractum. (See [1] for terminology and notation.) This note is to observe that there is no other such redex.

Proposition *The only term* $(\lambda xA)B$ *of* λ-*calculus such that*

(2) $(\lambda xA)B \equiv [B/x]A$

is the term D *of* (1).

Here, $M \equiv N$ means that M and N are identical except for changes in bound variables.

The proposition answers a small puzzle involved in Stenlund's book [2] where $A > B$ means that A reduces to B but $A \not\equiv B$, and where a term A is said to be strongly normalizable if every reduction sequence $A > A_1 > A_2 > \ldots$ is finite. Stenlund remarks that a strongly normalizable term has a normal form. The term D of (1) is not a counterexample to his remark since Stenlund is dealing with a theory of λ-calculus with types, and no type assignment to x can make D a well-formed term of that theory; but the possibility of other examples, valid for typed λ-calculus, was not considered. (The proposition of this note is not necessary for the application to Stenlund's theory since the difficulty there could be avoided by requiring reduction sequences either to end with a term in normal form, or to be infinite.)

The proof of the proposition involves only elementary comparisons of the two sides of the identity (2). We may assume that x does not occur free in B. Counting the number of λ's and atomic terms on the two sides of (2), we see that there are exactly two free occurrences of x in A. A can have no component of the form B since otherwise, the left-hand side of (2) would

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have fewer components B than the right-hand side. The right-hand side has exactly two components B, so that λxA of the left-hand side has B as a component. But B cannot be a component of A, so we conclude $\lambda xA \equiv B$. Finally, the left-hand side of (2) has the form MN, so that the term A has the form MN, and $[B/x]M \equiv \lambda xA \equiv B \equiv [B/x]N$. Then, $M \equiv N \equiv x$.

REFERENCES

[1] Curry, H. B., and R. Feys, *Combinatory Logic*, vol. I, North-Holland, Amsterdam (1958).

[2] Stenlund, S., *Combinators, λ-Terms, and Proof Theory*, Reidel, Dordrecht (1972).

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