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A NOTE ON DEFINING THE RUDIN-KEISLER ORDERING OF ULTRAFILTERS

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1 This note is prompted by a confusion which arises when one reads M. E. Rudin's initial paper [3] on the Rudin-Keisler order in conjunction with later papers [1], [2], and several others concerning this ordering.*

Let ω denote the natural numbers and let ${}^{\omega}\omega$ denote the set of all functions from ω to ω. Let p and q be ultrafilters (**u.f.**) on ω. Rudin [3] defines

$$p \leqslant q \text{ iff } \exists f \epsilon\, {}^{\omega}\omega p = fq$$

where, for $a \subseteq \omega$, $fa = \{fn \mid n \epsilon a\}$ and $fq = \{fa \mid a \epsilon q\}$.

In the later papers, we find a different definition. Let q be an **u.f.** on ω.

$$p \leqslant_* q \text{ iff } \exists f \epsilon\, {}^{\omega}\omega p = f_*q$$

where $f_*q = \{a \subseteq \omega \mid f^{-1}a \epsilon q\}$ and $f_a^{-1} = \bigcup_{n \epsilon a} f^{-1}n$.

Before considering the connections between $\leqslant$ and $\leqslant_*$ we present some further definitions.

Two **u.f.**s on ω are said to be of the same type iff they are isomorphic when viewed as partially ordered sets under inclusion. It is proved in W. Rudin ([4], Theorem 1.5) that p and q have the same type iff there exists a permutation $\pi \epsilon\, {}^{\omega}\omega$ such that $p = \pi q$. We denote the type of p by $p^{\sim}$.

Theorem III A of [3] establishes that $\leqslant$ is a partial order on types, i.e.,

$$p \leqslant q \wedge q \leqslant p \rightarrow p^{\sim} = q^{\sim}.$$

Kunen remarks in [2] that an easy modification of her argument can be used to show

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$$p \leqslant_* q \wedge q \leqslant_* p \rightarrow p^{\sim} = q^{\sim}$$

i.e., that $\leqslant_*$ is also a partial order on types.

Thus both $\leqslant$ and $\leqslant_*$ are preorderings and we can combine the preceding results together with their obvious converses to conclude $p \leqslant q \leqslant p$ iff $p \leqslant_* q \leqslant_* p$. This still does not answer the question whether the two preorderings are the same, since distinct preorderings $\leqslant_1$, $\leqslant_2$ can lead to the same collection of equivalence classes under $p \underset{i}{\sim} q$ iff $p \leqslant_i q \wedge q \leqslant_i p$. We show here that they are the same.

2 For an **u.f.** q on ω and an arbitrary $f \in {}^{\omega}\omega$, it is not in general true that fq is an **u.f.** on ω; it is not even immediate that fq is an **u.f.** The following proposition, however, has a routine proof.

Proposition *Let q be an* **u.f.** *on ω, let f be an arbitrary element of ${}^{\omega}\omega$ and let* R *denote the range of f; then fq is an* **u.f.** *on* R.

On the other hand, for an **u.f.** q on ω and an arbitrary $f \in {}^{\omega}\omega$, it is always the case that f_*q is an **u.f.** on ω. The following theorem establishes the connection between fq and f_*q.

Theorem *Let q be an* **u.f.** *on ω, $f \in {}^{\omega}\omega$ and let* R *denote the range of f; then*

$$f_*q = \{a \cup b \mid a \in fq \wedge b \subseteq \omega - \mathrm{R}\}.$$

Proof: Given $a \in fq$ and $b \subseteq \omega - \mathrm{R}$ let $a = fz$ where $z \in q$; then $f^{-1}(a \cup b) = f^{-1}a = f^{-1}(fz) \supseteq z \in q$ so $f^{-1}(a \cup b) \in q$, i.e., $a \cup b \in f_*q$. Conversely, given $z \in f_*q$, write $z = a \cup b$ where $a \subseteq R$, $b \subseteq \omega - \mathrm{R}$; now $f^{-1}z = f^{-1}(a \cup b) = f^{-1}a$, so $f(f^{-1}z) = f(f^{-1}z) = a$ since $a \subseteq \mathrm{R}$; thus $a \in fq$ since $f^{-1}z \in q$.

Corollary 1 *If q is an* **u.f.** *on ω and $f \in {}^{\omega}\omega$ is onto ω, then $fq = f_*q$.*

Corollary 2 *If p and q are* **u.f.**s *on ω and $p \leqslant q$ then $p \leqslant_* q$.*

Proof: The hypothesis implies that $\omega \in p = fq$, and hence that Range $f = \omega$; so $p = f_*q$ by Corollary 1.

One can give far simpler direct proofs of Corollaries 1 and 2; these particular proofs were given because the information contained in the theorem will also be used to establish the reverse implication between the two orderings.

Corollary 3 *If q is an* **u.f.** *on ω and $p \leqslant_* q$ then $p \leqslant q$.*

Proof: Case I. $p = f_*q$ for some $f \in {}^{\omega}\omega$ with finite range, R. By the proposition, fq is an **u.f.** on the finite set **R**, so fq is the principal **u.f.** on R generated by $\{i\}$ for some $i \in \mathbf{R}$. From the theorem we conclude that f_*q must be the principal **u.f.** on ω generated by $\{i\}$. Thus $p \leqslant q$ since the type of the principal **u.f.**s is least under $\leqslant$ among the types of **u.f.**s on ω ([3], Section III, C).

Case II. $p = f_*q$ for some $f \in {}^{\omega}\omega$ with infinite range, R. By the proposition, fq is an **u.f.** on R so we can pick an infinite set $E \in fq$ such that R - E

is infinite and define a map $h : \mathrm{R} \to \omega$ by letting $h \upharpoonright E$ = identity and letting $h \upharpoonright \mathrm{R} - E \to \omega - E$ be an arbitrary onto map. We will show $p \leqslant q$ by establishing that $p = (h \circ f)q$. Let $x \in fq$; then $x \cap E \in fq$ and $hx \supseteq h(x \cap E) = x \cap E$ since $h \upharpoonright E = \mathrm{id}$; thus $hx \in fq$. But $fq \subseteq f_* q$ by the theorem, so $hx \in p$. Conversely, invoking the theorem once again, let $a \cup b$, with $a \in fq$ and $b \subseteq \omega - \mathrm{R}$, be an arbitrary element of p; now $a \cup b \supseteq a \cap E \in fq$ and $a \cap E = h(a \cap E) \in (h \circ f)q$; thus $a \cup b \in (h \circ f)q$ since, by the proposition $(h \circ f)q$ is an **u.f.** on Range $(h \circ f) = \omega$.

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