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SHORTEST SINGLE AXIOMS FOR THE CLASSICAL EQUIVALENTIAL CALCULUS

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1 *Introduction* The first shortest single axioms for the classical equivalential calculus **EC** were found by Łukasiewicz [1], who published in 1939 the following three:

(1) *EEpqEErqEpr*, (2) *EEpqEEprErq*, (3) *EEpqEErpEqr*.

Łukasiewicz was believed to have shown that (1), (2), and (3) are the only shortest single axioms, but in 1963, Meredith ([4], pp. 185-186) proved that each of

(4) *EEEpqrEqErp*, (5) *EpEEqEprErq*

also possesses this property. In the same paper Meredith claimed further that each of

(6) *EpEEqErpEqr*, (7) *EEpEqrErEpq*, (8) *EEpqErEEqrp*
(9) *EEpqErEErqp*, (10) *EEEpEqrrEqp*, (11) *EEEpEqrqErp*

is a single axiom for **EC** (a misprint in (9) was corrected in [5], Appendix I, p. 307).

In this paper we shall prove Meredith's claim in respect of the axioms (7)-(11), but show that in fact (6) is not a single axiom for **EC**. Meredith ([4], pp. 185-186) showed in addition that (4) remains a single axiom for **EC** when the ordinary rule of detachment (the rule to infer β from $E\alpha\beta$ and α) is replaced by reverse detachment (the rule to infer α from $E\alpha\beta$ and β). We shall show that (7)-(11) also have this property, but that none of (1), (2), (3), (5), or (6) is a single axiom for **EC** under reverse detachment. The question, which of (1)-(11) is a single axiom for **EC** under ordinary detachment or under reverse detachment, is thus completely settled. Whether or not there exist any shortest single axioms other than (1)-(5) and (7)-(11) for **EC** under ordinary detachment remains unknown. The derivations given in this paper are simplified by the use of Meredith's condensed detachment operator **D** ([5], Appendix II, pp. 318-319). These derivations were found

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with the aid of a computer program based on iteration of an algorithm for the above **D**.

2 (7)-(11) *with ordinary detachment* We start with (7):

1.	*EEpEqrErEpq*	
2.	*EEpqEEpEqrr*	= D1.1
3.	*EpEEqrEqErp*	= D1.2
4.	*EEpqEpEqEErEstEtErs*	= D3.1
5.	*EEpEEEqrEqErpss*	= D2.3
6.	*EEpqEEEpEqrrEEsEtuEuEst*	= D4.2
7.	*EpEEqEEErsErEsqEtpt*	= D1.5
8.	*EEEpqEEEEpEqrrEEsEtuEuEstvv*	= D2.6
9.	*EpEEEqpEqrr*	= D8.7
10.	*EpEqEErqErp*	= D1.9
11.	*EEEpqEprErq*	= D1.10
12.	*EEpqEqp*	= D11.1
13.	*EEpqEErpErp*	= D12.11
14.	*EEpqEEprErq*	= D1.D1.13

This is Łukasiewicz's axiom (2).

We start with (9):

1.	*EEpqErEErqp*	
2.	*EpEEpEqEEqrsEsr*	= D1.1
3.	*EpEEpEEqErEErstEtsq*	= D1.2
4.	*EEEEpqErEErqpEsEEstuEut*	= D2.1
5.	*EEEEpqErEErqpEEsEtEEtuvEvus*	= D3.1
6.	*EEEEEpqrqpr*	= D4.4
7.	*EEEEEEEpqrqprEsEEstuEut*	= D2.6
8.	*EpEEpqEEEErsqsr*	= D1.6
9.	*EpEEpEEqrEEEEstrtsq*	= D1.8
10.	*EEEEpqErEErqpEEstEEEEuvtvus*	= D9.1
11.	*EpEEpEqrEEEEEEstutsuEvEEvrq*	= D1.7
12.	*EEEEpEEpqrsErqs*	= D5.7
13.	*EEEpqEEEErspsrq*	= D10.7
14.	*EEpEEpqErqr*	= D7.11
15.	*EEEpqpq*	= D14.1
16.	*EEpEEpqrErq*	= D5.14
17.	*EEEpEEpqrsEErqs*	= D12.14
18.	*EEpqEEEqrpr*	= D13.16
19.	*EEpEEpEEqrsqEsr*	= D5.18
20.	*EEEpqErEErsEpsq*	= D18.14
21.	*EEEpqEErprq*	= D18.15
22.	*EEpqEEEEErqpsrs*	= D17.18
23.	*EEpqEqp*	= D21.16
24.	*EEpEEqrErpq*	= D21.19
25.	*EEEpEEpqErqsEsr*	= D20.24

26. $EEEEpqprErq$ = D21.24
27. $EEEpqrEpErq$ = D24.22
28. $EEpqEEprEqr$ = D26.25
29. $EEEpqErqEpr$ = D23.28
30. $EEpqEEprErq$ = D27.29

This is Łukasiewicz's axiom (2).

We start with (11):

1. $EEEpEqrqErp$
2. $EpEqEEErqpr$ = D1.1
3. $EpEEEqpEEErEstsEtrq$ = D2.1
4. $EEEpEEEqErsrEsqEEEtEuvuEvtp$ = D3.1
5. $EEpqEEEqErprEEEsEtutEus$ = D1.4
6. $EEEEpEqrqErpEEEsEtutEus$ = D4.4
7. $EEEpqErEEqEspsr$ = D4.5
8. $EEEpEEEqprqEsrs$ = D7.5
9. $EEpEqEErsEpEEsEtrtq$ = D7.6
10. $EEpEqEEErqprEEEsEtutEus$ = D8.3
11. $EEEEpEEEqErsrEsqtpt$ = D1.10
12. $EEEEpEqEEErqprEEsEtutEvEusv$ = D9.10
13. $EEEpEEqErsrEsqp$ = D12.1
14. $EEEEpEqrqEEEsEtutEusErp$ = D13.11
15. $EEpEqEErEEEsrtsEptq$ = D7.10
16. $EEEEpEqEEErqprsEtEuEEEvusvt$ = D15.10
17. $EEEpqErEEEsrqsp$ = D16.1
18. $EEEEpqrpErq$ = D17.1
19. $EpEEqEprErq$ = D14.18

This is Meredith's axiom (5).

From the results above we prove that (8) and (10) are single axioms for **EC** using the notion of a dual.

The dual α^{d} of a formula α is defined as follows:

(i) $\alpha^{d} = \alpha$ if α is a variable,
(ii) $\alpha^{d} = E\gamma^{d}\beta^{d}$ if $\alpha = E\beta\gamma$.

Lemma *α is an axiom for* **EC** *with ordinary detachment iff α^{d} is an axiom for* **EC** *with reverse detachment.*

Proof: It is easily seen that β is derivable from α by ordinary detachment iff β^{d} is derivable from α^{d} by reverse detachment. Since β belongs to **EC** iff β^{d} belongs to **EC**, this proves the lemma.

From the lemma, (8) = $(11)^{d}$ and (10) = $(9)^{d}$ are axioms with reverse detachment. We show that from each of (8) and (10) with ordinary detachment we may deduce reverse detachment.

We start with (8), $E\alpha\beta$ and β:

1.	*EEpqErEEqrp*	
2.	$E\alpha\beta$	
3.	β	
4.	*EpEEEqEErqspEsr*	= D1.1
5.	*EEEpEEqprEsEEEtEEutvsEvuErq*	= D4.4
6.	*EEEpEqErprq*	= D5.1
7.	*EpEEqpEErEqEsrs*	= D1.6
8.	$EEp\beta EEqEpErqr$	= D7.3
9.	$EEpE\alpha Eqpq$	= D8.2
10.	α	= D6.2

and derive α, i.e., reverse detachment holds.

We start with (10), $E\alpha\beta$ and β:

1.	*EEEpEqrrEpq*	
2.	$E\alpha\beta$	
3.	β	
4.	*EpEqErEpErq*	= D1.1
5.	*EEpEqErEpErqEsEtEuEsEut*	= DDDD4.4.4.4.4
6.	*EEpEpqq*	= D1.5
7.	*EpEqEpq*	= D6.4
8.	$EEpEqEpqE\alpha\beta$	= DD7.7.2
9.	$E\beta\alpha$	= D1.8
10.	α	= D9.3

Thus (7)-(11) are single axioms for **EC** with ordinary detachment.

3 (6) *with ordinary detachment* The matrix

	E	0	1	2
*	0	0	2	1
	1	1	0	2
	2	2	1	0

satisfies (6) but does not satisfy (1). Thus (6) is not a single axiom for **EC** with ordinary detachment.

4 (1)-(11) *with reverse detachment* By the lemma and since $(8)^d$-$(11)^d$ (i.e., (11), (10), (9), and (8)) are single axioms with ordinary detachment, (8)-(11) are single axioms for **EC** with reverse detachment. Similarly $(4) = (4)^d$ is an axiom under either ordinary or reverse detachment as proved by Meredith [4].

It may be shown that *EEppEqq* is not derivable from $(1)^d$-$(3)^d$, $(5)^d$-$(7)^d$ by ordinary detachment (this may be verified by hand calculations for $(1)^d$, $(3)^d$, $(5)^d$, and $(6)^d$, but for $(2)^d$ and $(7)^d$ has been verified only by computer). Hence none of these is an axiom for **EC** with ordinary detachment. Thus none of (1)-(3), (5)-(7) is a single axiom for **EC** with reverse detachment.

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