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AN HERBRAND THEOREM FOR PRENEX FORMULAS OF LJ

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For formulas of the intuitionistic predicate calculus which are in prenex normal form there is a very simple analogue of the Herbrand Theorem for the classical calculus.* Let A be such a formula and let B be its (open) matrix. We assume that all the quantified variables in the prefix of A are mutually distinct (if not, one can always pass to a suitable equivalent variant of A). Let $x_1, \ldots, x_n (y_1, \ldots, y_n)$ be all the variables which are existentially (universally) quantified in the prefix of A. A *special instance* of B is a formula of the form

$$B_{x_1,\ldots,x_n}[a_1, \ldots, a_n],$$

where $a_1, \ldots, a_n$ are terms such that for $i = 1, \ldots, n$, a_i does not contain any of the variables $y_1, \ldots, y_n$ which occur to the right of $\exists x_i$ in the prefix of A. We will show that the sequent $\Rightarrow A$ is provable in **LJ**, *cf.* [1], if and only if for some special instance B' of B, the sequent $\Rightarrow B'$ is provable in **LJ**.

Lemma (*cf.* [3] and [4]) *The following hold:*

a) *If* $\Rightarrow \exists x A$ *is provable in* **LJ**, *then for some term* a, $\Rightarrow A_x[a]$ *is provable in* **LJ**.
b) *If* $\Rightarrow \forall x A$ *is provable in* **LJ**, *then for any variable* y *which is either* x *or does not occur free or bound in* A, $\Rightarrow A_x[y]$ *is provable in* **LJ**.

Proof: By Gentzen's Hauptsatz for **LJ**, if $\Rightarrow \exists x A$ is provable, it has a cut-free proof. Since sequents in **LJ** can contain at most one formula in the succedent, the only possible inferences (other than Cut) leading immediately to $\Rightarrow \exists x A$ are Thinning and $\exists$-IS (Note: we understand the rule $\exists$-IS to *be as stated* for the system G3 of [2]; i.e., in the $\mathfrak{F}\mathfrak{a}$ of $\exists$-IS of [1], $\mathfrak{a}$ may be a free variable or term). Since **LJ** is consistent, $\Rightarrow$ is not derivable, and hence $\Rightarrow \exists x A$ must have followed by an application of $\exists$-IS from a premiss

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of the form $\Rightarrow A_x[a]$, where a is some term. Similarly, if $\Rightarrow \forall xA$ is provable in **LJ**, the only possible rule (other than Cut) leading immediately to $\Rightarrow \forall xA$ is $\forall$-IS, and hence $\Rightarrow A_x[z]$ must be provable in **LJ** for some variable z. Since substitution is a derived rule in **LJ**, it follows that $\Rightarrow A_x[y]$ is provable for any y as described.

Note that the converses of a) and b) both obviously hold.

Theorem *If A is in prenex normal form with matrix B, then $\Rightarrow A$ is provable in* **LJ** *if and only if for some special instance B' of B, $\Rightarrow B'$ is provable in* **LJ**.

Proof: If $\Rightarrow B'$ is provable in **LJ** where B' is a special instance of B, then $\Rightarrow A$ follows from $\Rightarrow B'$ by successive applications of the rules $\forall$-IS and $\exists$-IS. On the other hand, if $\Rightarrow A$ is provable, then successive applications of parts a) and b) of the Lemma above yield the desired result.

Let $\mathbf{LJ}^=$ be the system **LJ** extended by adding each of the following open sequents as axioms, where p and f range over all n-ary (for any n) predicate and function letters to be used:

$$\begin{gathered} \Rightarrow x = x \\ x_1 = y_1, \ldots, x_n = y_n \Rightarrow fx_1 \ldots x_n = fy_1 \ldots y_n \\ x_1 = y_1, \ldots, x_n = y_n, px_1 \ldots x_n \Rightarrow py_1 \ldots y_n. \end{gathered} \tag{*}$$

By examining the original proof of the Hauptsatz for **LJ** in [1], one can verify that if a sequent is provable in $\mathbf{LJ}^=$, it is provable with a proof whose only cuts are on cut-formulas which occur in one of the sequents of (*) above. Thus in such a normal proof, no cut-formula can contain quantifiers. With this observation, it is easy to see that the Lemma and Theorem above extend to $\mathbf{LJ}^=$.

The usefulness of the Theorem of course is reduced by the fact that not all formulas of **LJ** or $\mathbf{LJ}^=$ possess prenex normal forms. The following are known to be provable (*cf.* [2], pp. 162-163) where A contains no occurrence of x.

$$\begin{gathered} \Rightarrow \neg\exists xB \equiv \forall x\neg B. \\ \Rightarrow A \wedge \forall xB \equiv \forall x[A \wedge B], \qquad \Rightarrow A \wedge \exists xB \equiv \exists x[A \wedge B]. \\ \Rightarrow A \vee \exists xB \equiv \exists x[A \vee B]. \\ \Rightarrow \forall x[A \supset B] \equiv A \supset \forall xB, \qquad \Rightarrow \forall x[B \supset A] \equiv \exists xB \supset A. \end{gathered} \tag{**}$$

None of the remaining classical equivalences for prenex normal form are provable in **LJ**. However, the following two implications hold:

$$(\dagger) \qquad \exists x[A \supset B] \Rightarrow A \supset \exists xB.$$
$$(\dagger\dagger) \qquad A \vee \forall xB \Rightarrow \forall x[A \vee B].$$

Surprisingly, these two implications can be reversed in the following weak sense:

($\dagger'$) *If $\Rightarrow A \supset \exists xB$ is provable in* **LJ** *and A has no strictly positive subformula beginning with $\exists$ in the sense of [4], then $\Rightarrow \exists x[A \supset xB]$ is provable in* **LJ**.

(††′) *If* $\Rightarrow A \vee \forall x B$ *is provable in* **LJ**, *then* $\Rightarrow \forall x[A \vee B]$ *is provable in* **LJ**.

Let us first argue for (††′); so assume $\Rightarrow A \vee \forall x B$ has been proved in **LJ**. As observed in [1], then either $\Rightarrow A$ or $\Rightarrow \forall x B$ must be provable in **LJ**. In the latter case, we must have $\Rightarrow B$ provable in **LJ** by the Lemma above. Then in each case we proceed:

$$\dfrac{\dfrac{\Rightarrow A}{\Rightarrow A \vee B}\ \vee\text{-IS}}{\Rightarrow \forall x[A \vee B]}\ \forall\text{-IS} \qquad \dfrac{\dfrac{\Rightarrow B}{\Rightarrow A \vee B}\ \vee\text{-IS}}{\Rightarrow \forall x[A \vee B]}\ \forall\text{-IS}$$

For (†′), we first observe that from the provability of $\Rightarrow A \supset \exists x B$, it must follow that $A \Rightarrow \exists x B$ is provable in **LJ**. If any terms occur in either A or $\exists x B$, then by Corollary 7(ii) of [4] (*cf.* also [3]), for some term a, $A \Rightarrow B_x[a]$ is provable in **LJ**. Then we proceed:

$$\dfrac{\dfrac{A \Rightarrow B_x[a]}{\Rightarrow A \supset B_x[a]}\ \supset\text{-IS}}{\Rightarrow \exists x[A \supset B]}\ \exists\text{-IS}$$

If neither A nor $\exists x B$ contains any terms, then by Corollary 7(iii) of [4], $A \Rightarrow \forall x B$ is provable in **LJ**. Then we proceed:

$$\dfrac{\dfrac{\dfrac{A \Rightarrow \forall x B}{\Rightarrow A \supset \forall x B}\ \supset\text{-IS} \qquad \begin{matrix}(**)\\ \vdots \\ A \supset \forall x B \Rightarrow \forall x[A \supset B]\end{matrix}}{\Rightarrow \forall x[A \supset B]}\ \text{Cut} \qquad \begin{matrix}{*}81[2]\\ \vdots \\ \forall x[A \supset B] \Rightarrow \exists x[A \supset B]\end{matrix}}{\Rightarrow \exists x[A \supset B]}\ \text{Cut}$$

Thus we have:

if A *contains no strictly positive subformula beginning with* $\exists$, *then* $\Rightarrow A \supset \exists x B$ *is provable in* **LJ** *iff* $\Rightarrow \exists x[A \supset B]$ *is provable in* **LJ**,

and

$\Rightarrow A \vee \forall x B$ *is provable in* **LJ** *iff* $\Rightarrow \forall x[A \vee B]$ *is provable in* **LJ**.

These principles somewhat extend the range of formulas which can be reduced to prenex normal form. That such a reduction, even of the latter weak type, is not possible for the classical equivalence $\forall x A \supset B \equiv \exists x[A \supset B]$ can be seen by constructing a counter-model using Kripke's semantics for **LJ**.

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