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PLEDGER LEMMA AND THE MODAL SYSTEM S3°

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1 In [8] I defined modal systems S3.02, S3.03, and S3.04 as the systems which are obtained by adding to S3 the respective axioms

Ł1 $\mathfrak{C}\mathfrak{C}\mathfrak{C}pLppCLMLpp$
Ł2 $\mathfrak{C}\mathfrak{C}\mathfrak{C}pLpp\mathfrak{C}LMLpp$
L1 $\mathfrak{C}LMLpCpLp$

Remark: It should be noted that either **Ł1** or **Ł2** can be accepted as a proper axiom of S4.02, *cf.* [6], and that **L1** is a proper axiom of S4.04, *cf.*, e.g., [9]. Obviously, these axioms are not consequences of S4.

1.1 In [8] it has been established:

(a) that each of the systems S3.02, S3.03, and S3.04 is a proper extension of S3 and that they do not contain S4.
(b) that system S3.04 is a subsystem neither of S3.02 nor of S3.03.

and

(c) that S3.02 is a subsystem of S3.03.

On the other hand, in [8] the following problems were left open:

(d) is S3.02 a proper subsystem of S3.03?

and

(e) does S.04 contain S3.02 or S3.03?

1.2 In [4] G. F. Schumm solved problem (d), proving metalogically that in the field of S3 axiom **Ł1** implies **Ł2**, and, therefore, S3.02 = S3.03. Independently, in [3], K. E. Pledger obtained the same result, but used, in some respects, a different method. Namely, he remarked that it is easy to prove metalogically that the following formula (called here the Pledger lemma):

PL $\mathfrak{C}\mathfrak{C}LpCLqr\mathfrak{C}Lp\mathfrak{C}Lqr$

is a thesis of system S3. Hence, it follows immediately from this fact that

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S3.02 = S3.03. Also, in the same paper, Pledger established that S3.04 does not contain S3.02 (S3.03), *cf.* problem (e) above.

1.3 In section **2** of this note I shall present a very short, but rather tricky logical proof that *PL* is provable in the field of S3° (for a definition of that system, *cf.* [5], pp. 52-53). The fact that the Pledger lemma is obtainable in the field of this proper subsystem of S3 yields several interesting results. Only some of them will be discussed in sections **3** and **4** below.

2 S3° ⊢ *PL* Let us assume S3°. Then:

Z1 $\mathfrak{C}\mathfrak{C}pqCLpLq$ [S1°]

Z2 $\mathfrak{C}CCpqpp$ [S1°]

Z3 $\mathfrak{C}\mathfrak{C}Cpqr\mathfrak{C}Crpp$ [S2°]

Z4 $\mathfrak{C}Lp\mathfrak{C}qp$ [S2°]

Z5 $\mathfrak{C}\mathfrak{C}pq\mathfrak{C}LpLq$ [S3°]

Z6 $\mathfrak{C}\mathfrak{C}pq\mathfrak{C}\mathfrak{C}qr\mathfrak{C}pr$ [S3°]

Z7 $\mathfrak{C}\mathfrak{C}pq\mathfrak{C}\mathfrak{C}\mathfrak{C}prs\mathfrak{C}\mathfrak{C}qrs$ [S3°]

Z8 $\mathfrak{C}Lp\mathfrak{C}LqLp$ [S3°]

Z9 $\mathfrak{C}\mathfrak{C}pCqr\mathfrak{C}\mathfrak{C}pq\mathfrak{C}pr$ [S3°]

Z10 $\mathfrak{C}\mathfrak{C}CCLqrvLpC\mathfrak{C}LpCLq\mathfrak{C}Lqr$ [*Z6*, $p/\mathfrak{C}CCLqrvLp$, $q/\mathfrak{C}CLpCLqrCLqr$, $r/C\mathfrak{C}LpCLqr\mathfrak{C}Lqr$; *Z3*, $p/CLqr$, q/v, r/Lp; *Z1*, $p/CLpCLqr$, $q/CLqr$]

Z11 $\mathfrak{C}\mathfrak{C}LqLpC\mathfrak{C}LpCLqr\mathfrak{C}Lqr$ [*Z7*, $p/CCLqrLq$, q/Lq, r/Lp, $s/C\mathfrak{C}LpCLqr\mathfrak{C}Lqr$; *Z2*, p/Lq, q/r; *Z10*, v/Lq]

Z12 $\mathfrak{C}LpC\mathfrak{C}LpCLqr\mathfrak{C}Lqr$ [*Z6*, p/Lp, $q/\mathfrak{C}LqLp$, $r/C\mathfrak{C}LpCLqr\mathfrak{C}Lqr$; *Z8*; *Z11*]

Z13 $\mathfrak{C}\mathfrak{C}Lp\mathfrak{C}LpCLqr\mathfrak{C}Lp\mathfrak{C}Lqr$ [*Z9*, p/Lp, $q/\mathfrak{C}LpCLqr$, $r/\mathfrak{C}Lqr$; *Z12*]

Z14 $\mathfrak{C}\mathfrak{C}pCLpCLqr\mathfrak{C}Lp\mathfrak{C}Lqr$ [*Z6*, $p/\mathfrak{C}pCLpCLqr$, $q/\mathfrak{C}Lp\mathfrak{C}LpCLqr$, $r/\mathfrak{C}Lp\mathfrak{C}Lqr$; *Z5*, $q/CLpCLqr$; *Z13*]

PL $\mathfrak{C}\mathfrak{C}LpCLqr\mathfrak{C}Lp\mathfrak{C}Lqr$ [*Z6*, $p/\mathfrak{C}LpCLqr$, $q/\mathfrak{C}pCLpCLqr$, $r/\mathfrak{C}Lp\mathfrak{C}Lqr$; *Z4*, $p/CLpCLqr$, q/p; *Z14*]

Thus, *PL* is a thesis of the modal system S3°.

3 Let us assume the formula *Z1*, *cf.* section **2** above, and *PL*. Then:

Z5 $\mathfrak{C}\mathfrak{C}pq\mathfrak{C}LpLq$ [*PL*, p/Cpq, q/p, r/Lq; *Z1*]

Hence in the axiomatization of S3° we can substitute its proper axiom, namely *Z5*, by *PL*. This fact shows also that *PL* is not provable in the field of system T of Feys-von Wright.

4 Let us define modal systems S3.02°, S3.03°, and S3.04° as the systems obtained by adding axioms **Ł1**, **Ł2**, and **L1** respectively to S3°. Then, we have:

(*α*) Since *PL* is a thesis of S3°, we know that in the field of S3°, **Ł1** implies **Ł2**. On the other hand, the following matrix:

𝔐1

C	1	2	3	4	N	M	L
*1	1	2	3	4	4	2	1
2	1	1	3	3	3	2	3
3	1	2	1	2	2	2	3
4	1	1	1	1	1	4	3

which is the familiar Group IV of Lewis-Langford (*cf.* [1], p. 494) and in which 1 is the designated value, verifies S3°, **Ⱡ2** and **L1**, but falsifies **Ⱡ1** for *p*/2: ℭℭℭ2*L*22*CLML*22 = ℭℭ*LC*232*CLM*32 = ℭℭ*L*32*CL*22 = ℭ*LC*32*C*32 = ℭ*L*2*C*32 = *LC*32 = *L*2 = 3. Hence, a formula ℭ**Ⱡ2Ⱡ1** is not a thesis of S3° and, therefore, system S3.03° is a proper subsystem of S3.02°.

(β) Since matrix 𝔐1 verifies S3°, **Ⱡ2**, and **Ⱡ1**, but falsifies a formula ℭ*Lpp* for *p*/2: ℭ*L*22 = *LC*32 = *L*2 = 3, we know that system S3 is contained neither in S3° nor in S3.03° nor in S3.04°. I have no proof that S3.02° does not contain S3, but it is rather obvious.

(γ) Since (in [8], pp. 416-417, section **3**) it has been proved that S3.04 is not contained in S3.02 (or S3.03), it follows *a fortiori* that system S3.04° is not deducible from S3.02° or S3.03°.

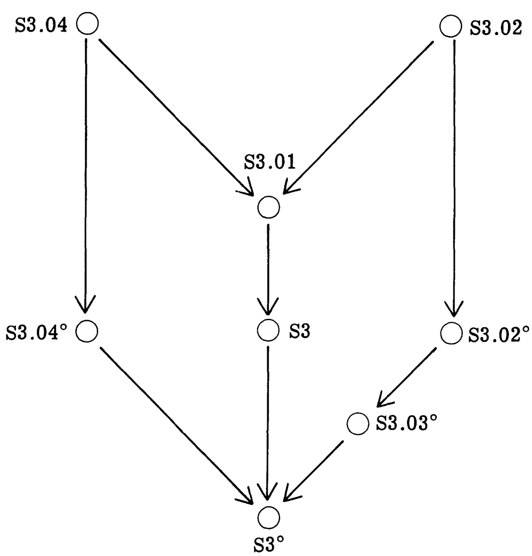
(δ) Since in [3] Pledger has shown that S3.02 (or S3.03) is not contained in S3.04, it follows *a fortiori* that S3.04° yields neither S3.02° nor S3.03°.

4 In [2] Pledger has shown that the addition of the following formula:

PS1 ℭ*LMLLMpLLMp*

which is an easy consequence of S4 to S3 as a new axiom, constructs a modal system which is a proper extension of S3 and a proper subsystem of S4. Pledger called this system 16s, but here, for reasons of uniformity, I shall call this system S3.01. In [3] Pledger proved that S3.01 is a proper subsystem of S3.02 and of S3.04.

The following diagram



in which Pledger's result, mentioned above, is included and, in which an arrow occurring between two systems indicates that a tail system contains an edge system, visualizes the relations existing among the discussed systems.

5 Open problems:

1. To prove that S3.02° does not contain S3, and, therefore, S3.02° is a proper subsystem of S3.02.
2. To investigate the effect of the addition of *PS1* to S3° as a new axiom.

Final remark: In [8] and [7] I either investigated or mentioned several formulas akin to **Ł1**, **Ł2**, and **L1**. I did not yet analyze these formulas in connection with the fact that *PL* is a thesis of S3°.

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