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THREE IDENTITIES FOR ORTHOLATTICES

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In a recent paper [1], B. Sobociński proved the following theorem:

(A) *Any algebraic system*

$$\mathfrak{A} = \langle A, \cup, \cap, {}^{\perp}\rangle$$

where $\cup$ *and* $\cap$ *are two binary operations and* ${}^{\perp}$ *is a unary operation defined on the carrier set* A, *is an ortholattice, if it satisfies the following four mutually independent postulates:*

$B1$ $[ab]:a, b \in A .\supset. a \cup b = b \cup a$
$B2$ $[ab]:a, b \in A .\supset. a = a \cap (a \cup b)$
$B3$ $[ab]:a, b \in A .\supset. a = a \cup (b \cap b^{\perp})$
$B4$ $[abc]:a, b, c \in A .\supset. (a \cup b) \cup c = ((c^{\perp} \cap b^{\perp}) \cap a^{\perp})^{\perp}$

In the present paper, we improve this result by showing that Sobociński's system of axioms can be replaced by a shorter one. We will presuppose acquaintance with the principal results of [1]; the reader is also asked to refer to [1] for definitions and notations not given here. Our result is as follows:

(a) *Any algebra* $\langle A, \cup, \cap, {}^{\perp}\rangle$ *with two binary operations* $\cup$, $\cap$ *and one unary operation* ${}^{\perp}$ *which satisfies the mutually independent axioms*

$b1$ $[abc]:a, b, c \in A .\supset. (a \cup b) \cup c = (c^{\perp} \cap b^{\perp})^{\perp} \cup a$
$b2$ $[ab]:a, b \in A .\supset. a = a \cap (a \cup b)$
$b3$ $[ab]:a, b \in A .\supset. a = a \cup (b \cap b^{\perp})$

is an ortholattice.

Proof:

1 It is enough to prove that $b1$, $b2$, $b3$ imply $B1$, $B2$, $B3$, and $B4$. Now, as consequences of $b1$-$b3$ we have

$b4$ $[a]:a \in A .\supset. a = a \cap a$ $\qquad [b2, b/b \cap b^{\perp}; b3]$

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b5 $[ab]:a, b \epsilon A .\supset. a = ((b \cap b^{\perp})^{\perp})^{\perp} \cup a$

$[b3; b3, a/a \cup (b \cap b^{\perp}); b1, b/b \cap b^{\perp}, c/b \cap b^{\perp}; b4, a/(b \cap b^{\perp})^{\perp}]$

b6 $[b]:b \epsilon A .\supset. b \cap b^{\perp} = ((b \cap b^{\perp})^{\perp})^{\perp}$ $[b3, a/((b \cap b^{\perp})^{\perp})^{\perp}; b5, a/b \cap b^{\perp}]$

b7 $[ab]:a, b \epsilon A .\supset. a = (b \cap b^{\perp}) \cup a$ $[b5; b6]$

b8 $[bc]:b, c \epsilon A .\supset. b \cup c = (c^{\perp} \cap b^{\perp})^{\perp}$

$[b1, a/a \cap a^{\perp}; b7, a/b, b/a; b3, a/(c^{\perp} \cap b^{\perp})^{\perp}, b/a]$

b9 $[abc]:a, b, c \epsilon A .\supset. a \cup b = ((c \cap c^{\perp})^{\perp} \cap b^{\perp})^{\perp} \cup a$

$[b1, c/c \cap c^{\perp}; b3, a/a \cup b, b/c]$

b10 $[bc]:b, c \epsilon A .\supset. b = ((c \cap c^{\perp})^{\perp} \cap b^{\perp})^{\perp}$

$[b7, b/a, a/b; b9, a/a \cap a^{\perp}; b3, a/((c \cap c^{\perp})^{\perp} \cap b^{\perp})^{\perp}, b/a]$

b11 $[ab]:a, b \epsilon A .\supset. a \cup b = b \cup a$ $[b9; b10]$

b12 $[ab]:a, b \epsilon A .\supset. (a^{\perp})^{\perp} = (a^{\perp} \cap (b^{\perp} \cap (a^{\perp})^{\perp})^{\perp})^{\perp}$ $[b2, a/a^{\perp}; b8, b/a^{\perp}, c/b]$

b13 $[a]:a \epsilon A .\supset. a = (a^{\perp})^{\perp}$ $[b12, b/a; b8, b/a^{\perp} \cap (a^{\perp})^{\perp}, c/a; b7, b/a^{\perp}]$

b14 $[abc]:a, b, c \epsilon A .\supset. (a \cup b) \cup c = ((c^{\perp} \cap b^{\perp}) \cap a^{\perp})^{\perp}$

$[b1; b11, a/(c^{\perp} \cap b^{\perp})^{\perp}, b/a; b8, b/a, c/(c^{\perp} \cap b^{\perp})^{\perp}; b13, a/c^{\perp} \cap b^{\perp}]$

2 The algebra $\mathfrak{M}2$ ([1], p. 143) verifies *b1*, *b3* and falsifies *b2*; the algebra $\mathfrak{M}3$ ([1], p. 143) verifies *b1*, *b2* and falsifies *b3*. Since the axioms *B1-B4* are mutually independent, we therefore conclude that *b1-b3* are also mutually independent.

This completes the proof ($\mathfrak{a}$).

REFERENCE

[1] Sobociński, B., "A short postulate-system for ortholattices," *Notre Dame Journal of Formal Logic*, vol. XVI (1975), pp. 141-144.

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