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A NOTE ON THE ADEQUACY OF TRANSLATIONS

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Given a theory which can be axiomatized in more than one way using different sets of constants as primitive, any two such axiomatizations, *sans* definitions, can be said to be deductively (semantically) synonymous, following de Bouvêre [1], in the sense that the addition of appropriate definitions to each and closure under provability (consequence) results in a single theory. The question then arises: Can the definitions of one axiomatization be used to provide another axiomatization by translating the first into the second?

For the analagous problem concerning formal systems, Hiż [3] signalled that this is not always the case. In [2], Halmos takes the Hilbert-Ackermann axioms for a sentential logic of $\sim$ and $\vee$, and the rule of inference

$$\frac{\begin{array}{c} p \vee q \\ \sim p \end{array}}{q}$$

and provides an axiom system for $\sim$ and & by means of the definition

$$p \vee q \leftrightarrow \sim(\sim p \; \& \; \sim q).$$

Hiż gives a model-theoretic demonstration that the translated system is incomplete.

The general theorem which could have predicted this result is as follows:

If $\mathrm{T}(A)$ *is the closure of a formal system in a language* $\mathcal{L}$, *with axioms* A1, . . ., AN; *and rules* R1, . . ., RM; *and* **t** *a rule of translation from* $\mathcal{L}$ *to* $\mathcal{L}'$, *then* T', *the closure of* **t**(A1), . . ., **t**(AN), **t**(R1), . . ., **t**(RM), *is equal to* **t**(T(A)).

To show this, note that if RJ is a k-place rule, it contains the $k+1$-tuple $\langle y_1, \ldots, y_k, x\rangle$ just when **t**(RJ) contains $\langle \mathbf{t}(y_1), \ldots, \mathbf{t}(y_k), \mathbf{t}(x)\rangle$. So

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$$\mathbf{t}(\mathrm{R}J\langle y_1, \ldots, y_k\rangle) = \mathbf{t}(x) = \mathbf{t}(\mathrm{R}J)\langle \mathbf{t}(y_1), \ldots, \mathbf{t}(y_k)\rangle$$

The result then follows by induction on the length of proofs.

Thus, if **t** is not an onto-mapping from $\mathcal{L}$ to $\mathcal{L}'$, (as the domain D is not, having as its range in the language containing ~ and & only sentences beginning with ~), a complete axiomatization in $\mathcal{L}$ will result in an incomplete one in $\mathcal{L}'$.

Of course, for any two theories there exists between them a bijection of some kind. The issues that remain are first, whether, and under what conditions, there exists a bijection that could properly be called translations. In general, one of the difficulties is that it is not definitions in T′ that would be useful in translating to T′, but rather definitions in T. In the case of D, we have a definition of v rather than of &. And second, noting that our theorem involves formal systems, we may ask what occurs in the case of proper mathematical systems, where the axioms are translated but the rules of inference remain the same. For example, taking **BD** to be the self-dual system of axioms for Boolean algebra, **BR** to be the Boolean ring axioms, and to be the translation function incorporating the appropriate definitions, **t**(T(**BD**)) is a proper subset of T(**BR**), but T(**t**(**BD**)) may well equal T(**BR**).

REFERENCES

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