222
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A SPECIES-ALGEBRAIC INTERPRETATION OF THE INTUITIONISTIC PROPOSITIONAL CALCULUS

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The topological and lattice-theoretical interpretations of the intuitionistic propositional calculus (see [4] and [5]) differ from the set-algebraic interpretation of the classical two-valued propositional calculus in that, in the former cases, the intuitionistic propositional calculus is interpreted by means of classical theories which are definable in the second order classical predicate calculus, but, in the latter, the classical propositional calculus is interpreted by means of a classical theory which is definable in the monadic classical predicate calculus of the first order.

The algebra of species is the intuitionistic analogy to the Boolean algebra of sets (for details, see [1] and [2]). The aim of this article is to give a species-algebraic interpretation of the intuitionistic propositional calculus analogous to the set-algebraic interpretation of the classical propositional calculus. By using the method of logical matrix, it will be shown that the intuitionistic propositional calculus is equivalent to the algebra of species, of all subspecies of any infinite species, in the sense that, if the intuitionistic propositional functors $\rightarrow$, $\vee$, $\wedge$, $\sim$, are interpreted as the corresponding species-algebraic operators, namely: species-implication $\Rrightarrow$, species-union $\cup$, species-intersection $\cap$, and species-complement $-$, then the formulae of the propositional calculus can be mapped one-to-one onto the formulae of the algebra of species, in such a way that a formula H of the intuitionistic propositional calculus is provable in the intuitionistic propositional calculus if and only if the corresponding formula $\mathfrak{H}$ of the algebra of species is valid in every algebra of species of all subspecies of any infinite species.

1 *The intuitionistic propositional calculus* In the formulae of the propositional calculus variables of only one kind occur, namely, propositional variables, the letters $P_1, P_2, \ldots, P_n$ will be used. In addition to the variables, four constants occur in the propositional calculus: the implication sign $\rightarrow$, the disjunction sign $\vee$, the conjunction sign $\wedge$, and the negation sign $\sim$, (a fifth constant, the equivalence sign $\leftrightarrow$, may also be used).

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Definition 1.1 The formulae of the propositional calculus will be defined inductively as follows:

1. Propositional variables are formulae.
2. a) If H is a formula, then $\neg H$ is also a formula.
b) If H_1 and H_2 are formulae, then $(H_1 \to H_2)$, $(H_1 \vee H_2)$, $(H_1 \wedge H_2)$, are also formulae.
3. A sequence of symbols is a formula of the propositional calculus if and only if it is the case according to 1 and 2.

Definition 1.2 A formula H is called an axiom of the intuitionistic propositional calculus if there are formulae H_1, H_2, H_3, such that H satisfies one of the following equalities or identities:

1. $H = H_1 \to (H_2 \to H_1)$
2. $H = H_1 \to (H_1 \to H_2). \to .H_1 \to H_2$
3. $H = H_1 \to H_2. \to .(H_2 \to H_3) \to (H_1 \to H_3)$
4. $H = H_1 \wedge H_2 \to H_1$
5. $H = H_1 \wedge H_2 \to H_2$
6. $H = H_1 \to H_2. \to .(H_1 \to H_3) \to (H_1 \to H_2 \wedge H_3)$
7. $H = H_1 \to H_1 \vee H_2$
8. $H = H_2 \to H_1 \vee H_2$
9. $H = H_1 \to H_3. \to .(H_2 \to H_3) \to (H_1 \vee H_2 \to H_3)$
10. $H = (H_1 \to \neg H_2) \to (H_2 \to \neg H_1)$
11. $H = H_1 \to (\neg H_1 \to H_2)$

Definition 1.3 If H_1, H_2, and H_3 are three formulae such that $H_1 = H_2 \to H_3$, then H_3 is said to be the result of the detachment of the formulae H_2 from the formula H_1.

2 *The algebra of species* Here only the basic concepts of the intuitionistic algebra of species will be given (for further details see [1]). In the algebra of species, we consider the relations and operations of the theory of species which are definable in the monadic intuitionistic predicate calculus of the first order.

As basic concepts we define, in the monadic intuitionistic predicate calculus of the first order, the following relations and operations:

1. The element relation ϵ: $a \epsilon A =_{Df} Aa$
2. The species-inclusion $\subset$: $A \subset B =_{Df} \forall a(a \epsilon A .\to. a \epsilon B)$
3. The identity: $A = B =_{Df} A \subset B \wedge B \subset A$
4. The species-implication $\Rightarrow$: $a \epsilon A \Rightarrow B =_{Df} a \epsilon A \to a \epsilon B$
5. The species-union $\cup$: $a \epsilon A \cup B =_{Df} a \epsilon A \vee a \epsilon B$
6. The species-intersection $\cap$: $a \epsilon A \cap B =_{Df} a \epsilon A \wedge a \epsilon B$
7. The species-complement $-$: $a \epsilon \overline{A} =_{Df} \neg(a \epsilon A)$
8. The universal species 1: $a \epsilon 1 =_{Df} a \epsilon A \to a \epsilon A$
9. The empty species $\emptyset$: $a \epsilon \emptyset =_{Df} \neg(a \epsilon A \to a \epsilon A)$

We shall define the terms of the algebra of species inductively as follows:

Definition 2.1

1. A species variable A_k $(k = 0, 1, 2, 3, \dots)$ is a term, the universal species 1, and the empty species $\emptyset$ are terms.
2. a) If T is a term then $\overline{T}$ is also a term.
b) If T_1 and T_2 are terms, then $T_1 \Rightarrow T_2$, $T_1 \cup T_2$, and $T_1 \cap T_2$ are also terms.
3. A sequence of symbols is a term if and only if it is the case according to 1 and 2.

Definition 2.2 The formulae of the algebra of species are formed, with the help of the terms, as follows:

Formulae of the first kind are of the form $T = 1$ (where T is a term).
Formulas of the second kind are formed by applying the propositional functors: $\rightarrow$, $\vee$, $\wedge$, $\sim$, and the quantifiers, to the formlae of the first kind.

Definition 2.3 A formula $\mathfrak{H} = (T = 1)$ of the algebra of species is called an axiom of the algebra of species, if there are terms T_1, T_2, T_3 such that $\mathfrak{H}$ satisfies one of the following equalities:

1. $\mathfrak{H} = T_1 \Rightarrow (T_2 \Rightarrow T_1) = 1$
2. $\mathfrak{H} = [T_1 \Rightarrow (T_1 \Rightarrow T_2)] \Rightarrow (T_1 \Rightarrow T_2) = 1$
3. $\mathfrak{H} = (T_1 \Rightarrow T_2) \Rightarrow [(T_2 \Rightarrow T_3) \Rightarrow (T_1 \Rightarrow T_3)] = 1$
4. $\mathfrak{H} = (T_1 \cap T_2) \Rightarrow T_1 = 1$
5. $\mathfrak{H} = (T_1 \cap T_2) \Rightarrow T_2 = 1$
6. $\mathfrak{H} = (T_1 \Rightarrow T_2) \Rightarrow [(T_1 \Rightarrow T_3) \Rightarrow (T_1 \Rightarrow (T_1 \cap T_2))] = 1$
7. $\mathfrak{H} = T_1 \Rightarrow (T_1 \cup T_2) = 1$
8. $\mathfrak{H} = T_2 \Rightarrow (T_1 \cup T_2) = 1$
9. $\mathfrak{H} = (T_1 \Rightarrow T_3) \Rightarrow [(T_2 \Rightarrow T_3) \Rightarrow (T_1 \cup T_2) \Rightarrow T_3)] = 1$
10. $\mathfrak{H} = (T_1 \Rightarrow \overline{T}_2) \Rightarrow (T_2 \Rightarrow \overline{T}_1) = 1$
11. $\mathfrak{H} = T_1 \Rightarrow (\overline{T}_1 \Rightarrow T_2) = 1$
12. $\mathfrak{H} = ((\overline{T_1 \Rightarrow T_1}) \Rightarrow \emptyset) \cap (\emptyset \Rightarrow (\overline{T_1 \Rightarrow T_1})) = 1$

Definition 2.4 If A is a species with at least one element, S a system of subspecies of A, such that S is closed with respect to the species-algebraic operations of species implication $\Rightarrow$, union $\cup$, intersection $\cap$, and complement $-$, then the sextuple

$$\mathfrak{A} = \langle S, A, \Rightarrow, \cup, \cap, - \rangle$$

is an algebra of species. Here the term variables range over subspecies of A, and the universal species 1 is the species A itself.

Corollary 2.5 If A is a species with at least one element, and S is a system of all subspecies of A, then

$$\mathfrak{A} = \langle S, A, \Rightarrow, \cup, \cap, - \rangle$$

is an algebra of species of all subspecies of A.

Definition 2.6 If

$$\mathfrak{A}_1 = \langle S_1, A, \Rightarrow, \cup, \cap, - \rangle$$

and

$$\mathfrak{A}_2 = \langle S_2, A, \Rightarrow, \cup, \cap, -\rangle$$

are two algebras of species such that $S_1 \subset S_2$, then $\mathfrak{A}_1$ is said to be a subalgebra of $\mathfrak{A}_2$ and $\mathfrak{A}_2$ is said to be an extension of $\mathfrak{A}_1$.

In an algebra of species $\mathfrak{A} = \langle S, A, \Rightarrow, \cup, \cap, -\rangle$ of subspecies of a given species A, the following lemmas can be proved.

Lemma 2.7

a) *For any* $A_i, A_k \in S$, $A_i \Rightarrow A_k = A . \leftrightarrow . A_i \subseteq A_k$;
b) *For any* $A_j \in S$, $A_j \Rightarrow A = A$; $A \Rightarrow A_j = A_j$; $A = A_j \cup A$; $A \cap A_j = A_j$.

Lemma 2.8 *Let* S_1 *be a system of subspecies of a species* $B \subset A$ *such that* $\mathfrak{A} = \langle S_1, B, \Rightarrow, \cup, \cap, -\rangle$ *is an algebra of species, then*:

a) $\mathfrak{A} = \langle S_2, A, \Rightarrow, \cup, \cap, -\rangle$ *is also an algebra of species, where* $S_2 = S_1 \cup \{A\}$;
b) *If the species algebraic operations in* $\mathfrak{A}_1$ *and* $\mathfrak{A}_2$ *are denoted* $\overset{B}{\Rightarrow}, \overset{B}{\cup}, \overset{B}{\cap}, {}^{-B}$ *and* $\overset{A}{\Rightarrow}, \overset{A}{\cup}, \overset{A}{\cap}, {}^{-A}$ *respectively, then we have the following relations*:

(i) *if* $A_k, A_j \in S_1$ *and* $A_k \overset{A}{\Rightarrow} A_j \neq B$, *then* $A_k \overset{B}{\Rightarrow} A_j = A_k \overset{A}{\Rightarrow} A_j$; *if* $A_k \overset{B}{\Rightarrow} A_j = B$, *then* $A_k \overset{A}{\Rightarrow} A_j = A$; $A_i \overset{A}{\Rightarrow} A = A$, *for every* $A_i \in S_2$; $A \overset{A}{\Rightarrow} A_j = A_j$, *for every* $A_j \in S_2$.

(ii) $A_j \overset{A}{\cup} A_k = A_j \overset{B}{\cup} A_k$, *for every* $A_j, A_k \in S_1$; $A_i \overset{A}{\cup} A = A \overset{A}{\cup} A_i$, *for every* $A_i \in S_2$.

(iii) $A_j \overset{A}{\cap} A_k = A_j \overset{B}{\cap} A_k$, *for every* $A_j, A_k \in S_1$.

(iv) *if* $A_i \in S_1$ *and* $A_i^{-B} \neq B$, *then* $A_i^{-A} = A_i^{-B}$; *if* $A_i \in S_1$ *and* $A_i^{-B} \neq B$, *then* $A_i^{-A} = A$; $A^{-A} = B^{-B}$.

Proof: a) Since $\mathfrak{A}$ is an algebra of species, S_1 is closed under $\Rightarrow, \cup, \cap$, and $-$, and since $S_2 = S_1 \cup \{A\}$, it suffices to show that for any $A_i \in S_1$, $A_i \overset{A}{\Rightarrow} A$, $A \overset{A}{\cup} A_i$, $A \overset{A}{\cap} A_i$, $A_i \Rightarrow A$, and A^{-A} are elements of S_2; then the proof follows from the fact that $B \subset A$, and Lemma 2.7.

b) Lemmas 2.7 and 2.8a imply points (i), (ii), and (iii) at once. Since $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are algebras of species, $B^{-B} = \emptyset = A^{-A}$, and since if $A_i^{-B} \neq B$, then $A_i \neq \emptyset$, by Lemma 2.8a, point (iv) follows.

Lemma 2.9 *If* a *and* b, $(a \neq b)$, *are any two positive prime numbers, then for any natural number* n *and* m, $a^n \neq b^m$.

Proof: Since for any two natural numbers c and d it can be decided whether $c = d$ or $c \neq d$, and for any natural number n it can be decided whether n is prime or not, we can apply indirect proof.

(i) Suppose that for some two positive prime numbers a and b, $(a \neq b)$, and for some two natural numbers k and j, $a^k = b^j$, then for $k = j$ we have

$a^k = b^k$, $\frac{a^k}{b^k} = 1 = \frac{a^k}{b} = 1^k$, $\frac{a}{b} = 1$, $a = b$, which contradicts the hypothesis that $a \neq b$.

(ii) Suppose that $k \neq j$, say $k > j$, then it follows from $a^k = b^j$ that $a^{(k-j)}a^j = b^j$, thus $a^{k-j} = \left(\frac{b}{a}\right)^j$. Since a^{k-j} is an integer, it follows that $\left(\frac{b}{a}\right)^j$, and $\frac{b}{a}$ are also integers, hence b is divisible by a, which contradicts the hypothesis that b is a prime number. This concludes the proof.

Lemma 2.10 *Let $\mathcal{N}$ be the species of all natural numbers, then for every natural number n there are n infinite pairwise disjoint subspecies of $\mathcal{N}$.*

Proof: Let $a_1, \ldots, a_n$ be any positive prime numbers and A_1 the species of all the natural numbers of the form a_1^i with $i \epsilon N$ (that is all the numbers $a_1^1, a_1^2, a_1^3, a_1^4, a_1^5, \ldots$, etc.), and A_j the species of all the natural numbers of the form a_j^i with $j \epsilon \mathcal{N}$, then, by Lemma 2.9, the species $A_1 \ldots, A_n$ are pairwise disjoint infinite subspecies of $\mathcal{N}$ as required.

Lemma 2.11 *For any infinite species A and any natural number n, there are n pairwise disjoint infinite subspecies A.*

The proof of the lemma follows from Lemma 2.9 and the fact that every infinite species has a countable infinite subspecies.

3 *The matrix method*

Definition 3.1 Let $\mathcal{W}$ be any given species, an element $A \epsilon \mathcal{W}$, three binary operations $\twoheadrightarrow$, $\curlyvee$, and $\curlywedge$, and one unary operation $\sim$, such that $\mathcal{W}$ is closed under the above operations and that the following holds: *if $Y \epsilon \mathcal{W}$ and $A \twoheadrightarrow Y = A$, then $Y = A$.* Under these assumptions, the ordered sextuple

$$\mathfrak{M} = \langle \mathcal{W}, A, \twoheadrightarrow, \curlyvee, \curlywedge, \sim \rangle$$

is called a (normal logical) matrix.

Definition 3.2 Two matrices:

$$\mathfrak{M}_1 = \langle \mathcal{W}_1, A_1, \underset{1}{\twoheadrightarrow}, \underset{1}{\curlyvee}, \underset{1}{\curlywedge}, \underset{1}{\sim} \rangle$$

and

$$\mathfrak{M}_2 = \langle \mathcal{W}_2, A_2, \underset{2}{\twoheadrightarrow}, \underset{2}{\curlyvee}, \underset{2}{\curlywedge}, \underset{2}{\sim} \rangle$$

are said to be isomorphic if there is a function F which maps $\mathcal{W}_1$ one-to-one onto $\mathcal{W}_2$, and is such that $F(A_1) = A_2$, $F(X \underset{1}{\twoheadrightarrow} Y) = F(X) \underset{2}{\twoheadrightarrow} F(Y)$, $F(X \underset{1}{\curlyvee} Y) = F(X) \underset{2}{\curlyvee} F(X)$, $F(X \underset{1}{\curlywedge} Y) = F(X) \underset{2}{\curlywedge} F(Y)$ and $F(\underset{1}{\sim} X) = \underset{2}{\sim} F(X)$, for all $X, Y \epsilon \mathcal{W}_1$. (The isomorphism is reflexive and transitive.)

Definition 3.3 Let $\mathfrak{M} = \langle \mathcal{W}, A, \twoheadrightarrow, \curlyvee, \curlywedge, \sim \rangle$ be a matrix and H a formula of the intuitionistic propositional calculus. The following formulae define (recursively) a function $F_{H,\mathfrak{M}}$ which correlates an element $F_{H,\mathfrak{M}}(X_1, \ldots, X_n, \ldots) \epsilon \mathcal{W}$ with every infinite sequence $X_1, \ldots, X_n, \ldots \epsilon \mathcal{W}$:

a) $F_{H,\mathfrak{M}}(X_1, \ldots, X_n \ldots) = X_p$ if $H = H_p (p = 1, 2 \ldots)$.

b) $F_{H,\mathfrak{M}}(X_1, \ldots, X_n, \ldots) \twoheadrightarrow F_{H_2,\mathfrak{M}}(X_1, \ldots, X_n, \ldots)$, if $H = H_1 \rightarrow H_2$ (where H_1, H_2 are formulae of the propositional calculus).

c), d) Analogously for the operations $\curlyvee$ and $\vee$, or $\curlywedge$ and $\wedge$.

e) $F_{H,\mathfrak{M}}(X_1, \ldots, X_n, \ldots) = F_{H_i,\mathfrak{M}}(X_1, \ldots, X_n, \ldots)$ if $H = \sim H_i$ (where H_i is a formula of the propositional calculus).

We say that a formula H is verified by the matrix $\mathfrak{M}$, in symbols $H \in \mathrm{E}(\mathfrak{M})$, if $F_{H,\mathfrak{M}}(X_1, \ldots, X_n, \ldots) = A$ for all $X_1 \ldots, X_n$. The matrix $\mathfrak{M}$ is said to be adequate for the system S of propositional formulae if $\mathrm{E}(\mathfrak{M}) = S$.

Corollary 3.4 *If the matrices* $\mathfrak{M}_1$ *and* $\mathfrak{M}_2$ *are isomorphic then* $\mathrm{E}(\mathfrak{M}_1) = \mathrm{E}(\mathfrak{M}_2)$. [By 3.1, 3.2, 3.3]

Definition 3.5 If $\mathfrak{M}_1 = \langle \mathcal{W}_1, A, \twoheadrightarrow, \curlyvee, \curlywedge, \sim\rangle$ and $\mathfrak{M}_2 = \langle \mathcal{W}_2, A, \twoheadrightarrow, \curlyvee, \curlywedge, \sim\rangle$ are two matrices and if $\mathcal{W}_1 \subseteq \mathcal{W}_2$, then $\mathfrak{M}_1$ is called a submatrix of $\mathfrak{M}_2$.

Corollary 3.6 *If* $\mathfrak{M}_1$ *is a submatrix of* $\mathfrak{M}_2$, *then* $\mathrm{E}(\mathfrak{M}_2) \subseteq \mathrm{E}(\mathfrak{M}_1)$.

Definition 3.7 We denote by ZK the ordered sextuple

$$\langle \mathcal{W}, 1, \twoheadrightarrow, \curlyvee, \curlywedge, \sim\rangle$$

where $\mathcal{W} = \{0, 1\}$, $x \twoheadrightarrow y = 1 - x + x \cdot y$, $x \curlyvee y = x + y - x \cdot y$, $x \curlywedge y = x \cdot y$, and $\sim x = 1 - x$, for all $x, y \in \mathcal{W}$.

Definition 3.8 Let $\mathfrak{M} = \langle \mathcal{W}, B, \twoheadrightarrow, \curlyvee, \curlywedge, \sim\rangle$ be a matrix and A any element which does not belong to $\mathcal{W}$. We put:

a) $\mathcal{W}^* = \mathcal{W} \cup \{A\}$.
b) then $X \twoheadrightarrow^* Y = X \twoheadrightarrow Y$ if $X, Y \in \mathcal{W}$ and $X \twoheadrightarrow Y \neq B$, if $X, Y \in \mathcal{W}$ and $X \twoheadrightarrow Y = B$, then $X \twoheadrightarrow^* Y = A$; $X \twoheadrightarrow^* A = A$, for all $X \in \mathcal{W}^*$; $A \twoheadrightarrow^* Y = Y$, for all $Y \in \mathcal{W}^*$.
c) $X \curlyvee^* Y = X \curlyvee Y$, for all $X, Y \in \mathcal{W}^*$; $Z \curlyvee^* A = A \curlyvee^* Z = A$, for $Z \in \mathcal{W}^*$.
d) $X \curlywedge^* Y = X \curlywedge Y$, for all $X, Y \in \mathcal{W}$; $Z \curlywedge^* A = A \curlywedge^* Z = Z$, for $Z \in \mathcal{W}^*$.
e) if $X \in \mathcal{W}$ and $\sim X \neq B$, then $\sim^* X = A$; if $X \in \mathcal{W}$ and $\sim X = B$, then $\sim^* X = A$; $\sim^* A = \sim B$.

The ordered sextuple $\langle \mathcal{W}^*, A, \twoheadrightarrow^*, \curlyvee^*, \curlywedge^*, \sim^*\rangle$ is denoted by $\mathfrak{M}^*$.

Corollary 3.9 *If* $\mathfrak{M}$ *is a matrix, then* $\mathfrak{M}^*$ *is also a matrix and* $\mathrm{E}(\mathfrak{M}^*) \subseteq \mathrm{E}(\mathfrak{M})$; *if the matrices* $\mathfrak{M}_1$ *and* $\mathfrak{M}_2$ *are isomorphic, then* $\mathfrak{M}_1^*$ *and* $\mathfrak{M}_2^*$ *are also isomorphic.* [By 3.1, 3.2, 3.4, 3.7]

Definition 3.10 Let n be a natural number and $\mathfrak{M} = \langle \mathcal{W}, A, \twoheadrightarrow, \curlyvee, \curlywedge, \sim\rangle$ a matrix. We put:

(i) $\mathcal{W}^n$ = the system of ordered n-tuples: $X_1, \ldots, X_n$ with $X_1, \ldots, X_n \in \mathcal{W}$;
(ii) $A^n = \langle X_1, \ldots, X_n\rangle$, where $X_1 =, \ldots, X_n = A$;
(iii) $\langle X_1, \ldots, X_n\rangle \overset{n}{\twoheadrightarrow} \langle Y_1, \ldots, Y_n\rangle = \langle X_1 \twoheadrightarrow Y_1, \ldots, X_n \twoheadrightarrow Y_n\rangle$, for all $X_1, \ldots, X_n, Y_1, \ldots, X_n \in \mathcal{W}$;

(iv), (v) analogously for $\curlyvee$ and $\curlywedge$;
(vi) $\overset{n}{\sim}\langle X_1, \ldots, X_n\rangle = \langle \sim X_1, \ldots, \sim X_n\rangle$, for all $X_1, \ldots, X_n \in \mathcal{W}$

$$\langle \mathcal{W}^n, A^n, \overset{n}{\twoheadrightarrow}, \overset{n}{\curlyvee}, \overset{n}{\curlywedge}, \overset{n}{\sim}\rangle$$

is called the n'th power of the matrix $\mathfrak{M}$ and is denoted by '$\mathfrak{M}^n$'.

Corollary 3.10 *If n is a natural number and $\mathfrak{M}$ is a matrix, then $\mathfrak{M}^n$ is also a matrix and we have* $\mathsf{E}(\mathfrak{M}^n) = \mathsf{E}(\mathfrak{M})$; *if $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are isomorphic matrices, then $\mathfrak{M}_1^n$ and $\mathfrak{M}_2^n$ are also isomorphic.* [By 3.1, 3.2, 3.3, 3.9]

Definition 3.12 (i) $\mathsf{IK}_1 = \mathsf{ZK}$, and (ii) $\mathsf{IK}_{n+1} = ((\mathsf{IK}_n)^n)^*$.

Theorem 3.13 *Let $\mathfrak{A} = \langle S, A, \Rightarrow, \cup, \cap, -\rangle$ be an algebra of species of subspecies of A, then $\mathfrak{A}$ is a matrix.* [By 3.1, 2.4, 2.2]

We shall denote by $\mathsf{E}(\mathfrak{A})$ the set of formulae verified by the matrix $\mathfrak{A}$.

Theorem 3.14 *Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be two algebras of species as defined in* 2.3. *If $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are two matrices such that $\mathfrak{M}_1$ and $\mathfrak{A}_1$ and $\mathfrak{M}_2$ and $\mathfrak{A}_2$ are isomorphic respectively, then $\mathfrak{M}_1 = \mathfrak{M}_2^*$.* [By 3.8, 2.3]

Lemma 3.15 *If:*

(i) *A is an infinite species;*
(ii) *$B_1, \ldots, B_n$ are pairwise disjoint subspecies of B;*
(iii) *$B_1, \ldots, B_n = B \subset A$;*
(iv) *S_p is a system of subspecies of B_p ($p = 1, \ldots, n$) and for each p*

$$\mathfrak{A}_{B_p} = \langle S_p, B_p, \overset{B_p}{\Rightarrow}, \overset{B_p}{\cup}, \overset{B_p}{\cap}, \overset{B_p}{-}\rangle$$

is an algebra of species isomorphic with the matrix

$$\mathfrak{M} = \langle \mathcal{W}, A, \twoheadrightarrow, \curlyvee, \curlywedge, \sim\rangle;$$

(v) *$\mathfrak{A}_B$ and $\mathfrak{A}_{B_p}$ satisfy the conditions in* 2.9;

and

(vi) *S is a system of all species $X = X_1 \cup \ldots \cup X_n$ where $X_1 \in S_1, \ldots, X_2 \in S_2, \ldots, X_n \in S_n$ and the algebra*

$$\mathfrak{A}_B = \langle S_B, B, \overset{B}{\Rightarrow}, \overset{B}{\cup}, \overset{B}{\cap}, \overset{B}{-}\rangle$$

is isomorphic with a matrix P;

then $\mathfrak{P}$ is a matrix isomorphic with $\mathfrak{M}^n$, and if $S_A = S_B \cup \{A\}$ then the species algebra

$$\mathfrak{A}_A = \langle S_A, A, \overset{A}{\Rightarrow}, \overset{A}{\cup}, \overset{A}{\cap}, \overset{A}{-}\rangle$$

is isomorphic with the matrix $\mathfrak{P}^*$ which is isomorphic with $(\mathfrak{M}^n)^*$.

Proof: Since $\mathfrak{M}$ is isomorphic with each $\mathfrak{A}_{B_p}$, by 3.2, there are isomorphisms $F_1, \ldots, F_n$ which map $\mathcal{W}$ one-to-one onto $S_1, \ldots, S_n$. Then:

1. We put $F(U) = F_1(U_1) \cup \ldots \cup F_n(U_n)$ for $U = \langle U_1, \ldots, U_n\rangle \in \mathcal{W}^n$. Since

$F_p(U_p) \in S_p$ ($p = 1, 2, \ldots, n$), $F_p(U_p) \subseteq B_p$ and since the species $B_1, \ldots, B_n$ are pairwise disjoint, by (v), we conclude that F maps $\mathcal{W}^n$ one-to-one onto S.

By 3.2, 3.10,

2. $F(A^n) = F_1(A) \cup \ldots \cup F_n(A) = B_1 \cup \ldots \cup B_n = B$.
3. Further, let $U = \langle U_1, \ldots, U_n\rangle \in \mathcal{W}^n$ and $V = \langle V_1, \ldots, V_n\rangle \in \mathcal{W}^n$.

By 3.2, 3.10, 3.15.1, and 2.3, we get:

4. $F(U \overset{n}{\twoheadrightarrow} V) = F(\langle U_1 \twoheadrightarrow V_1, \ldots, U_n \twoheadrightarrow V_n\rangle)$
$= F_1(U_1 \twoheadrightarrow V_1) \cup \ldots \cup F_n(U_n \twoheadrightarrow V_n)$
$= (F_1(U_1) \overset{B_1}{\Rightarrow} F_1(V_1)) \cup \ldots \cup (F_n(U_n) \overset{B}{\Rightarrow} F_n(V_n))$
$= (F_1(U_1) \overset{B}{\Rightarrow} F_1(V_1)) \ldots (F_n(U_n) \overset{B}{\Rightarrow} F_n(V_n))$.

By 2.3 and 3.15.1:

$F(U \overset{n}{\twoheadrightarrow} V) = F(U) \overset{B}{\Rightarrow} F(V)$.

Thus for all $U, V \in \mathcal{W}^n$, $F(U \overset{n}{\Rightarrow} V) = F(U) \overset{B}{\Rightarrow} F(V)$.

In an analogous way we obtain the formulae:

5. $F(U \overset{n}{\curlyvee} V) = F(U) \overset{B}{\cup} F(V)$ and $F(U \overset{n}{\curlywedge} V) = F(U) \overset{B}{\cap} F(V)$, for all $U, V \in \mathcal{W}^n$.
6. $F(\sim^n U) = F(U)^{-B}$, for all $U \in \mathcal{W}^n$.

By 3.10 the matrix $\mathfrak{P}$ is isomorphic with the matrix $\mathfrak{M}^n$, and, by 2.3, 3.8, and 3.2, $\mathfrak{A}_A$ is isomorphic with $\mathfrak{P}^*$ and since $\mathfrak{P}$ is isomorphic with $\mathfrak{M}^n$ it follows, by 3.9, that $\mathfrak{P}^*$ is isomorphic with $(\mathfrak{M}^n)^*$ as required.

Lemma 3.16 *Let A be any species with at least one element, and $S = \{A, \emptyset\}$. Then*

$$\mathfrak{A} = \langle S, A, \Rightarrow, \cup, \cap, -\rangle$$

is an algebra of species.

Proof: Since $A \Rightarrow \emptyset = \emptyset \in S$, $\emptyset \Rightarrow A = A \in S$, $A \cup A = A \in S$, $A \cap \emptyset = \emptyset \in S$, $A \cup \emptyset = A \in S$, and $\overline{A} = \emptyset \in S$ and $\overline{\emptyset} = A$, hence, by 2.4, $\mathfrak{A}$ is an algebra of species.

Lemma 3.17 *Let $\mathfrak{A} = \langle S, A, \Rightarrow, \cup, \cap, -\rangle$ be an algebra of species of all subspecies of an infinite species A, then for every natural number $n \geqslant 1$, there is a subalgebra $\mathfrak{A}_n$ of $\mathfrak{A}$ which is isomorphic with the matrix $\mathbb{K}_n$.*

Proof: We shall prove the lemma by giving a method of constructing $\mathfrak{A}_n$ for any given natural number $n \geqslant 1$. To construct $\mathfrak{A}_n$ for the given natural number $n \geqslant 1$ we proceed as follows:

a) We construct $n!$ subalgebras which are isomorphic to $\mathbb{K}_1$. Then for each of the natural numbers m ($2 \leqslant m < n$) we construct $\frac{n!}{m!}$ algebras which are (starting with $m = 2$) isomorphic with $\mathbb{K}_m$. After constructing $\frac{n!}{(n-1)!}$

algebras which are isomorphic with IK_{n-1} we then construct $\mathfrak{A}_n$ in accordance with 3.7, 3.2, and 3.14.
b) If $n = 1$, then, by 3.12, $\mathsf{IK}_1 = \mathsf{ZK}$, and, by 3.7 and 3.2, ZK is isomorphic with $\mathfrak{A} = \langle S, A, \Rightarrow, \cup, \cap, - \rangle$ where $S = \{A, \emptyset\}$.
c) If $n = 2$; then, by 2.5, there are two pairwise disjoint infinite subspecies B_1, B_2, of A. Let $S_1 = \{B_1, \emptyset\}$, $S_2 = \{B_2, \emptyset\}$. Then the algebras

$$\mathfrak{A}_{11} = \langle S_1, B_1, \overset{B_1}{\Rightarrow}, \overset{B_1}{\cup}, \overset{B_1}{\cap}, \overset{B_1}{-} \rangle$$

and

$$\mathfrak{A}_{12} = \langle S_2, B_2, \overset{B_2}{\Rightarrow}, \overset{B_2}{\cup}, \overset{B_2}{\cap}, \overset{B_2}{-} \rangle$$

are both isomorphic with IK_1 (by 3.7, 3.2). Further, let $B_1 \cup B_2 = B$ and S a system of subspecies $X = X_1 \cup X_2$, with $X_1 \in S_1$ and $X_2 \in S_2$. Then

$$\mathfrak{A}_1 = \langle S, B, \overset{B}{\Rightarrow}, \overset{B}{\cup}, \overset{B}{\cap}, \overset{B}{-} \rangle$$

and, by 3.14, $\mathfrak{A}_1$ is isomorphic with IK_2. Moreover, if $S' = S \cup \{A\}$, then, by 3.14,

$$\mathfrak{A}_2 = \langle S', A, \overset{A}{\Rightarrow}, \overset{A}{\cup}, \overset{A}{\cap}, \overset{A}{-} \rangle$$

is isomorphic with IK_2.

We have shown that the construction is valid for $n = 1$ and $n = 2$. Now suppose that the construction is valid for any number n, then we construct $\frac{(n+1)!}{n!}$ subalgebras of $\mathfrak{A}$ which are isomorphic with IK_n. Then we construct $\mathfrak{A}_{n+1}$ which is isomorphic with IK_{n+1} in accordance with 3.14 and 3.14c as required.

1. Let $H = H_1 \rightarrow (\neg H_1 \rightarrow H_2)$ where H_1, H_2 are propositional formulae. We construct, in accordance with 3.3 and with the help of 3.13, the functions $F_{H,\mathfrak{A}}$, $F_{H_1,\mathfrak{A}}$, $F_{H_2,\mathfrak{A}}$. We consider further an arbitrary sequence of sets $X_1, \ldots, X_n \in S$ and put

2. $F_{H,\mathfrak{A}}(X_1, \ldots, X_n) = X$,
$F_{H_1,\mathfrak{A}}(X_1, \ldots, X_n, \ldots) = Y$,
$F_{H_2,\mathfrak{A}}(X_1, \ldots, X_n, \ldots) = Z$.

By 3.13(ii), 3.13(v), and from 3.17.1 and 3.17.2 we have $X = Y \Rightarrow (\overline{Y} \Rightarrow Z)$; and, by 2.3.11, $X = 1$; from 2.4, $X = 1 = A$. Thus we have $F_{H,\mathfrak{A}}(X_1, \ldots, X_n, \ldots) = A$, for all $X_1, \ldots, X_n, \ldots \in S$. Hence, by 3.3, $H \in \mathsf{E}(\mathfrak{A})$.

Lemma 4.2 *Let A be any algebra of species of subspecies of a given species A, and let H, H_1, H_2 be formulae of the intuitionistic propositional calculus such that $H = H_1 \rightarrow H_2$. If $H, H_1 \in \mathsf{E}(\mathfrak{A})$, then also $H_2 \in \mathsf{E}(\mathfrak{A})$. In other words, $\mathsf{E}(\mathfrak{A})$ is closed under the operation of detachment.*

Proof: In accordance with 3.3 and with the help of 3.13, we construct functions $F_{H,\mathfrak{A}}$, $F_{H_1,\mathfrak{A}}$, $F_{H_2,\mathfrak{A}}$. Then we have: for all species $X_1, \ldots, X_n, \ldots \epsilon S$

$$F_{H,\mathfrak{A}}(X_1, \ldots, X_n, \ldots) = F_{H_1,\mathfrak{A}}(X_1, \ldots, X_n, \ldots) \Rightarrow F_{H_2,\mathfrak{A}}(X_1, \ldots, X_n, \ldots)$$

and, since H_1, $H_2 \epsilon \mathsf{E}(\mathfrak{A})$,

$$F_{H,\mathfrak{A}}(X_1, \ldots, X_n, \ldots) = A = F_{H_1,\mathfrak{A}}(X_1, \ldots, X_n),$$

for all $X_1, \ldots, X_n, \ldots \epsilon S$, and since, by 2.2,

$$F_{H_2,\mathfrak{A}}(X_1, \ldots, X_n) = A \text{ for all } X_1, \ldots, X_n \epsilon S,$$

we have, by 3.3, $H_2 \epsilon \mathsf{E}(\mathfrak{A})$. Thus $\mathsf{E}(\mathfrak{A})$ is closed under the operation of detachment.

Theorem 4.5 *Let* IK *be the species of all provable formulae of the intuitionistic propositional calculus. Then for any species algebra* $\mathfrak{A}$, IK $\subseteq \mathsf{E}(\mathfrak{A})$.
[By 1.5, 4.2]

The following theorem was proved by Jaškowski (see [3] and [5]):

Theorem 4.6 *In order that* $H \epsilon$ IK *it is necessary and sufficient that* $H \epsilon \mathsf{E}(\mathsf{IK}_n)$ *for every natural number n, in other words:*

$$\bigcap_{n=1}^{\infty} \mathsf{E}(\mathsf{IK}_n) = \mathsf{IK}.$$

Corollary 4.7 *An algebra of species* $\mathfrak{A}$ *of all subspecies of any infinite species A is an adequate matrix for the system of all provable formulae of the intuitionistic propositional calculus.*

Proof: By 3.16, 4.6, 3.3(v), and 3.4.

Definition 4.8 Let $\mathfrak{A} = \langle S, A, \Rightarrow, \cup, \cap, - \rangle$ be an algebra of species and T a term of the algebra of species. The following formulae define (recursively) a function F_T, which correlates an element $F_{T,\mathfrak{A}}(X_1, \ldots, X_n, \ldots) \epsilon S$ with every infinite sequence of elements $X_1, \ldots, X_n \epsilon S$:

(i) $F_{T,\mathfrak{A}}(X_1, \ldots, X_n, \ldots) = X_p$ if $T = T_p (p = 1, 2, \ldots)$;
(ii) $F_{T,\mathfrak{A}}(X_1, \ldots, X_n, \ldots) = F_{T_1,\mathfrak{A}}(X_1, \ldots, X_n, \ldots) \Rightarrow F_{T_2,\mathfrak{A}}(X_1, \ldots, X_n, \ldots)$, if $T = T_1 \Rightarrow T_2$ (where T_1, and T_2 are terms);
(iii) and (iv) Analogously for the operations $\cup$ and $\cap$;
(v) $F_{T,\mathfrak{A}}(X_1, \ldots, X_n, \ldots) = (\overline{F_{T_1,\mathfrak{A}}(X_1, \ldots, X_n, \ldots)})$, if $T = \overline{T}_1$.

We say that a term T is verified by the algebra $\mathfrak{A}$, in symbol $T \epsilon \mathsf{E}(\mathfrak{A})$, if $F_{T,\mathfrak{A}}(X_1, \ldots, X_n, \ldots) = A$ for all $X_1, \ldots, X_n \epsilon S$.

Note 4.9 Definition 4.8 means that if the algebra of species is considered as a matrix, then the definition of verifiability of a term by $\mathfrak{A}$ is the same as the verifiability of a propositional formula by $\mathfrak{A}$, and the resulting theorems and corollaries in section 3 hold in both cases.

Definition 4.10 A formula $\mathfrak{H} = (T = 1)$ of the algebra of species is said to be valid in an algebra $\mathfrak{A}$, if T is verified by $\mathfrak{A}$.

Theorem 4.11 (i) φ *is a function which assigns a formula* $\mathfrak{H}$ *of the algebra of species to every formula H of the propositional calculus;* (ii) $\varphi(H) = \mathfrak{H} = (T = 1)$ *where T is a term obtained from H as follows: every propositional variable* p_i *and every propositional functor* $\rightarrow, \vee, \wedge, \sim$ *occurring in H is replaced by the corresponding element* a_i *and the corresponding species-algebraic operation* $\Rightarrow, \cup, \cap, -$.

Then under the above conditions φ *maps the formulae of the propositional calculus one-to-one onto the formulae of the algebra of species in such a way that a formula H of the intuitionistic propositional calculus is provable in the intuitionistic propositional calculus, if and only if the corresponding formula* $\mathfrak{H}$ *of the algebra of species is valid in every algebra of species* $\mathfrak{A}$ *of all subspecies of any infinite species A.*

Proof: Let H be a formula of the intuitionistic propositional calculus and $\mathfrak{A}$ an algebra of all subspecies of any infinite species. If H is provable in the intuitionistic propositional calculus then, by Corollary 4.7, $H \in E(\mathfrak{A})$ and, by 4.8, 4.1(i), and 4.1(ii), $\varphi(H) = \mathfrak{H}$ is valid in $\mathfrak{A}$. Conversely, if $\varphi(H) = \mathfrak{H}$ is valid in $\mathfrak{A}$, then, by 4.10, 4.8, and 3.3, $H \in E(\mathfrak{A})$ and, by 4.7, H is provable in the propositional calculus as required.

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