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A FIRST-ORDER LOGIC OF KNOWLEDGE AND BELIEF WITH IDENTITY. II

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Part I* presented a formal first-order, Gentzen-type system of the logic of knowledge and belief based closely upon the informal work of Hintikka [1]. In Part II, this system is shown to be semantically complete.

8 *Completeness: Preliminary Definitions* Formula-sequences can be regarded as finite sequences of formulae, and sequents as ordered pairs of such sequences. In what follows we shall accordingly deal with sequences rather than with expressions, though we shall retain the expression symbolism already introduced.

If σ is a finite sequence of length n (an n-tuple) and $0 \leqslant k < n$, then $(\sigma)_k$ shall be the $k + 1$'st element of σ; if α and β are such that $((\sigma)_\alpha)_\beta$ is defined, we write $(\sigma)_{\alpha*\beta} = ((\sigma)_\alpha)_\beta$; as before, $\alpha*\beta*\gamma = (\alpha*\beta)*\gamma$. Let $\mathrm{D}(\Gamma, i) \equiv \forall j[([j < i] \supset [((\Gamma)_{j*0} = \mathbf{K}) \vee ((\Gamma)_{j*0} = \mathbf{B})]) \;\&\; ([i \leqslant j] \supset [((\Gamma)_{j*0} \neq \mathbf{K}) \;\&\; ((\Gamma)_{j*0} \neq \mathbf{B})])]$. If $X = \{\Gamma | \exists i\, \mathrm{D}(\Gamma, i)\}$, define $\delta: X \to \omega$ by $\delta(\Gamma) = i$ iff $\mathrm{D}(\Gamma, i)$. Let $Y = \{S | -\exists i \exists j[(S)_{0*i} = (S)_{1*j}] \;\&\; -\exists i[((S)_{1*i*0} = \mathsf{I}) \;\&\; ((S)_{1*i*1} = (S)_{1*i*2})]\}$ and $Z = \{S \in Y | ((S)_0 \in X) \;\&\; ((S)_1 \in X)\}$. Finally, if Γ has length $n + 1$ and $0 \leqslant j \leqslant n$, then $\Phi(\Gamma, i)$ is the sequence of formulae defined by $\forall k[([k < i] \supset [(\Gamma)_k = (\Phi(\Gamma, i))_k]) \;\&\; ([i \leqslant k] \supset [(\Gamma)_{k+1} = (\Phi(\Gamma, i))_k])]$.

We now define a number of functions f from sequents to sequents such that $f(S)$ is related to S as premiss to conclusion by a rule of inference, modulo applications of the enabling rules. These functions will be used to construct the 'proof trees' used in the completeness argument. We assume throughout this section that a_k is the k'th free individual variable.

1. f_0 shall be the identity function on the set of sequents.
2. If $Z_{\mathbf{N},0} = \{S \in Z | \exists i((S)_{0*i*0} = \mathbf{N})\}$, then $f_{\mathbf{N},0}$ is defined on $Z_{\mathbf{N},0}$ by $f_{\mathbf{N},0}(S) = \langle \Phi((S)_0, i), \Gamma\rangle$, where

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a. $i = \mu j((S)_{0*j*0} = \mathbf{N})$.
b. Γ is defined by

i. if $j < \delta((S)_1)$, then $(\Gamma)_j = (S)_{1*j}$,
ii. $(\Gamma)_{\delta((S)_1)} = (S)_{0*i*1}$,
iii. if $\delta((S)_1) \leq j$, then $(\Gamma)_{j+1} = (S)_{1*j}$.

3. If $Z_{\mathbf{N},1} = \{S \in Z \mid \exists i((S)_{1*i*0} = \mathbf{N})\}$, then $f_{\mathbf{N},1}$ is defined on $Z_{\mathbf{N},1}$ by $f_{\mathbf{N},1}(S) = \langle \Gamma, \Phi((S)_1, i)\rangle$, where

a. $i = \mu j((S)_{1*j*0} = \mathbf{N})$.
b. Γ is defined by

i. if $j < \delta((S)_0)$, then $(\Gamma)_j = (S)_{0*j}$,
ii. $(\Gamma)_{\delta((S)_0)} = (S)_{1*i*1}$,
iii. if $\delta((S)_0) \leq j$, then $(\Gamma)_{j+1} = (S)_{0*j}$.

4. If $Z_{\mathbf{C},0} = \{S \in Z \mid \exists i((S)_{0*i*0} = \mathbf{C})\}$, then $f^0_{\mathbf{C},0}$ and $f^1_{\mathbf{C},0}$ are defined on $Z_{\mathbf{C},0}$ by $f^0_{\mathbf{C},0}(S) = \langle \Gamma_0, (S)_1\rangle$ and $f^1_{\mathbf{C},0}(S) = (\Phi((S)_0, i), \Gamma_1)$, where

a. $i = \mu j((S)_{0*j*0} = \mathbf{C})$.
b. Γ_0 is defined by

i. if $j < \delta((S)_0)$, then $(\Gamma_0)_j = (S)_{0*j}$,
ii. $(\Gamma_0)_{\delta((S)_0)} = (S)_{0*i*2}$,
iii. if $\delta((S)_0) \leq j$, then $(\Gamma_0)_{j+1} = (\Phi((S)_0, i))_j$.

c. Γ_1 is defined by

i. if $j < \delta((S)_1)$, then $(\Gamma_1)_j = (S)_{1*j}$,
ii. $(\Gamma_1)_{\delta((S)_1)} = (S)_{0*i*1}$,
iii. if $\delta((S)_1) \leq j$, then $(\Gamma_1)_{j+1} = (S)_{1*j}$.

5. If $Z_{\mathbf{C},1} = \{S \in Z \mid \exists i((S)_{1*i*0} = \mathbf{C})\}$, then $f_{\mathbf{C},1}$ is defined on $Z_{\mathbf{C},1}$ by $f_{\mathbf{C},1}(S) = \langle \Gamma_0, \Gamma_1\rangle$, where

a. $i = \mu j((S)_{1*j*0} = \mathbf{C})$.
b. Γ_0 is defined by

i. if $j < \delta((S)_0)$, then $(\Gamma_0)_j = (S)_{0*j}$,
ii. $(\Gamma_0)_{\delta((S)_0)} = (S)_{1*i*1}$,
iii. if $\delta((S)_0) \leq j$, then $(\Gamma_0)_{j+1} = (S)_{0*j}$.

c. Γ_1 is defined by

i. if $j < \delta((S)_1)$, then $(\Gamma_1)_j = (S)_{1*j}$,
ii. $(\Gamma_1)_{\delta((S)_1)} = (S)_{1*i*2}$,
iii. if $\delta((S)_1) \leq j$, then $(\Gamma_1)_{j+1} = (\Phi((S)_1, i))_j$.

6. If $Z_{\mathbf{E},0} = \{S \in Z \mid \exists i((S)_{0*i*0} = \mathbf{E})\}$, then for each free individual variable a, $f_{\mathbf{E},0}(a)$ is defined on $Z_{\mathbf{E},0}$ by $f_{\mathbf{E},0}(a)(S) = \langle \Gamma, (S)_1\rangle$, where

a. $i = \mu j((S)_{0*j*0} = \mathbf{E})$.
b. Γ is defined by

i. if $j < \delta((S)_0)$, then $(\Gamma)_j = (S)_{0*j}$,
ii. $(\Gamma)_{\delta((S)_0)} = (S)_{0*i*2}(a/(S)_{0*i*1})$,
iii. $(\Gamma)_{\delta((S)_0)+1} = \mathsf{P}a$,
iv. if $\delta((S)_0) \leqslant j$, then $(\Gamma)_{j+2} = (\Phi((S)_0, i))_j$.

7. Let $\Gamma \in I_{\mathbf{E}}(n)$ iff there exists a strictly increasing function $\iota_{\mathbf{E}}(\Gamma): n \to \omega$ such that (i) if $i \in n$, then $(\Gamma)_{\iota_{\mathbf{E}}(\Gamma)(i)*0} = \mathbf{E}$, and (ii) if $(\Gamma)_{i*0} = \mathbf{E}$, then $i = \iota_{\mathbf{E}}(\Gamma)(j)$ for some $j \in n$. Obviously $\iota_{\mathbf{E}}(\Gamma)$ is unique. If $Z_{\mathbf{E},1,m} = \{S \in Z \mid (S)_1 \in I_{\mathbf{E}}(m)\}$ then for each free individual variable a, $f^0_{\mathbf{E},1,n}(a)$ and $f^1_{\mathbf{E},1,n}(a)$ are defined on $\sum_{n \in m} Z_{\mathbf{E},1,m}$ by $f^0_{\mathbf{E},1,n}(a)(S) = ((S)_0, \Gamma_0)$ and $f^1_{\mathbf{E},1,n}(a)(S) = \langle (S)_0, \Gamma_1 \rangle$, where

a. $i = \iota_{\mathbf{E}}((S)_1)(n)$.
b. Γ_0 is defined by

i. if $j < \delta((S)_1)$, then $(\Gamma_0)_j = (S)_{1*j}$,
ii. $(\Gamma_0)_{\delta((S)_1)} = (S)_{1*i*2}(a/(S)_{1*i*1})$,
iii. if $\delta((S)_1) \leqslant j$, then $(\Gamma_0)_{j+1} = (S)_{1*j}$.

c. Γ_1 is defined by

i. if $j < \delta((S)_1)$, then $(\Gamma_1)_j = (S)_{1*j}$,
ii. $(\Gamma_1)_{\delta((S)_1)} = \mathsf{P}a$,
iii. if $\delta((S)_1) \leqslant j$, then $(\Gamma_1)_{j+1} = (S)_{1*j}$.

8. If $k \neq j$ and $Z_{\mathsf{I},k,j} = \{S \in Z \mid \exists i(\mathsf{I}a_k a_j = (S)_{0*i})\}$, then $f_{\mathsf{I},k,j}$ is defined on $Z_{\mathsf{I},k,j}$ by $f_{\mathsf{I},k,j}(S) = (\Gamma, (S)_1)$, where Γ is defined by

a. if $i < \delta((S)_0)$, then $(\Gamma)_i = (S)_{0*i}$.
b. $(\Gamma)_{\delta((S)_0)} = \mathsf{I}a_j a_k$.
c. if $\delta((S)_0) \leqslant i$, then $(\Gamma)_{i+1} = (S)_{0*i}$.

9. Let $\Gamma \in I_{F,i,k}(n)$ iff there exists a strictly increasing function $\iota_{F,i,k}(\Gamma): n \to \omega$ such that (i) if $m \in n$, then $(\Gamma)_{\iota_{F,i,k}(\Gamma)(m)}$ is atomic and $(\Gamma)_{\iota_{F,i,k}(\Gamma)(m)*i} = a_k$, and (ii) if $(\Gamma)_p$ is atomic and $(\Gamma)_{p*i} = a_k$, then $p = \iota_{F,i,k}(\Gamma)(m)$ for some $m \in n$. Obviously $\iota_{F,i,k}(\Gamma)$ is unique. If $Z_{F(i,k,j),0,m} = \{S \in Z_{\mathsf{I},k,j} \mid (S)_0 \in I_{F,i,k}(m)\}$, then $f_{F(i,k,j),0,n}$ is defined on $\sum_{n \in m} Z_{F(i,k,j),0,m}$ by $f_{F(i,k,j),0,n}(S) = \langle \Gamma, (S)_1 \rangle$, where

a. $p = \iota_{F,i,k}((S)_0)(n)$.
b. Γ is defined by

i. if $q < \delta((S)_0)$, then $(\Gamma)_q = (S)_{0*q}$,
ii. $((\Gamma)_{\delta((S)_0)})_i = a_j$, and if $q \neq i$, $((\Gamma)_{\delta((S)_0)})_q = (S)_{0*p*q}$,
iii. if $\delta((S)_0) \leqslant q$, then $(\Gamma)_{q+1} = (S)_{0*q}$.

10. If $Z_{F(i,k,j),1,m} = \{S \in Z_{\mathsf{I},k,j} \mid (S)_1 \in I_{F,i,k}(m)\}$, then $f_{F(i,k,j),1,n}$ is defined on $\sum_{n \in m} Z_{F(i,k,j),1,m}$ by $f_{F(i,k,j),1,n}(S) = \langle (S)_0, \Gamma \rangle$, where

a. $p = \iota_{F,i,k}((S)_1)(n)$.
b. Γ is defined by

i. if $q < \delta((S)_1)$, then $(\Gamma)_q = (S)_{1*q}$,

ii. $((\Gamma)_{\delta((S)_1)})_i = a_j$, and if $q \neq i$, $((\Gamma)_{\delta((S)_1)})_q = (S)_{1*p*q}$,
iii. if $\delta((S)_1) \leq q$, then $(\Gamma)_{q+1} = (S)_{1*q}$.

11. Let $\Gamma \in I_{\mathbf{K},k}(n)$ iff there exists a strictly increasing function $\iota_{\mathbf{K},k}(\Gamma):n \to \omega$ such that (i) if $i \in n$, then $(\Gamma)_{\iota_{\mathbf{K},k}(\Gamma)(i)*0} = \mathbf{K}$ and $(\Gamma)_{\iota_{\mathbf{K},k}(\Gamma)(i)*1} = a_k$, and (ii) if $(\Gamma)_{i*0} = \mathbf{K}$ and $(\Gamma)_{i*1} = a_k$, then $i = \iota_{\mathbf{K},k}(\Gamma)(j)$ for some $j \in n$. Obviously $\iota_{\mathbf{K},k}(\Gamma)$ is unique. If $Z_{\mathbf{K}(k,j),0,m} = \{S \in Z_{\mathbf{I},k,j} \mid (S)_0 \in I_{\mathbf{K},k}(m)\}$, then $f_{\mathbf{K}(k,j),0,n}$ is defined on $\sum_{n \in m} Z_{\mathbf{K}(k,j),0,m}$ by $f_{\mathbf{K}(k,j),0,n}(S) = \langle \Gamma, (S)_1 \rangle$, where

a. $p = \iota_{\mathbf{K},k}((S)_0)(n)$.
b. Γ is defined by

i. if $q < \delta((S)_0)$, then $(\Gamma)_q = (S)_{0*q}$,
ii. $((\Gamma)_{\delta((S)_0)})_0 = \mathbf{K}$, $((\Gamma)_{\delta((S)_0)})_1 = a_j$, and $((\Gamma)_{\delta((S)_0)})_2 = (S)_{0*p*2}$,
iii. if $\delta((S)_0) \leq q$, then $(\Gamma)_{q+1} = (S)_{0*q}$.

12. If $Z_{\mathbf{K}(k,j),1,m} = \{S \in Z_{\mathbf{I},k,j} \mid (S)_1 \in I_{\mathbf{K},k}(m)\}$, then $f_{\mathbf{K}(k,j),1,n}$ is defined on $\sum_{n \in m} Z_{\mathbf{K}(k,j),1,m}$ by $f_{\mathbf{K}(k,j),1,n}(S) = \langle (S)_0, \Gamma \rangle$, where

a. p = $\iota_{\mathbf{K},k}((S)_1)(n)$.
b. Γ is defined by

i. if $q < \delta((S)_1)$, then $(\Gamma)_q = (S)_{1*q}$,
ii. $((\Gamma)_{\delta((S)_1)})_0 = \mathbf{K}$, $((\Gamma)_{\delta((S)_1)})_1 = a_j$, and $((\Gamma)_{\delta((S)_1)})_2 = (S)_{1*p*2}$,
iii. if $\delta((S)_1) \leq q$, then $(\Gamma)_{q+1} = (S)_{1*q}$.

13. Replacing '$\mathbf{K}$' by '$\mathbf{B}$' in (11), we obtain a definition of $f_{\mathbf{B}(k,j),0,n}$.
14. Replacing '$\mathbf{K}$' by '$\mathbf{B}$' in (12), we obtain a definition of $f_{\mathbf{B}(k,j),1,n}$.
15. Let $\Gamma \in I_{\mathbf{K}}(n)$ iff there exists a strictly increasing function $\iota_{\mathbf{K}}(\Gamma):n \to \omega$ such that (i) if $i \in n$, then $(\Gamma)_{\iota_{\mathbf{K}}(\Gamma)(i)*0} = \mathbf{K}$, and (ii) if $(\Gamma)_{i*0} = \mathbf{K}$, then $i = \iota_{\mathbf{K}}(\Gamma)(j)$ for some $j \in n$. Obviously $\iota_{\mathbf{K}}(\Gamma)$ is unique. If $Z_{\mathbf{K},0,m} = \{S \in Z \mid (S)_0 \in I_{\mathbf{K}}(m)\}$, then $f_{\mathbf{K},0,n}$ is defined on $\sum_{n \in m} Z_{\mathbf{K},0,m}$ by $f_{\mathbf{K},0,n}(S) = \langle \Gamma, (S)_1 \rangle$, where

a. $p = \iota_{\mathbf{K}}((S)_0)(n)$.
b. Γ is defined by

i. if $q < \delta((S)_0)$, then $(\Gamma)_q = (S)_{0*q}$,
ii. $(\Gamma)_{\delta((S)_0)} = (S)_{0*p*2}$,
iii. if $\delta((S)_0) \leq q$, then $(\Gamma)_{q+1} = (S)_{0*q}$.

16. If $Z_{\mathbf{K},k,1,m} = \{S \in Z \mid (S)_1 \in I_{\mathbf{K},k}(m)\}$, then $f_{\mathbf{K},k,1,n}$ is defined on $\sum_{n \in m} Z_{\mathbf{K},k,1,m}$ by $f_{\mathbf{K},k,1,n}(S) = \langle \Gamma, ((S)_{1*i*2}) \rangle$, where

a. $i = \iota_{\mathbf{K},k}((S)_1)(n)$.
b. $(\Gamma)_q = (S)_{0*\iota_{\mathbf{K},k}((S)_0)(q)}$.

17. Let $n^0_{\mathbf{B},\mathbf{K},k}(\Gamma) \in \omega$ be such that $\iota^0_{\mathbf{B},\mathbf{K},k}(\Gamma):n^0_{\mathbf{B},\mathbf{K},k}(\Gamma) \to \omega$ is a strictly increasing function such that (i) if $j \in n^0_{\mathbf{B},\mathbf{K},k}(\Gamma)$, then $(\Gamma)_{\iota^0_{\mathbf{B},\mathbf{K},k}(\Gamma)(j)*0} = \mathbf{K}$ or $(\Gamma)_{\iota^0_{\mathbf{B},\mathbf{K},k}(\Gamma)(j)*0} = \mathbf{B}$, and $(\Gamma)_{\iota^0_{\mathbf{B},\mathbf{K},k}(\Gamma)(j)*1} = a_k$, and (ii) if $(\Gamma)_{i*0} = \mathbf{K}$ or $(\Gamma)_{i*0} = \mathbf{B}$, and $(\Gamma)_{i*1} = a_k$, then $i = \iota^0_{\mathbf{B},\mathbf{K},k}(\Gamma)(j)$ for some $j \in n^0_{\mathbf{B},\mathbf{K},k}(\Gamma)$. Let $n^1_{\mathbf{B},\mathbf{K},k}(\Gamma) \in \omega$ be

such that $\iota^1_{\mathbf{B},\mathbf{K},k}(\Gamma): n^1_{\mathbf{B},\mathbf{K},k}(\Gamma) \to \omega$ is a strictly increasing function such that (i) if $j \in n^1_{\mathbf{B},\mathbf{K},k}(\Gamma)$, then $(\Gamma)_{\iota^1_{\mathbf{B},\mathbf{K},k}(\Gamma)(j)*0} \neq \mathbf{K}$ and $(\Gamma)_{\iota^1_{\mathbf{B},\mathbf{K},k}(\Gamma)(j)*0} \neq \mathbf{B}$, or $(\Gamma)_{\iota^1_{\mathbf{B},\mathbf{K},k}(\Gamma)(j)*1} \neq a_k$, and (ii) if $(\Gamma)_{i*0} \neq \mathbf{K}$ and $(\Gamma)_{i*0} \neq \mathbf{B}$, or $(\Gamma)_{i*1} \neq a_k$, then $i = \iota^1_{\mathbf{B},\mathbf{K},k}(\Gamma)(j)$ for some $j \in n^1_{\mathbf{B},\mathbf{K},k}(\Gamma)$. Obviously $\iota^0_{\mathbf{B},\mathbf{K},k}(\Gamma)$ and $\iota^1_{\mathbf{B},\mathbf{K},k}(\Gamma)$ are unique. If $Z_{\mathbf{B},k,0,n} = \{S \in Z \mid (S)_0 \in I_{\mathbf{B},k}(n)\}$, then $f_{\mathbf{B},k,0}$ is defined on $\sum_{0 \in n} Z_{\mathbf{B},k,0,n}$ by $f_{\mathbf{B},k,0}(S) = \langle \Gamma, \emptyset \rangle$, where

a. Γ' is defined by

i. if $q < n^0_{\mathbf{B},\mathbf{K},k}((S)_0)$, then $(\Gamma')_q = (S)_{0*\iota^0_{\mathbf{B},\mathbf{K},k}((S)_0)(q)}$,
ii. $(\Gamma')_{n^0_{\mathbf{B},\mathbf{K},k}((S)_0)+q} = (S)_{0*\iota_{\mathbf{B},k}((S)_0)(q)*2}$.

b. Γ is defined by

i. if $q < n^0_{\mathbf{B},\mathbf{K},k}(\Gamma')$, then $(\Gamma)_q = (\Gamma')_{\iota^0_{\mathbf{B},\mathbf{K},k}(\Gamma')(q)}$,
ii. $(\Gamma)_{n^0_{\mathbf{B},\mathbf{K},k}(\Gamma')+q} = (\Gamma')_{\iota^1_{\mathbf{B},\mathbf{K},k}(\Gamma')(q)}$.

18. If $Z_{\mathbf{B},k,1,m} = \{S \in Z \mid (S)_1 \in I_{\mathbf{B},k}(m)\}$, then $f_{\mathbf{B},k,1,n}$ is defined on $\sum_{n \in m} Z_{\mathbf{B},k,1,m}$ by $f_{\mathbf{B},k,1,n}(S) = \langle \Gamma, ((S)_{1*p*2}) \rangle$, where

a. $p = \iota_{\mathbf{B},k}((S)_1)(n)$.
b. Γ is defined as in (17).

$\mathcal{W}$*-functions* are those defined by (1), (16), (17), and (18); $\mathcal{B}$*-functions* are those defined by (4) and (7). $\mathcal{B}$-functions are said to *correspond* if their names differ only in superscript; f^0 and f^1 will be used to represent corresponding $\mathcal{B}$-functions with the obvious intent.

Let σ be a finite sequence of functions defined by (1) through (18), say $\sigma = \langle g_0, \ldots, g_n \rangle$. If $n > 0$, $\mathrm{pd}(\sigma) = (g_0, \ldots, g_{n-1})$. $\sigma(S) = g_n(\ldots g_0(S) \ldots)$, provided the functions g_i are all defined at the indicated arguments. $\mathbf{s}_1(\sigma)$ is the subsequence of σ consisting of the $\mathcal{W}$- and $\mathcal{B}$-functions of σ. $\mathbf{s}_2(\sigma)$ is the subsequence of σ consisting of the $\mathcal{W}$-functions of σ. $\mathbf{e}_1(\sigma)$ is the subsequence of σ consisting of the functions following the last $\mathcal{W}$- or $\mathcal{B}$-function of σ. $\mathbf{e}_2(\sigma)$ is the subsequence of σ consisting of the functions following the last $\mathcal{W}$-function of σ. $\mathrm{pd}_2(\sigma)$ is the initial segment τ of σ such that $\mathbf{e}_2(\tau) = \emptyset$ and $\mathbf{s}_2(\tau) = \mathbf{s}_2(\sigma)$. $\mathbf{r}(\sigma)$ is the set of initial segments of σ. $\mathbf{r}'(\sigma)$ is the set of proper initial segments of σ. $\mathbf{t}(\sigma)$ is the set of sequences differing from σ at most by insertion of finite sequences of $\mathcal{B}$-functions between $\mathcal{B}$-functions and immediately following $\mathcal{W}$-functions or between $\mathcal{W}$-functions and immediately following $\mathcal{W}$-functions. $\tau_1 * \tau_2 \in \mathbf{t}'(\sigma)$ iff $\tau_1 \in \mathbf{t}(\sigma)$ and τ_2 is a finite sequence of $\mathcal{B}$-functions.

9 *Completeness: The Basic Construction* Our aim is to construct, from a given unprovable sequent S, a kb'-model system Ω such that $|S| \subset \mu$ for some $\mu \in \Omega$. In view of Theorems 1 and 3 this will suffice to establish the semantic completeness of $\mathcal{J}\langle \mathbf{K}, \mathbf{B} \rangle$. The classical construction must be ramified to take account of the operators $\mathbf{K}$ and $\mathbf{B}$; essentially we must carry out the usual construction in each possible world. Instead of constructing a single 'tree' we must construct a series of successively better approximations thereto from which we derive the desired model system.

We now define inductively, for any given sequent S, a sequence of sets $\mathbf{P}_n(S)$ of finite sequences of the functions defined by (1) through (18) of section **8**. The construction requires that we simultaneously define, for each $\sigma \in \mathbf{P}_n(S)$, two registers $\mathbf{V}_n(S)(\sigma)$ and $\mathbf{V}'_n(S)(\sigma)$ of free individual variables and a well-ordering $\mathbf{R}_n(S)(\sigma)$ of $\mathbf{V}'_n(S)(\sigma)$. We shall also need to establish along the way that $\mathbf{P}_n(S)$ is finite and that for each $\sigma \in \mathbf{P}_n(S)$, $\mathbf{V}_n(S)(\sigma)$ and $\mathbf{V}'_n(S)(\sigma)$ are finite. To simplify the notation we shall suppress the reference to S.

I. Construction of $\mathbf{P}_0$, $\mathbf{V}_0$, $\mathbf{V}'_0$, and $\mathbf{R}_0$.

$\mathbf{P}_0 = \{\langle f_0\rangle\}$
$\mathbf{V}_0(\langle f_0\rangle) = \mathbf{V}'_0(\langle f_0\rangle) = \mathbf{v}(|S|)$
$\mathbf{R}_0(\langle f_0\rangle)$ is the alphabetical order of the free individual variables restricted to $\mathbf{V}'_0(\langle f_0\rangle)$.

Obviously $\mathbf{P}_0$ is finite and if $\sigma \in \mathbf{P}_0$, then $\mathbf{V}_0(\sigma)$ and $\mathbf{V}'_0(\sigma)$ are finite.

II. Construction of $\mathbf{P}_{n+1}$, $\mathbf{V}_{n+1}$, $\mathbf{V}'_{n+1}$, and $\mathbf{R}_{n+1}$. Suppose that we have constructed $\mathbf{P}_n$, $\mathbf{V}_n$, $\mathbf{V}'_n$, and $\mathbf{R}_n$; suppose that $\mathbf{P}_n$ is finite and if $\sigma \in \mathbf{P}_n$, then $\mathbf{V}_n(\sigma)$ and $\mathbf{V}'_n(\sigma)$ are finite. Let $\mathbf{V}_n = \sum_{\sigma\in\mathbf{P}_n} \mathbf{V}_n(\sigma)$; since $\mathbf{P}_n$ is finite and $\mathbf{V}_n(\sigma)$ is finite for each $\sigma \in \mathbf{P}_n$, $\mathbf{V}_n$ is also finite. We now define $\mathbf{P}_{n+1}$ inductively.

A. The initial element.

$$\langle f_0\rangle \in \mathbf{P}_{n+1}$$
$$\mathbf{V}_{n+1}(\langle f_0\rangle) = \mathbf{V}_n$$
$$\mathbf{V}'_{n+1}(\langle f_0\rangle) = \mathbf{v}(|S|)$$
$$\mathbf{R}_{n+1}(\langle f_0\rangle) = \mathbf{R}_0(\langle f_0\rangle)$$

Obviously $\mathbf{V}_{n+1}(\langle f_0\rangle)$ and $\mathbf{V}'_{n+1}(\langle f_0\rangle)$ are finite.

B. The inductive step. We suppose that $\sigma \in \mathbf{P}_{n+1}$ and that $\mathbf{V}_{n+1}(\sigma)$ and $\mathbf{V}'_{n+1}(\sigma)$ are finite.

1. Suppose $\mathbf{e}_2(\sigma) = \emptyset$ and suppose there exists $\tau \in \mathbf{P}_n$ such that $\mathbf{e}_2(\tau) \neq \emptyset$ and $\mathbf{s}_1(\sigma * \mathbf{e}_2(\tau)) \in \mathbf{t}(\mathbf{s}_1(\tau))$. Then for any such τ:

$\sigma * \mathbf{e}_2(\tau) \in \mathbf{P}_{n+1}$
$\mathbf{V}_{n+1}(\sigma * \mathbf{e}_2(\tau)) = \mathbf{V}_{n+1}(\sigma)$
$\mathbf{V}'_{n+1}(\sigma * \mathbf{e}_2(\tau)) = \mathbf{V}'_{n+1}(\sigma) \cup \mathbf{V}_n(\tau)$
$\langle a_1, a_2\rangle \in \mathbf{R}_{n+1}(\sigma * \mathbf{e}_2(\tau))$ iff $[(a_1 \in \mathbf{V}'_n(\tau))$ & $(a_2 \in \mathbf{V}'_n(\tau))$ & $(\langle a_1, a_2\rangle \in \mathbf{R}_n(\tau))] \vee [(a_1 \in (\mathbf{V}'_{n+1}(\sigma) - \mathbf{V}'_n(\tau)))$ & $(a_2 \in (\mathbf{V}'_{n+1}(\sigma) - \mathbf{V}'_n(\tau)))$ & $(\langle a_1, a_2\rangle \in \mathbf{R}_{n+1}(\sigma))] \vee [(a_1 \in \mathbf{V}'_n(\tau))$ & $(a_2 \in (\mathbf{V}'_{n+1}(\sigma) - \mathbf{V}'_n(\tau)))]$.

Obviously $\mathbf{V}_{n+1}(\sigma * \mathbf{e}_2(\tau))$ and $\mathbf{V}'_{n+1}(\sigma * \mathbf{e}_2(\tau))$ are finite.

2. Suppose either (a) $\mathbf{e}_2(\sigma) = \emptyset$, $n \neq 0$, there is no $\tau \in \mathbf{P}_n$ such that $\mathbf{e}_2(\tau) \neq \emptyset$ and $\mathbf{s}_1(\sigma * \mathbf{e}_2(\tau)) \in \mathbf{t}(\mathbf{s}_1(\tau))$, and there is some $\tau \in \mathbf{P}_n$ such that $\mathbf{s}_1(\mathrm{pd}(\sigma)) \in \mathbf{t}'(\mathbf{s}_1(\tau))$,

or (b) there is some $\tau \in \mathbf{P}_n$ such that $\mathbf{s}_1(\sigma) \in \mathbf{t}(\mathbf{s}_1(\tau))$, $\mathbf{e}_2(\sigma) = \mathbf{e}_2(\tau)$, and if $\rho \in \mathbf{P}_n$ and $\tau \in \mathbf{r}'(\rho)$, then $\mathbf{s}_2(\rho) \neq \mathbf{s}_2(\tau)$. The construction in this case is described in stages.

a. The propositional construction. Partially order the functions defined by (2), (3), (4), and (5) of section **8** according to $f_{\mathbf{N},0} < f_{\mathbf{N},1} < f_{\mathbf{C},1}$, $f_{\mathbf{C},1} < f^0_{\mathbf{C},0}$, and $f_{\mathbf{C},1} < f^1_{\mathbf{C},0}$. If $\{g_j\}$ is a sequence of these functions such that (i) g_0 is a least function defined at $\sigma(S)$, and (ii) for each j, g_{j+1} is a least function defined at $\sigma * g_0 * \ldots * g_j(S)$, then for each j:

$$\sigma * g_0 * \ldots * g_j \in \mathbf{P}_{n+1}$$
$$\mathbf{V}_{n+1}(\sigma * g_0 * \ldots * g_j) = \mathbf{V}_{n+1}(\sigma)$$
$$\mathbf{V}'_{n+1}(\sigma * g_0 * \ldots * g_j) = \mathbf{V}'_{n+1}(\sigma)$$
$$\mathbf{R}_{n+1}(\sigma * g_0 * \ldots * g_j) = \mathbf{R}_{n+1}(\sigma).$$

If the complexity $\mathbf{c}(S)$ of a sequent S is defined as the sum of the complexities of the component formulae, each instance of a formula counting as a formula, then since $\mathbf{c}(\sigma * g_0 * \ldots * g_{j+1}(S)) < \mathbf{c}(\sigma * g_0 * \ldots * g_j(S))$, $\mathbf{c}(\sigma(S))$ is an upper bound on the length of the sequences $\{g_j\}$, so $\mathbf{c}(\sigma(S)) \cdot 2^{\mathbf{c}(\sigma(S))}$ is an upper bound on their number. Therefore the propositional construction adds only a finite number of elements to $\mathbf{P}_{n+1}$.

b. The quantifier construction.

i. Let τ be a maximal element resulting from (a), i.e., there is no element ρ added to $\mathbf{P}_{n+1}$ by (a) such that $\tau \in \mathbf{r}'(\rho)$. Let $\{a_k\}$ be the sequence of free individual variables such that (α) a_0 is the first free individual variable not in $\mathbf{V}_{n+1}(\tau)$, (β) for each k, a_{k+1} is the first free individual variable not in $\mathbf{V}_{n+1}(\tau) \cup \{a_j \mid j \leq k\}$, ($\gamma$) $f_{\mathbf{E},0}(a_0)$ is defined at $\tau(S)$, and (δ) for each k, $f_{\mathbf{E},0}(a_{k+1})$ is defined at $\tau * f_{\mathbf{E},0}(a_0) * \ldots * f_{\mathbf{E},0}(a_k)(S)$. Since $\mathbf{V}_{n+1}(\tau)$ is assumed finite, the length of $\{a_j\}$ is precisely $\sum_i n(i)$, where $n(i)$ is the number of initially placed existential quantifiers in $(\tau(S))_{0*i}$. Then for each k:

$$\tau * f_{\mathbf{E},0}(a_0) * \ldots * f_{\mathbf{E},0}(a_k) \in \mathbf{P}_{n+1}$$
$$\mathbf{V}_{n+1}(\tau * f_{\mathbf{E},0}(a_0) * \ldots * f_{\mathbf{E},0}(a_k)) = \mathbf{V}_{n+1}(\tau) \cup \{a_j \mid j \leq k\}$$
$$\mathbf{V}'_{n+1}(\tau * f_{\mathbf{E},0}(a_0) * \ldots * f_{\mathbf{E},0}(a_k)) = \mathbf{V}'_{n+1}(\tau) \cup \{a_j \mid j \leq k\}$$
$$\langle a, a'\rangle \in \mathbf{R}_{n+1}(\tau * f_{\mathbf{E},0}(a_0) * \ldots * f_{\mathbf{E},0}(a_k)) \text{ iff } [(a \in \mathbf{V}'_{n+1}(\tau)) \;\&\; (a' \in \mathbf{V}'_{n+1}(\tau)) \;\&\; (\langle a, a'\rangle \in \mathbf{R}_{n+1}(\tau))] \vee [(a \in \mathbf{V}'_{n+1}(\tau)) \;\&\; \exists i(a' = a_i)] \vee \exists i \exists j[(a = a_i) \;\&\; (a' = a_j) \;\&\; (i < j)].$$

Since $\{a_k\}$ is finite, we add only a finite number of elements to $\mathbf{P}_{n+1}$ and only a finite number of elements to $\mathbf{V}_{n+1}(\tau)$ and $\mathbf{V}'_{n+1}(\tau)$ to obtain $\mathbf{V}_{n+1}(\tau * f_{\mathbf{E},0}(a_0) * \ldots * f_{\mathbf{E},0}(a_k))$ and $\mathbf{V}'_{n+1}(\tau * f_{\mathbf{E},0}(a_0) * \ldots * f_{\mathbf{E},0}(a_k))$.

ii. Let τ be a maximal element resulting from (i). Partially order the functions $f^0_{\mathbf{E},1,n}(a)$ and $f^1_{\mathbf{E},1,n}(a)$ with $n \in \omega$ and $a \in \mathbf{V}'_{n+1}(\tau)$ as follows: if $p \in 2$, $p' \in 2$, $a \in \mathbf{V}'_{n+1}(\tau)$, $a' \in \mathbf{V}'_{n+1}(\tau)$, $m \in \omega$, and $m' \in \omega$, then $f^p_{\mathbf{E},1,m}(a) < f^{p'}_{\mathbf{E},1,m'}(a')$ iff $m < m'$ or $m = m'$ and $\langle a, a'\rangle \in \mathbf{R}_{n+1}(\tau)$. Let $\{g_k\}$ be a sequence of these functions such that (α) g_0 is a least function $f^p_{\mathbf{E},1,m}(a)$ defined at $\tau(S)$ such that $\mathrm{D}(\tau, m, a, \tau, S)$, and ($\beta$) for each k, g_{k+1} is a least function $f^p_{\mathbf{E},1,m}(a)$

defined at $\tau * g_0 * \ldots * g_k(S)$ such that $\mathrm{D}(\tau * g_0 * \ldots * g_k, m, a, \tau, S)$, where $\mathrm{D}(\rho, m, a, \tau, S) \equiv -\exists \tau' \exists p' \exists m' [(p' \in 2) \,\&\, (m' \in \omega) \,\&\, (\tau' * f^{p'}_{\mathbf{E},1,m'}(a) \in \mathbf{r}(\rho)) \,\&\, (\mathbf{s}_2(\tau') = \mathbf{s}_2(\tau)) \,\&\, ((\tau'(S))_{1*\iota\mathbf{E}((\tau'(S))_1)(m')*2}(a/(\tau'(S))_{1*\iota\mathbf{E}((\tau'(S))_1)(m')*1}) = (\rho(S))_{1*\iota\mathbf{E}((\rho(S))_1)(m)*2}(a/(\rho(S))_{1*\iota\mathbf{E}((\rho(S))_1)(m)*1}))]$. Then for each k:

$$\begin{aligned}
&\tau * g_0 * \ldots * g_k \in \mathbf{P}_{n+1}\\
&\mathbf{V}_{n+1}(\tau * g_0 * \ldots * g_k) = \mathbf{V}_{n+1}(\tau)\\
&\mathbf{V}'_{n+1}(\tau * g_0 * \ldots * g_k) = \mathbf{V}'_{n+1}(\tau)\\
&\mathbf{R}_{n+1}(\tau * g_0 * \ldots * g_k) = \mathbf{R}_{n+1}(\tau).
\end{aligned}$$

We easily see that this step adds only a finite number of elements to $\mathbf{P}_{n+1}$, since if $n(i)$ is the number of initially placed existential quantifiers in $(\tau(S))_{1*i}$, then $m = \overline{\overline{\mathbf{V}'_{n+1}(\tau)}}^{\mathsf{N}}$, where $\mathsf{N} = \sum_i n(i)$, is an upper bound on the length of the sequences $\{g_k\}$, so $m \cdot 2^m$ is an upper bound on their number.

c. The identity construction.

i. Let τ be a maximal element resulting from (b). Well-order the functions defined by (8) of section **8** by $f_{\mathsf{I},k,j} < f_{\mathsf{I},k',j'}$ iff $k < k'$ or $k = k'$ and $j < j'$. Let $\{g_m\}$ be the sequence of these functions such that (α) g_0 is the least function $f_{\mathsf{I},k,j}$ defined at $\tau(S)$ such that $\mathrm{D}(k, j, \tau, \tau)$, and ($\beta$) for each m, g_{m+1} is the least function $f_{\mathsf{I},k,j}$ defined at $\tau * g_0 * \ldots * g_m(S)$ such that $\mathrm{D}(k, j, \tau, \tau * g_0 * \ldots * g_m)$, where $\mathrm{D}(k, j, \tau, \rho) \equiv -\exists \tau' [(\tau' * f_{\mathsf{I},k,j} \in \mathbf{r}(\rho)) \,\&\, (\mathbf{s}_2(\tau') = \mathbf{s}_2(\tau))]$. Then for each m:

$$\begin{aligned}
&\tau * g_0 * \ldots * g_m \in \mathbf{P}_{n+1}\\
&\mathbf{V}_{n+1}(\tau * g_0 * \ldots * g_m) = \mathbf{V}_{n+1}(\tau)\\
&\mathbf{V}'_{n+1}(\tau * g_0 * \ldots * g_m) = \mathbf{V}'_{n+1}(\tau)\\
&\mathbf{R}_{n+1}(\tau * g_0 * \ldots * g_m) = \mathbf{R}_{n+1}(\tau).
\end{aligned}$$

Since the number of identity formulae in $(\tau(S))_0$ is an upper bound on the length of the sequence $\{g_m\}$, this step adds only a finite number of elements to $\mathbf{P}_{n+1}$.

ii. Let τ be a maximal element resulting from (i). Well-order the functions defined by (9) and (10) of section **8** by $f_{F(i,k,j),p,m} < f_{F(i',k',j'),p',m'}$ iff $k < k'$ or $k' = k$ and $j < j'$ or $k = k'$, $j = j'$, and $p < p'$ or $k = k'$, $j = j'$, $p = p'$, and $m < m'$ or $k = k'$, $j = j'$, $p = p'$, $m = m'$, and $i < i'$. Let $\{g_q\}$ be the sequence of these functions such that (α) g_0 is the least function $f_{F(i,k,j),p,m}$ defined at $\tau(S)$ such that $\mathrm{D}(\tau, i, k, j, p, m, \tau, S)$, and ($\beta$) for each q, g_{q+1} is the least function $f_{F(i,k,j),p,m}$ defined at $\tau * g_0 * \ldots * g_q(S)$ such that $\mathrm{D}(\tau * g_0 * \ldots * g_q, i, k, j, p, m, \tau, S)$, where $\mathrm{D}(\rho, i, k, j, p, m, \tau, S) \equiv -\exists \tau' \exists m' [(m' \in \omega) \,\&\, (\tau' * f_{F(i,k,j),p,m'} \in \mathbf{r}(\rho)) \,\&\, (\mathbf{s}_2(\tau') = \mathbf{s}_2(\tau)) \,\&\, ((\tau'(S))_{p*\iota_{F,i,k}((\tau'(S))_p)(m')} = (\rho(S))_{p*\iota_{F,i,k}((\rho(S))_p)(m)})]$. Then for each q:

$$\begin{aligned}
&\tau * g_0 * \ldots * g_q \in \mathbf{P}_{n+1}\\
&\mathbf{V}_{n+1}(\tau * g_0 * \ldots * g_q) = \mathbf{V}_{n+1}(\tau)\\
&\mathbf{V}'_{n+1}(\tau * g_0 * \ldots * g_q) = \mathbf{V}'_{n+1}(\tau)\\
&\mathbf{R}_{n+1}(\tau * g_0 * \ldots * g_q) = \mathbf{R}_{n+1}(\tau).
\end{aligned}$$

The sequence $\{g_q\}$ is finite, for if m_1 is the number of atomic formulae in $\tau(S)$, m_2 is the maximum index of the atomic formulae in $\tau(S)$, and m_3 is the number of identity formulae in $(\tau(S))_0$, then $m_1 \cdot m_2 \cdot m_3$ is an upper bound on the length of $\{g_q\}$. Therefore this step adds only a finite number of elements to $\mathbf{P}_{n+1}$.

iii. Let τ be a maximal element resulting from (ii). Well-order the functions defined by (11), (12), (13), and (14) of section **8** by $f_{Z(k,j),p,m} < f_{Z'(k',j'),p',m'}$ iff $k < k'$ or $k = k'$ and $j < j'$ or $k = k'$, $j = j'$, and $p < p'$ or $k = k'$, $j = j'$, $p = p'$, and $m < m'$ or $k = k'$, $j = j'$, $p = p'$, $m = m'$, $Z = \mathbf{K}$, and $Z' = \mathbf{B}$. Let $\{g_i\}$ be the sequence of these functions such that (α) g_0 is the least function $f_{Z(k,j),p,m}$ defined at $\tau(S)$ such that $\mathrm{D}(\tau, Z, k, j, p, m, \tau, S)$, and ($\beta$) for each i, g_{i+1} is the least function $f_{Z(k,j),p,m}$ defined at $\tau * g_0 * \ldots * g_i(S)$ such that $\mathrm{D}(\tau * g_0 * \ldots * g_i, Z, k, j, p, m, \tau, S)$, where $\mathrm{D}(\rho, Z, k, j, p, m, \tau, S) \equiv -\exists\tau'\exists m'[(m' \epsilon\, \omega)\ \&\ (\tau' * f_{Z(k,j),p,m'} \epsilon\, \mathbf{r}(\rho))\ \&\ (\mathbf{s}_2(\tau') = \mathbf{s}_2(\tau))\ \&\ ((\tau'(S))_{p*\iota_{Z,k}((\tau'(S))_p)(m')} = (\rho(S))_{p*\iota_{Z,k}((\rho(S))_p)(m)})]$. Then for each i:

$$\begin{aligned}
&\tau * g_0 * \ldots * g_i \,\epsilon\, \mathbf{P}_{n+1}\\
&\mathbf{V}_{n+1}(\tau * g_0 * \ldots * g_i) = \mathbf{V}_{n+1}(\tau)\\
&\mathbf{V}'_{n+1}(\tau * g_0 * \ldots * g_i) = \mathbf{V}'_{n+1}(\tau)\\
&\mathbf{R}_{n+1}(\tau * g_0 * \ldots * g_i) = \mathbf{R}_{n+1}(\tau).
\end{aligned}$$

The sequence $\{g_i\}$ is finite, for if m_1 is the number of initial occurrences of **K** and **B** in the formulae of $\tau(S)$ and m_2 is the number of identity formulae in $(\tau(S))_0$, then $m_1 \cdot m_2$ is an upper bound on the length of $\{g_i\}$; therefore the number of elements added to $\mathbf{P}_{n+1}$ in this step is finite.

d. The operator construction.

i. Let τ be a maximal element resulting from (c). Well-order the functions defined by (15) of section **8** by $f_{\mathbf{K},0,m} < f_{\mathbf{K},0,m'}$ iff $m < m'$. Let $\{g_k\}$ be the sequence of these functions such that (α) g_0 is the least function $f_{\mathbf{K},0,m}$ defined at $\tau(S)$ such that $\mathrm{D}(\tau, m, \tau, S)$, and ($\beta$) for each k, g_{k+1} is the least function $f_{\mathbf{K},0,m}$ defined at $\tau * g_0 * \ldots * g_k(S)$ such that $\mathrm{D}(\tau * g_0 * \ldots * g_k, m, \tau, S)$, where $\mathrm{D}(\rho, m, \tau, s) \equiv -\exists\tau'\exists m'[(m' \epsilon\, \omega)\ \&\ (\tau' * f_{\mathbf{K},0,m'} \epsilon\, \mathbf{r}(\rho))\ \&\ (\mathbf{s}_2(\tau') = \mathbf{s}_2(\tau))\ \&\ ((\tau'(S))_{0*\iota\mathbf{K}((\tau'(S))_0)(m')} = (\rho(S))_{0*\iota\mathbf{K}((\rho(S))_0)(m)})]$. Then for each k:

$$\begin{aligned}
&\tau * g_0 * \ldots * g_k \,\epsilon\, \mathbf{P}_{n+1}\\
&\mathbf{V}_{n+1}(\tau * g_0 * \ldots * g_k) = \mathbf{V}_{n+1}(\tau)\\
&\mathbf{V}'_{n+1}(\tau * g_0 * \ldots * g_k) = \mathbf{V}'_{n+1}(\tau)\\
&\mathbf{R}_{n+1}(\tau * g_0 * \ldots * g_k) = \mathbf{R}_{n+1}(\tau).
\end{aligned}$$

The sequence $\{g_k\}$ is finite, for if $n(i)$ is the number of initially placed operators **K** in $(\tau(S))_{0*i}$, then $\sum_i n(i)$ is an upper bound on the length of $\{g_k\}$; therefore this step adds only a finite number of elements to $\mathbf{P}_{n+1}$.

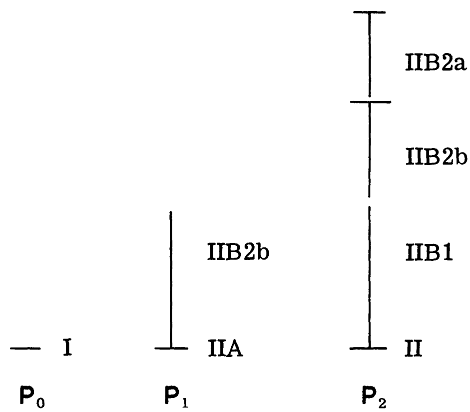
ii. Let τ be a maximal element resulting from (i). Then if g is defined by (16), (17), or (18) of section **8** and g is defined at $\tau(S)$,

$$\begin{aligned} &\tau * g \in \mathsf{P}_{n+1} \\ &\mathsf{V}_{n+1}(\tau * g) = \mathsf{V}_{n+1}(\tau) \\ &\mathsf{V}'_{n+1}(\tau * g) = \mathsf{v}(|\tau * g(S)|) \\ &\mathsf{R}_{n+1}(\tau * g) = \mathsf{R}_{n+1}(\tau) \,|\, \mathsf{V}'_{n+1}(\tau * g). \end{aligned}$$

This step adds only a finite number of elements to P_{n+1}, since if m_1 is the number of formulae in $(\tau(S))_0$ with initially placed operator **B**, and m_2 is the number of formulae in $(\tau(S))_1$ with initially placed operator **K** or **B**, then $m_1 + m_2$ is an upper bound on the number of functions defined by (16), (17), or (18) of section **8** which are defined at $\tau(S)$.

If $h(n)$ is the supremum of the lengths of $\mathsf{s}_2(\sigma)$ for $\sigma \in \mathsf{P}_n$, then it is easy to establish by induction that $h(n) \leq n + 1$. Suppose $\sigma \in \mathsf{P}_n$ and $\mathsf{e}_2(\sigma) = \emptyset$. By construction there are only finitely many $\tau \in \mathsf{P}_n$ such that either $\sigma \in \mathsf{r}'(\tau)$, $\mathsf{s}_2(\mathsf{pd}(\tau)) = \mathsf{s}_2(\sigma)$, and $\mathsf{s}_2(\mathsf{pd}(\tau)) \neq \mathsf{s}_2(\tau)$ or $\sigma \in \mathsf{r}(\tau)$, $\mathsf{s}_2(\tau) = \mathsf{s}_2(\sigma)$, and τ is maximal in P_n. Therefore there are only finitely many maximal elements, hence only finitely many elements, in P_n.

To aid the understanding we illustrate below the construction of maximal elements in P_0, P_1, and P_2. The horizontal bars represent $\mathcal{W}$-functions; the vertical segments represent sequences of functions other than $\mathcal{W}$-functions; the conditions sanctioning the various component constructions are indicated alongside.



Finally, we observe that the structure of P_n is determined by $\{\mathsf{s}_1(\sigma) \,|\, \sigma \in \mathsf{P}_n\}$ in the sense that if τ is a finite sequence of $\mathcal{B}$- and $\mathcal{W}$-functions whose last element is, say, g, then there is at most one element $\sigma \in \mathsf{P}_n$ such that the last element of σ is g and $\mathsf{s}_1(\sigma) = \tau$.

10 *Completeness: The Main Lemma* As noted, we have had to replace the single, typically infinite completeness construction with a typically infinite sequence of finite constructions: since we generally do not actually complete the construction of any world, we cannot begin the construction of other derivative worlds with a complete set of materials. Unfortunately, we are now rather far from being able to apply the usual completeness arguments and have to engage in some rather messy preparations.

Let $\sigma \epsilon \mathbf{L}_1(n)$ iff $\sigma \epsilon \mathbf{P}_n$ and $\mathbf{e}_2(\sigma) = \emptyset$. We may think of $\mathbf{L}_1(n)$ as the set of worlds defined by $\mathbf{P}_n$, or the set of 'n-worlds'. If $\sigma \epsilon \mathbf{L}_1(n)$, define the set $\mathbf{c}_1(\sigma, n)$ of $n+1$-worlds 'corresponding to σ' by $\mathbf{c}_1(\sigma, n) = \{\tau \epsilon \mathbf{L}_1(n+1) \mid \mathbf{s}_1(\tau) \epsilon \mathbf{t}(\mathbf{s}_1(\sigma))\}$. If $\sigma \epsilon \mathbf{L}_1(n)$, the set $\mathbf{c}_1^m(\sigma, n)$ of $n+m$-worlds 'corresponding to σ' is defined inductively by $\mathbf{c}_1^0(\sigma, n) = \{\sigma\}$ and $\mathbf{c}_1^{m+1}(\sigma, n) = \mathbf{c}_1(\mathbf{c}_1^m(\sigma, n), n+m)$.

Some easy consequences of these definitions are the following. (1) If $\Gamma \epsilon X$, let Γ^1 be the initial segment of Γ of length $\delta(\Gamma)$ and let Γ^2 be such that $\Gamma = \Gamma^1 * \Gamma^2$. If $\sigma \epsilon \mathbf{L}_1(n)$ and $\tau \epsilon \mathbf{c}_1(\sigma, n)$, then by construction, $(\sigma(S))_1 = (\tau(S))_1$, $((\sigma(S))_0)^1 \epsilon \mathbf{r}(((\tau(S))_0)^1)$, and $((\sigma(S))_0)^2 \epsilon \mathbf{r}((\tau(S))_0)^2)$. (2) If $\sigma \epsilon \mathbf{P}_n$ and $\sigma \epsilon \mathbf{P}_{n+1}$, then $\sigma \epsilon \mathbf{P}_m$ for each $m > n$; if $\sigma \epsilon \mathbf{L}_1(n)$ and $\sigma \epsilon \mathbf{P}_{n+1}$, then $\mathbf{c}_1(\sigma, n) = \mathbf{c}_1^m(\sigma, n) = \{\sigma\}$. (3) If $\sigma \epsilon \mathbf{L}_1(n+1)$, then there exists at most one $\tau \epsilon \mathbf{L}_1(n)$ such that $\sigma \epsilon \mathbf{c}_1(\tau, n)$. (4) If $\sigma \epsilon \mathbf{L}_1(n)$, then $\mathbf{c}_1(\sigma, n) = \emptyset$ iff $\forall \tau \{[(\tau \epsilon \mathbf{P}_{n+1})\ \&\ \exists \rho([\rho \epsilon \mathbf{t}(\mathbf{s}_1(\sigma))] \& [\mathbf{s}_1(\tau) \epsilon \mathbf{r}'(\rho)])] \supset \exists \pi [(\pi \epsilon \mathbf{P}_{n+1}) \& (\tau \epsilon \mathbf{r}(\pi)) \& \exists \rho([\rho \epsilon \mathbf{t}(\mathbf{s}_1(\sigma))]\ \&\ [\mathbf{s}_1(\pi) \epsilon \mathbf{r}'(\rho)])\ \&\ (\pi(S) \not\epsilon Y)]\}$.

Let $\sigma \epsilon \mathbf{L}_2(n)$ iff $\sigma \epsilon \mathbf{L}_1(n)$ and $\mathbf{c}_1^m(\sigma, n) \neq \emptyset$ for all m. If $\sigma \epsilon \mathbf{L}_2(n)$, define $\mathbf{c}_2(\sigma, n) = \mathbf{c}_1(\sigma, n) \cap \mathbf{L}_2(n+1)$ and $\mathbf{c}_2^m(\sigma, n) = \mathbf{c}_1^m(\sigma, n) \cap \mathbf{L}_2(n+m)$. If $\sigma \epsilon \mathbf{L}_2(\omega)$ and $n = \mu m(\sigma \epsilon \mathbf{L}_2(m))$, let $\mathbf{c}_2(\sigma) = \mathbf{c}_2(\sigma, n)$, $\mathbf{c}_2^m(\sigma) = \mathbf{c}_2^m(\sigma, n)$, and $\mathbf{c}(\sigma) = \sum_{m \epsilon \omega} \mathbf{c}_2^m(\sigma)$.

If $\sigma \epsilon \mathbf{L}_2(n)$, define the set $\mathbf{F}(\sigma, n)$ of 'ways of proceeding from σ in $\mathbf{P}_n$' by $\mathbf{F}(\sigma, n) = \{\mathbf{s}_1(\mathbf{e}_2(\tau)) \mid (\tau \epsilon \mathbf{P}_n)\ \&\ (\sigma \epsilon \mathbf{r}(\tau))\ \&\ (\mathbf{s}_2(\tau) = \mathbf{s}_2(\sigma))\}$ and the set $\mathbf{F}^m(\sigma, n)$ of 'ways of proceeding from the $n+m$-worlds corresponding to σ' by $\mathbf{F}^m(\sigma, n) = \mathbf{F}(\mathbf{c}_2^m(\sigma, n), n+m)$. If $\mathbf{G}'(\sigma, n) = \sum_{m \epsilon \omega} \mathbf{F}^m(\sigma, n)$, then $\mathbf{G}(\sigma, n)$ shall be the set of finite or infinite sequences ρ such that (i) each finite initial segment of ρ is an element of $\mathbf{G}'(\sigma, n)$ and (ii) there is no element of $\mathbf{G}'(\sigma, n)$ of which ρ is a proper initial segment. If $\sigma \epsilon \mathbf{L}_2(\omega)$ and $n = \mu m(\sigma \epsilon \mathbf{L}_2(m))$, let $\mathbf{F}(\sigma) = \mathbf{F}(\sigma, n)$ and $\mathbf{G}(\sigma) = \mathbf{G}(\sigma, n)$.

Some easy consequences of these definitions are the following. (1) If $\sigma \epsilon \mathbf{L}_2(n)$ and $\sigma \epsilon \mathbf{P}_{n+1}$, then $\mathbf{G}'(\sigma, n) = \mathbf{G}'(\sigma, n+1)$ and $\mathbf{G}(\sigma, n) = \mathbf{G}(\sigma, n+1)$. (2) If $\sigma \epsilon \mathbf{L}_2(n)$ and $\tau \epsilon \mathbf{c}_2(\sigma, n)$, then $\mathbf{G}'(\tau, n+1) \subset \mathbf{G}'(\sigma, n)$ and $\mathbf{G}(\tau, n+1) \subset \mathbf{G}(\sigma, n)$. (3) If $\sigma \epsilon \mathbf{L}_2(n)$ and $\rho \epsilon \mathbf{G}(\sigma, n)$, then $\rho \epsilon \mathbf{G}(\mathbf{c}_2(\sigma, n), n+1)$. (4) If $\sigma \epsilon \mathbf{L}_2(n)$ and $\tau \epsilon \mathbf{G}'(\sigma, n)$, then τ is an initial segment of some $\rho \epsilon \mathbf{G}(\sigma, n)$.

If $\sigma \epsilon \mathbf{L}_2(n)$, $\tau \epsilon \mathbf{c}_2^m(\sigma, n)$, and $\rho \epsilon \mathbf{G}(\sigma, n) \cap \mathbf{G}(\tau, n+m)$, let $\lambda(\rho, \tau, m, \sigma, n)$ be the largest initial segment of ρ in $\mathbf{F}(\tau, n+m)$ and let $\pi(\rho, \tau, m, \sigma, n)$ be the element π of $\mathbf{P}_{n+m}$ such that (i) $\mathbf{s}_1(\pi) = \mathbf{s}_1(\tau) * \lambda(\rho, \tau, m, \sigma, n)$ and (ii) if $\kappa \epsilon \mathbf{P}_{n+m}$ and $\mathbf{s}_1(\kappa) = \mathbf{s}_1(\tau) * \lambda(\rho, \tau, m, \sigma, n)$, then $\kappa \epsilon \mathbf{r}(\pi)$. If $\sigma \epsilon \mathbf{L}_2(\omega)$, let $\pi(\rho, \tau, m, \sigma) = \pi(\rho, \tau, m, \sigma, \mu n(\sigma \epsilon \mathbf{L}_2(n)))$.

Let $\sigma \epsilon \mathbf{L}_2(n)$ and $\rho \epsilon \mathbf{G}(\sigma, n)$. Define the set $\mathbf{W}^m(\sigma, n, \rho)$ of $n+m$-worlds 'reachable by ρ from the $n+m$-worlds corresponding to σ' by $\mathbf{W}^m(\sigma, n, \rho) = \{\pi \epsilon \mathbf{L}_2(n+m) \mid \exists \tau [\mathrm{pd}(\pi) = \pi(\rho, \tau, m, \sigma, n)]\}$ and the set $\mathbf{W}(\sigma, n, \rho)$ of worlds 'reachable by ρ from the worlds corresponding to σ' by $\mathbf{W}(\sigma, n, \rho) = \sum_{m \epsilon \omega} \mathbf{W}^m(\sigma, n, \rho)$. If $\sigma \epsilon \mathbf{L}_2(\omega)$ and $\rho \epsilon \mathbf{G}(\sigma)$, let $\mathbf{W}(\sigma, \rho) = \mathbf{W}(\sigma, \mu n(\sigma \epsilon \mathbf{L}_2(n)), \rho)$. Note that if $\tau \epsilon \mathbf{W}(\sigma, p)$, then for each $m \epsilon \omega$ there exists exactly one $\kappa \epsilon \mathbf{c}_2^m(\tau)$ such that $\kappa \epsilon \mathbf{W}(\sigma, \rho)$.

We now describe the elements h of $\mathbf{H}$. To do this we inductively describe a subset $\mathbf{W}_h$ of $\mathbf{L}_2(\omega)$ and a partition $[\mathbf{W}_h]$ of $\mathbf{W}_h$; the domain of h

shall be $[\mathbf{W}_h]$ and if $[\sigma]$ is the element of $[\mathbf{W}_h]$ to which an element σ of $\mathbf{W}_h$ belongs, then $h([\sigma])$ shall be an element of $\mathbf{G}(\sigma)$. First, $\langle f_0\rangle \in \mathbf{W}_h$ and $[\langle f_0\rangle] = \{\langle f_0\rangle\}$; $h([\langle f_0\rangle])$ shall be any element of $\mathbf{G}(\langle f_0\rangle)$. Now suppose $\sigma \in \mathbf{W}_h$ and if $\kappa \in [\sigma]$, then $h([\sigma]) \in \mathbf{G}(\kappa)$. If $\tau \in \mathbf{W}([\sigma], h([\sigma]))$, then $\tau \in \mathbf{W}_h$ and $[\tau] = \{\kappa \in \mathbf{W}([\sigma], h([\sigma])) \mid (\tau \in \mathbf{c}(\kappa)) \vee (\kappa \in \mathbf{c}(\tau))\}$. $[\tau]$ is well-ordered; if $\kappa_m([\tau])$ is the $m + 1$'th element of $[\tau]$, then $\kappa_{n+1}([\tau]) \in \mathbf{c}_2(\kappa_n([\tau]))$ for each $n \in \omega$. Since $\mathbf{F}(\kappa_n([\tau])) \subset \mathbf{F}(\kappa_{n+1}([\tau]))$, $\bigcap_{n\in\omega} \mathbf{G}(\kappa_n([\tau])) \neq \emptyset$; accordingly, $h([\tau])$ shall be any element of $\bigcap_{n\in\omega} \mathbf{G}(\kappa_n([\tau]))$.

If $\mathbf{W}([\sigma], h([\sigma])) = \emptyset$, then $[\sigma]$ is an *h-hypothesis*. If $[\sigma]$ is an h-hypothesis then $[\sigma]$ is an *h-axiom* iff $\pi(h([\sigma]), \kappa, 0, \kappa)(S)$ is an axiom for some $\kappa \in [\sigma]$. Of course, if $[\sigma]$ is an h-axiom, then there is some $m \in \omega$ such that if $n \geq m$, then $\pi(h([\sigma]), \kappa_n([\sigma]), 0, \kappa_n([\sigma]))(S) = \pi(h([\sigma]), \kappa_n([\sigma]), n, \kappa_0([\sigma]))(S)$ is an axiom. If n is the least such m, let $\mathbf{a}([\sigma]) = \pi(h([\sigma]), \kappa_n([\sigma]), n, \kappa_0([\sigma]))$.

The main lemma for completeness can now be stated.

Theorem 6 *If S is not provable, then there is some $h \in H$ such that no h-hypothesis is an h-axiom.*

Proof: Suppose, on the contrary, that if $h \in \mathbf{H}$, then the set $\mathbf{A}(h)$ of h-axioms is not empty. We construct a proof of S. Well-order $\mathbf{A}(h)$ and let $\alpha(h)$ be the least element. Define $\alpha'(h) = \mathbf{a}(\alpha(h))$ and let $\mathbf{A} = \{\mathbf{s}_1(\alpha'(h)) \mid h \in \mathbf{H}\}$.

Lemma 1 *Suppose π is a maximal element of $\mathbf{P}_n$, $\rho \in \mathbf{P}_m$, $\mathbf{s}_1(\pi) \in \mathbf{r}'(\mathbf{s}_1(\rho))$, $\mathrm{pd}_2(\pi) \in \mathbf{L}_2(n)$, and $\mathrm{pd}_2(\rho) \in \mathbf{L}_2(m)$. Then $m = n + p$ for some $p > 0$, and if $\tau \in \mathbf{r}'(\rho)$ is defined by $\mathbf{s}_1(\tau) = \mathbf{s}_1(\mathrm{pd}_2(\pi))$ and $\mathbf{e}_2(\tau) = \emptyset$, then $\mathbf{c}_2^p(\mathrm{pd}_2(\pi), n) = \{\tau\}$.*

Proof: Suppose $m = n$. There is just one maximal element τ of $\mathbf{P}_n$ such that $\mathbf{s}_1(\tau) = \mathbf{s}_1(\pi)$, viz. π. But $\mathbf{s}_1(\pi) \in \mathbf{r}'(\mathbf{s}_1(\rho))$ implies that π is not maximal. Suppose that $n = m + p$, where $p > 0$. $\mathbf{s}_1(\pi) \in \mathbf{r}'(\mathbf{s}_1(\rho))$ implies that $\mathbf{s}_1(\mathrm{pd}_2(\pi)) \in \mathbf{r}(\mathbf{s}_1(\mathrm{pd}_2(\rho)))$. Let $\kappa(\pi)$ be the element $\kappa \in \mathbf{r}'(\rho)$ such that $\mathbf{s}_1(\kappa) = \mathbf{s}_1(\mathrm{pd}_2(\pi))$ and $\mathbf{e}_2(\kappa) = \emptyset$. Then $\mathrm{pd}_2(\rho) \in \mathbf{L}_2(m)$ implies $\kappa(\pi) \in \mathbf{L}_2(m)$; since $\mathbf{s}_1(\mathrm{pd}_2(\pi)) = \mathbf{s}_1(\kappa(\pi))$, $\mathrm{pd}_2(\pi) \in \mathbf{c}_1^p(\kappa(\pi), m)$. $\mathrm{pd}_2(\pi) \in \mathbf{L}_2(n)$, so $\mathrm{pd}_2(\pi) \in \mathbf{c}_2^p(\kappa(\pi), m)$, and in fact $\mathbf{c}_2^p(\kappa(\pi), m) = \{\mathrm{pd}_2(\pi)\}$, as is easily verified. Suppose $\mathbf{s}_1(\mathrm{pd}_2(\pi)) = \mathbf{s}_1(\mathrm{pd}_2(\rho))$. Then $\kappa(\pi) = \mathrm{pd}_2(\rho)$ and $\mathbf{c}_2^p(\mathrm{pd}_2(\rho), m) = \{\mathrm{pd}_2(\pi)\}$. By construction, there is an element τ of $\mathbf{P}_n$ such that $\tau = \mathrm{pd}_2(\pi) * \mathbf{e}_2(\rho)$, so $\pi \in \mathbf{r}'(\tau)$, contradicting the maximality of π. Suppose $\mathbf{s}_1(\mathrm{pd}_2(\pi)) \in \mathbf{r}'(\mathbf{s}_1(\mathrm{pd}_2(\rho)))$. $\mathrm{pd}_2(\rho) \in \mathbf{L}_2(m)$, so $\mathbf{c}_2^p(\mathrm{pd}_2(\rho), m) \neq \emptyset$. But by construction, if $\tau \in \mathbf{c}_2^p(\mathrm{pd}_2(\rho), m)$, then $\pi \in \mathbf{r}'(\tau)$, contradicting the maximality of π. The other assertion follows from the same kind of considerations. Q.E.D.

Lemma 2 *If $\sigma \in \mathbf{A}$ and $\tau \in \mathbf{A}$, then $\sigma \notin \mathbf{r}'(\tau)$.*

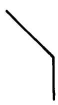
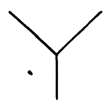
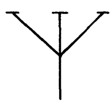
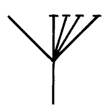
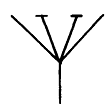
Proof: Suppose, on the contrary, that $\sigma \in \mathbf{A}$, $\tau \in \mathbf{A}$, and $\sigma \in \mathbf{r}'(\tau)$. If $\gamma \in \mathbf{A}$, let $\mathbf{H}(\gamma) = \{h \in \mathbf{H} \mid \gamma = \mathbf{s}_1(\alpha'(h))\}$ and $\mathbf{q}(\gamma) = \min_{h \in \mathbf{H}(\gamma)} \{n \mid \alpha'(h) \in \mathbf{P}_n\}$; let $h_\gamma \in \mathbf{H}$ be such that $\alpha'(h_\gamma) \in \mathbf{P}_{\mathbf{q}(\gamma)}$. Then $\alpha'(h_\sigma)$ is a maximal element of $\mathbf{P}_{\mathbf{q}(\sigma)}$, $\alpha'(h_\tau)$ is a maximal element of $\mathbf{P}_{\mathbf{q}(\tau)}$, $\mathrm{pd}_2(\alpha')h_\sigma)) \in \mathbf{L}_2(\mathbf{q}(\sigma))$, $\mathrm{pd}_2(\alpha'(h_\tau)) \in \mathbf{L}_2(\mathbf{q}(\tau))$, and $\mathbf{s}_1(\alpha'(h_\sigma)) = \sigma \in \mathbf{r}'(\tau) = \mathbf{r}'(\mathbf{s}_1(\alpha'(h_\tau)))$. By Lemma 1, $\mathbf{q}(\tau) = \mathbf{q}(\sigma) + p$, where

$p > 0$, and $\mathbf{c}_2^p(\mathrm{pd}_2(\alpha'(h_\sigma)), \mathbf{q}(\sigma)) = \{\kappa\}$, where κ is the initial segment of $\alpha'(h_\tau)$ such that $\mathbf{s}_1(\mathrm{pd}_2(\alpha'(h_\sigma))) = \mathbf{s}_1(\kappa)$ and $\mathbf{e}_2(\kappa) = \emptyset$. But then $\kappa * \mathbf{e}_2(\alpha'(h_\sigma))(S)$ is an axiom, while $\mathbf{s}_1(\kappa * \mathbf{e}_2(\alpha'(h_\sigma))) = \mathbf{s}_1(\mathrm{pd}_2(\alpha'(h_\sigma))) * \mathbf{s}_1(\mathbf{e}_2(\alpha'(h_\sigma))) = \sigma \in \mathbf{r}'(\tau) = \mathbf{r}'(\mathbf{s}_1(\alpha'(h_\tau)))$. Thus by construction $\alpha'(h_\tau) \notin \mathbf{P}_{\mathbf{q}(\tau)}$, a contradiction. Q.E.D.

Lemma 3 *If f and g are $\mathscr{B}$-functions but not corresponding $\mathscr{B}$-functions and $\sigma * f \in \mathbf{A}' = \mathbf{r}(\mathbf{A})$, then $\sigma * g \notin \mathbf{A}'$.*

Proof: We first note that the assertion is true if $\mathbf{A}'$ is replaced by $\mathbf{P}_n$. Suppose that $\sigma * f \in \mathbf{A}'$ and $\sigma * g \in \mathbf{A}'$. If $\gamma \in \mathbf{A}'$, let $\mathbf{H}(\gamma) = \{h \in \mathbf{H} \mid \gamma \in \mathbf{r}(\mathbf{s}_1(\alpha'(h)))\}$ and $\mathbf{q}(\gamma) = \min\limits_{h \in \mathbf{H}(\gamma)} \{n \mid \alpha'(h) \in \mathbf{P}_n\}$. Let $\tau(f)$ be the element of $\mathbf{P}_{\mathbf{q}(\sigma * f)}$ with $\mathbf{s}_1(\tau(f)) = \sigma * f$ and $\mathbf{e}_1(\tau(f)) = \emptyset$, and $\tau(g)$ the element of $\mathbf{P}_{\mathbf{q}(\sigma * g)}$ with $\mathbf{s}_1(\tau(g)) = \sigma * g$ and $\mathbf{e}_1(\tau(g)) = \emptyset$. By our note, $\mathbf{q}(\sigma * g) \neq \mathbf{q}(\sigma * f)$. Suppose $\mathbf{q}(\sigma * g) = \mathbf{q}(\sigma * f) + p$, where $p > 0$. $\mathrm{pd}_2(\tau(f)) \in \mathbf{L}_2(\mathbf{q}(\sigma * f))$ and $\mathrm{pd}_2(\tau(g)) \in \mathbf{L}_2(\mathbf{q}(\sigma * g))$. It is easy to verify that $\mathbf{c}_2^p(\tau(f), \mathbf{q}(\sigma * f)) = \{\mathrm{pd}_2(\tau(g))\}$. But then by construction, $\mathrm{pd}_2(\tau(g)) * \mathbf{e}_2(\tau(f)) \in \mathbf{P}_{\mathbf{q}(\sigma * g)}$. Since $\mathrm{pd}(\mathrm{pd}_2(\tau(g)) * \mathbf{e}_2(\tau(f))) = \mathrm{pd}(\tau(g))$, this is impossible by our note. Q.E.D.

By Lemma 3, the 'branch points' of $\mathbf{A}'$ are of the following kinds:

1. $\mathscr{B}$-function 
2. Corresponding $\mathscr{B}$-functions 
3. $\mathscr{W}$-functions 
4. $\mathscr{B}$-function, $\mathscr{W}$-functions 
5. Corresponding $\mathscr{B}$-functions, $\mathscr{W}$-functions 

Define $\chi \in 2^{\mathbf{A}'}$ as follows: (i) if $\alpha \in \mathbf{A}$, then $\chi(\alpha) = 1$; (ii) if $\alpha \notin \mathbf{A}$, then $\chi(\alpha) = 1$ iff either (a) $\chi(\alpha * g) = 1$ for some $\mathscr{W}$-function g such that $\alpha * g \in \mathbf{A}'$, or (b) $\chi(\alpha * f^0) = \chi(\alpha * f^1) = 1$ for some corresponding $\mathscr{B}$-functions f^0 and f^1 such that $\alpha * f^0 \in \mathbf{A}'$ and $\alpha * f^1 \in \mathbf{A}'$. By Lemma 2, χ is well-defined.

Lemma 4 $\chi(\langle f_0 \rangle) = 1$.

Proof: Suppose, on the contrary, that $\chi(\langle f_0 \rangle) = 0$. Define $\mathbf{A}''$ inductively as follows:

i. $\langle f_0 \rangle \in \mathbf{A}''$.

ii. Suppose $\sigma \in \mathbf{A}''$ and $\chi(\sigma) = 0$.

a. Suppose $\sigma * f \in \mathbf{A}'$ and $\chi(\sigma * f) = 0$ for some $\mathscr{B}$-function f. If f is the first of

the pair $\langle f^0, f^1\rangle$ of corresponding $\mathscr{B}$-functions, then $\sigma * f \in \mathbf{A}''$; if f is the second of the pair and $\sigma * f^0 \notin \mathbf{A}'$ or $\chi(\sigma * f^0) = 1$, then $\sigma * f \in \mathbf{A}''$.
b. Suppose for each $\mathscr{B}$-function f that either $\sigma * f \notin \mathbf{A}'$ or $\chi(\sigma * f) = 1$. If g is a $\mathscr{W}$-function and $\sigma * g \in \mathbf{A}'$, then $\sigma * g \in \mathbf{A}''$.

Evidently, if $\sigma \in \mathbf{A}''$, then $\chi(\sigma) = 0$.

We now define an element h of H. If $\sigma * f \in \mathbf{A}'$, let $\kappa(\sigma * f)$ be the element κ of $\mathbf{P}_{\mathbf{q}(\sigma * f)}$ such that $\mathbf{s}_1(\kappa) = \sigma * f$ and $\mathbf{e}_1(\kappa) = \emptyset$. Let σ be a maximal element of $\mathbf{A}''$. Then $\sigma \notin \mathbf{A}$ and there is a $\mathscr{B}$-function f such that $\sigma * f \in \mathbf{A}' - \mathbf{A}''$ and if f' is the corresponding $\mathscr{B}$-function, then $\sigma * f' \notin \mathbf{A}'$. $\mathrm{pd}_2(\kappa(\sigma * f)) \in \mathbf{W}_{h_{\sigma * f}}$ and $\mathbf{e}_2(\sigma) * f' \in \mathbf{F}(\mathrm{pd}_2(\kappa(\sigma * f)), \mathbf{q}(\sigma * f))$. If $\pi \in \mathbf{r}'(\mathrm{pd}_2(\kappa(\sigma * f)))$ and $\mathbf{e}_2(\pi) = \emptyset$, then $h([\pi]) = h_{\sigma * f}([\pi])$; $\mathbf{e}_2(\sigma) * f'$ shall be an initial segment of $h([\mathrm{pd}_2(\kappa(\sigma * f))])$; elsewhere h shall be defined arbitrarily.

This kind of assignment is possible because (1) if σ and τ are maximal elements of $\mathbf{A}''$, then $\mathrm{pd}_2(\sigma) \neq \mathrm{pd}_2(\tau)$, and (2) if $\sigma * f \in \mathbf{A}''$ and $\tau * g \in \mathbf{A}''$ for $\mathscr{W}$-functions f and g, then $\kappa(\sigma * f) \in \mathbf{c}(\kappa(\tau * g))$ or $\kappa(\tau * g) \in \mathbf{c}(\kappa(\sigma * f))$ iff $\sigma * f = \tau * g$.

Now $\mathbf{s}_1(\alpha'(h)) \in \mathbf{A}$. Some $\sigma \in \mathbf{r}(\mathbf{s}_1(\alpha'(h)))$ is a maximal element of $\mathbf{A}''$. Thus there is a $\mathscr{B}$-function f such that $\sigma * f \in \mathbf{A}' - \mathbf{A}''$ and if f' is the corresponding $\mathscr{B}$-function, then $\sigma * f' \notin \mathbf{A}'$. But by the definition of h, $\sigma * f' \in \mathbf{r}(\mathbf{s}_1(\alpha'(h)))$, so $\sigma * f' \in \mathbf{A}'$, RAA. Q.E.D.

We define B' inductively as follows:

i. $\langle f_0\rangle \in \mathrm{B}'$.
ii. Suppose $\sigma \in \mathrm{B}'$ and $\chi(\sigma) = 1$.

a. Suppose $\sigma * g \in \mathbf{A}'$ and $\chi(\sigma * g) = 1$ for some $\mathscr{W}$-function g. Then $\sigma * g \in \mathrm{B}'$ for the least such $\mathscr{W}$-function g (the $\mathscr{W}$-functions are countable).
b. Suppose that if g is a $\mathscr{W}$-function such that $\sigma * g \in \mathbf{A}'$, then $\chi(\sigma * g) = 0$. Then if $\sigma \notin \mathbf{A}$, there exist corresponding $\mathscr{B}$-functions f^0 and f^1 such that $\sigma * f^0 \in \mathbf{A}'$, $\sigma * f^1 \in \mathbf{A}'$, and $\chi(\sigma * f^0) = \chi(\sigma * f^1) = 1$; in this case, $\sigma * f^0 \in \mathrm{B}'$ and $\sigma * f^1 \in \mathrm{B}'$.

Evidently, if $\sigma \in \mathrm{B}'$, then $\chi(\sigma) = 1$. We let B be the maximal elements of B'.

We now derive from B' a set Pf' of sequences such that $(f_0) \in Pf'$ and if $\sigma \in Pf'$, then $\sigma(S)$ is provable. B' is essentially a tree with at most two branches at each node; since each branch is finite, B' is finite. Accordingly, if $\sigma \in \mathrm{B}'$, let $\mathbf{i}(\sigma) = \max_{\tau \in \mathrm{B}(\sigma)} \mathbf{q}(\tau)$, where $\mathrm{B}(\sigma) = \{\tau \in \mathrm{B} \mid \sigma \in \mathbf{r}(\tau)\}$. We see that if $\sigma_1 \in \mathrm{B}'$, $\sigma_2 \in \mathrm{B}'$, and $\sigma_1 \in \mathbf{r}(\sigma_2)$, then $\mathbf{i}(\sigma_1) \leq \mathbf{i}(\sigma_2)$.

If $\sigma \in \mathrm{B}'$, then $\zeta(\sigma)$ shall be a finite sequence of functions other than $\mathscr{W}$-functions and $\mathscr{B}$-functions defined as follows:

i. If $\sigma \in \mathrm{B}$, there is just one maximal element ζ of $\mathbf{P}_{\mathbf{i}(\sigma)}$ such that $\mathbf{s}_1(\zeta) = \sigma$; then $\zeta(\sigma) = \mathbf{e}_1(\zeta)$.
ii. If $\sigma * f \in \mathrm{B}'$, there is just one $\zeta \in \mathbf{P}_{\mathbf{i}(\sigma)}$ such that $\mathbf{s}_1(\zeta) = \sigma * f$ and $\mathbf{e}_1(\zeta) = \emptyset$; then $\zeta(\sigma) = \mathbf{e}_1(\mathrm{pd}(\zeta))$.

If $\sigma = \langle g_1, \ldots, g_k\rangle \in \mathrm{B}'$, define $\Pi(\sigma) = g_1 * \zeta(g_1) * g_2 * \zeta(g_1 * g_2) * \ldots * g_k *$

$\zeta(\sigma)$. Let $Pf' = \mathbf{r}(\Pi(B'))$. Since B$'$ is finite, so is Pf'. Pf shall be the maximal elements of Pf'.

Lemma 5 *If* $\tau \in Pf'$, *then* $\tau(S)$ *is provable.*

Proof: The proof is by induction on $g(\tau)$, where $g \in \omega^{Pf'}$ is defined by (i) if $\tau \in Pf$, then $g(\tau) = 0$, and (ii) if $\tau \in Pf' - Pf$, then $g(\tau)$ is the least integer greater than $\max_{\tau * f \in Pf'} g(\tau * f)$. g is well-defined by Lemma 2.

We verify first that if $g(\tau) = 0$, then $\tau(S)$ is an axiom. Let σ be the element of $\mathbf{A}$ such that $\sigma = \mathbf{s}_1(\tau)$. $\alpha = \alpha'(h_\sigma)$ differs from τ at most in having finite sequences of functions other than $\mathcal{W}$-functions or $\mathcal{B}$-functions inserted before $\mathcal{W}$-functions. If f is a $\mathcal{W}$-function, $\alpha' * f \in \mathbf{r}(\alpha)$, $\tau' * f \in \mathbf{r}(\tau)$, and $\mathbf{s}_1(\alpha') = \mathbf{s}_1(\tau')$, then $(\alpha' * f(S))_1 = (\tau' * f(S))_1$, $((\alpha' * f(S))_0)^1 \in \mathbf{r}(((\tau' * f(S))_0)^1)$, and $((\alpha' * f(S))_0)^2 \in \mathbf{r}(((\tau' * f(S))_0)^2)$. But then $\mathbf{e}_2(\alpha) = \mathbf{e}_2(\tau)$ and $\tau(S)$ is an axiom.

As for the inductive step, it is easy to verify that the functions defined by (1) through (18) of section **8** generate premisses from conclusions according to the rules of inference, modulo applications of the enabling rules. We note with respect to (E_0) that in constructing $\sigma * f$ from σ in $\mathbf{P}_n$, where f is a function defined by (6) of section **8**, the instantiating variable does not occur in $\sigma(S)$. Q.E.D.

To complete the proof of Theorem 6 we need only note that since $\langle f_0 \rangle \in Pf'$, $(f_0)(S) = S$ is provable by Lemma 5, RAA. Q.E.D.

Theorem 7 *If S is valid, then S is provable.*

Proof: In view of Theorems 1 and 3, it suffices to prove that if S is not provable, then $|S|$ is kb'-defensible.

By Theorem 6, there is some $h \in H$ such that no h-hypothesis is an h-axiom. If $\sigma \in \mathbf{W}_h$ and $m \in \omega$, let $C(\tau, \sigma, m) \equiv [\tau \in \mathbf{r}(\pi(h([\sigma]), \kappa_m([\sigma]), m, \kappa_0([\sigma]))))] \;\&\; [\mathbf{s}_2(\tau) = \mathbf{s}_2(\pi(h([\sigma]), \kappa_m([\sigma]), m, \kappa_0([\sigma])))]$, and define $\nu([\sigma]) = \sum_{m \in \omega} \sum_{C(\tau,\sigma,m)} |\tau(S)|$. It is easy to verify that $\nu([\sigma])$ is a model set for each $\sigma \in \mathbf{W}_h$. Let $\Omega = \{\nu([\sigma]) \mid \sigma \in \mathbf{W}_h\}$.

If $\tau \in \mathbf{W}([\sigma], h([\sigma]))$, then there is exactly one $\mathcal{W}$-function f such that if $\rho \in [\tau]$, then $\mathbf{s}_1(\mathrm{pd}(\rho) * f) = \mathbf{s}_1(\rho)$. Let this $\mathcal{W}$-function be $f([\tau])$. We define $R_k, R_b \in (\mathcal{P}(\Omega^2))^{\mathbf{v}(\Sigma\Omega)}$ as follows:

i. If $f([\tau]) = f_{\mathbf{B},k,0}$ or $f([\tau]) = f_{\mathbf{B},k,1,m}$, then $\langle \nu([\sigma]), \nu([\tau]) \rangle \in R_b(a_k)$.
ii. If $f([\tau]) = f_{\mathbf{K},k,1,m}$, then $\langle \nu([\sigma]), \nu([\tau]) \rangle \in R_k(a_k)$.
iii. If $a \in \mathbf{v}(\Sigma\Omega)$ and $R_{k/b}$ is not defined at a by (i) or (ii), then $R_{k/b}$ shall be the empty relation on Ω.

It is a simple if tedious matter to verify from the definitions of the functions of section **8** and the construction of section **9** that $\langle \Omega, R_k, R_b \rangle$ is a kb'-model system. But of course $|S| \subset \nu([(f_0)])$, so $|S|$ is kb'-defensible. Q.E.D.

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