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SIGNIFICANCE LOGICS

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1 *Introduction** Over the last few years, there have been a number of papers supporting the adoption of a 3-valued significance logic. Some of these papers carry the logic further and look into theories of classes and relations. Goddard [2] and Routley [7] give justification for a 3-valued significance logic and delve into its effects on predicates, relations, and classes. Routley [7], p. 188-189, also gives an axiomatisation of a functionally complete predicate logic. Goddard [3] contains an account of two sentential 3-valued significance logics, **T1** and **T2**. Routley, in [8], develops a number of significance logics. Routley [9] and Goddard [4] give further evidence in favour of a 3-valued significance logic in reply to a criticism from Lambert [6]. Philosophical problems regarding significance, especially that concerning sentences and statements, are sorted out in Part I of Goddard and Routley's book, [5].

In this paper I want to take for granted the 3-valuedness of significance logic and the characterisation of the three values given by Goddard and Routley for atomic sentences, i.e., sentences with no logical connectives or quantifiers. Although the assignment of values to atomic sentences is not always clear, I propose to examine the assignment of values to compound sentences, taking for granted some assignment of values to atomic sentences.

I want to follow up Goddard's systems **T1** and **T2** in [3] and Routley's significance logic in [7] and present a sequence of significance logics, each with a characterisation of its own. The problem of determining the value to assign to a compound sentence leads to the problem of determining a subset of the set of all 3-valued connectives and quantifiers such that these and only these connectives and quantifiers are used in determining the values of compound sentences. One then needs criteria to characterise the

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particular subset of the set of all 3-valued connectives and quantifiers. I will present various criteria and determine which subsets they characterise and prove the functional completeness of a set of primitives with respect to each characterisation.

2 *The classical system* **S0** Goddard, in [3], p. 239 introduced the criteria:

II (a) Any compound expression with a non-significant component is non-significant.
(b) Any compound expression in which all the components are significant is itself significant.

III The definitions of the three-valued connectives should "contain" the classical connectives of the two-valued sentential calculus.

These are necessary and sufficient to define the *classical*[1] connectives. The main ones are:

$\sim$			$\&$	1	0	n		$\dot{\vee}$	1	0	n		$\dot{\supset}$	1	0	n		$\dot{\equiv}$	1	0	n
1	0		1	1	0	n		1	1	1	n		1	1	0	n		1	1	0	n
0	1		0	0	0	n		0	1	0	n		0	1	1	n		0	0	1	n
n	n		n	n	n	n		n	n	n	n		n	n	n	n		n	n	n	n

Here '1' represents truth, '0' represents falsity, and 'n' represents non-significance.

If one adopts criterion **IV**, [3], p. 240, "A formula expresses a logical law if, and only if, it comes out true for all possible values of the variables", then one takes '1' as being the only designated value. This yields the sentential part of the classical significance logic, which I wish to consider.

However, if one adopts criterion **IV***, [3], p. 240, "A formula expresses a logical law, if, and only if, it does not come out false for any values of the variables (but may be either true or non-significant)" then '1' and 'n' would both be designated values. In this case unintuitive results follow, as Goddard points out. It seems a rash thing to designate non-significance anyway.

The connective $C(p_1, \cdots, p_n)$ belongs to the set of classical connectives iff (i) '1' or '0' is substituted for all of $p_1, \ldots, p_n$ then $C(p_1, \cdots, p_n)$ takes a value '1' or '0' and (ii) 'n' is substituted for any of $p_1, \ldots, p_n$ then $C(p_1, \cdots, p_n)$ takes the value 'n'.

Theorem *The set of all connectives*[2] *which can be defined from the primitive set,* $\{\sim, \&\}$, *is the set of all classical connectives.*

Proof: In 2-valued sentential calculus the primitive set, $\{\sim, \&\}$, is a

1. This term was introduced in [7], p. 189.

2. I will be using the word 'connective' instead of 'wff' throughout this paper for all the wffs definable from a primitive set of connectives.

functionally complete set, all 2-valued connectives being definable from it. Given a classical connective $C(p_1, \ldots, p_n)$ there is a 2-valued connective $C'(p_1, \ldots, p_n)$, definable from $\{\sim, \&\}$ and taking the same values as $C(p_1, \ldots, p_n)$ for 2-valued substitutions for $p_1, \ldots, p_n$. If C' is extended so that $p_1, \ldots, p_n$ can take the value n as well then whenever n is substituted for some p_i then $C'(p_1, \ldots, p_n)$ takes the value n, by induction on connectives $\sim$ and &, where if A takes the value n then $\sim A$ and A & B both take the value n. Hence $C'(p_1, \ldots, p_n)$ takes the same values as $C(p_1, \ldots, p_n)$, $C'(p_1, \ldots, p_n)$ is defined using $\sim$ and & only, and $C(p_1, \ldots, p_n)$ is definable using $\sim$ and & only.

Conversely, if $\sim$ and & are both classical connectives and if A and B are classical then $\sim A$ and A & B are classical. Hence all connectives definable from $\{\sim, \&\}$ are classical.

Also clearly, $\sim$ and & are independent connectives. To these connectives, one can add the quantifier, **A**, such that $(\mathbf{A}x)f(x)$ takes the value 1 if $f(x)$ takes the value 1 for all x, $(\mathbf{A}x)f(x)$ takes the value n, if $f(x)$ takes the value n for some x, and $(\mathbf{A}x)f(x)$ takes the value 0 if $f(x)$ takes the value 0 for some x and does not take the value n for any x. The domain over which the variable x ranges can be taken as any non-empty domain of individuals. The quantifier satisfies extended versions of the criteria, **II**(a), **II**(b), and **III**, where $f(x)$ is regarded as a component of $(\mathbf{A}x)f(x)$, and hence is taken as a "classical" quantifier. Also, if $A(x)$ is classical then $(\mathbf{A}x)A(x)$ is classical. In this paper, I will regard 'for all x' and 'for some x' as exhausting the possible quantifiers, and since $(\mathbf{E}x)A(x) =_{Df} \sim(\mathbf{A}x) \sim A(x)$ in 2-valued predicate calculus, it can be extended to a definition of 'for some x' in the 3-valued predicate logic. Hence the primitive set $\{\sim, \&, \mathbf{A}\}$ exhausts the connectives and quantifiers of the classical 3-valued predicate logic.

But the trouble with this classical significance logic is that it has no valid wffs, as Goddard points out in [3], p. 240. Hence the system is useless as it is, and other connectives (and perhaps quantifiers) must be added to the classical ones. Goddard, [3], pp. 240-244, adds the operators, T, F, and S and the connective $\equiv$, and develops a 2-sorted system **T1**. I will consider a 1-sorted form of **T1**, for the sake of uniformity. Routley in [8] also develops a 2-sorted system with primitives $\{\sim, \&, T\}$.

3 *The extended classical system* **S1** Goddard, in [3], p. 237, presents the criterion:

I To say of any sentence that it is true, that it is false or that it is *non-significant, is to make a significant statement; and in particular, to say* of a non-significant sentence that it is true or that it is false, is to make a false statement.

This contradicts criterion **II**(a) in that a sentence like 'It is true that Saturday is in bed', which contains a non-significant component, 'Saturday is in bed', is significant because it is false. Criteria **II**(b) and **III** are very plausible ones as they ensure that the 2-valued logic goes through as a subsystem of the 3-valued logic. Also the assignment of values according to criterion I is also very plausible because of the metatheoretic nature of T, F, S, and $\equiv$ and the fact that metatheoretic sentences like "'Saturday is in bed' is true' are obviously 2-valued. The matrices for T, F, S, $\equiv$ are as follows:

T	
1	1
0	0
n	0

F	
1	0
0	1
n	0

S	
1	1
0	1
n	0

$\equiv$	1	0	n
1	1	0	0
0	0	1	0
n	0	0	1

'Tp' is to be interpreted as 'It is true that p.'
'Fp' is to be interpreted as 'It is false that p.'
'Sp' is to be interpreted as 'It is significant that p.'
'$p \equiv q$' is to be interpreted as 'p and q have the same significance value, i.e., the same value in a 3-valued significance logic.'

Note that: $Fp =_{Df} T \sim A$, $SA =_{Df} TA \dot{\vee} FA$, $A \equiv B =_{Df} (TA \; \& \; TB) \dot{\vee} (FA \; \& \; FB) \dot{\vee} (\sim SA \; \& \sim SB)$.

Hence the addition of the extra primitive T to $\{\sim, \&, A\}$ seems to suffice. Moreover, the following can be proved:

Theorem *The set of all connectives which can be defined from the primitive set* $\{\sim, \&, T\}$ *is the set of all connectives* $C(p_1, \ldots, p_n)$ *satisfying the property* $\mathfrak{P}$*: for some subset* S *of all the variables in* $C(p_1, \ldots, p_n)$, $C(p_1, \ldots, p_n)$ *takes the value* n *iff at least one of the variables in* S *takes the value* n.

(*The variables in* S *have at least one place uncovered and the variables not in* S *have all their places covered*[3] (i.e., *uncovered or covered by* T.). *The subset* S *can be empty and also can consist of all the variables*, $p_1, \ldots, p_n$. *In the latter case*, $C(p_1, \ldots, p_n)$ *is a classical connective.*)

Proof: $\sim$, &, and T are connectives with the property $\mathfrak{P}$ and if $C(p_1, \ldots, p_n)$ and $D(q_1, \ldots, q_m)$ are connectives with property $\mathfrak{P}$ then $\sim C(p_1, \ldots, p_n)$ (with the same subset of the variables), $C(p_1, \ldots, p_n) \; \& \; D(q_1, \ldots, q_m)$ (with the subset being the union of the subsets for C and D), and $TC(p_1, \ldots, p_n)$ (with the empty subset) are all connectives with the property $\mathfrak{P}$.

Conversely, let $C(p_1, \ldots, p_n)$ be a connective with the property $\mathfrak{P}$. Let S be the corresponding subset and let T be the subset of the variables which is the complement of S. Let $q_1, \ldots, q_i$ be the variables in S and $r_1, \ldots, r_m$ the variables in T.

Note that $p \dot{\vee} \sim p$ has the matrix:

1	1
0	1
n	n

3. The terms 'covered' and 'uncovered' are from [8], p. 5.

Also $p \,\&\, {\sim}p$ has the matrix:

1	0
0	0
n	n

Let $j_1, \dots, j_i$ be the values of a particular assignment to $q_1, \dots, q_i$, where $q_1, \dots, q_i$ take the values 1 or 0 only, and let $k_1, \dots, k_m$ be the values of a particular assignment to $r_1, \dots, r_m$, where $r_1, \dots, r_m$, can take any value. Let $L_{j_1,\dots,j_i,\, k_1,\dots,k_m}$ be $(q_1 \dot{\vee} {\sim} q_1) \,\&\, \dots \,\&\, (q_i \dot{\vee} {\sim} q_i)$ if $C(p_1, \dots, p_n)$ takes the value 1 under the above assignment, and let $L_{j_1,\dots,j_i,\, k_1,\dots,k_m}$ be $(q_1 \,\&\, {\sim} q_1) \,\&\, \dots \,\&\, (q_i \,\&\, {\sim} q_i)$ if $C(p_1, \dots, p_n)$ takes the value 0 under the above assignment. For each assignment of values, form the formula $J_1 q_1 \,\&\, \dots \,\&\, J_i q_i \,\&\, K_1 r_1 \,\&\, \dots \,\&\, K_m r_m \supset L_{j_1,\dots,j_i,\, k_1,\dots,k_m}$, where $J_1, \dots, J_i$ are T or F according to the value 1 or 0, respectively, of the $j_1, \dots, j_i$, and where $K_1, \dots, K_m$ are T, F, or ${\sim}S$ according to the value 1, 0, or n, respectively, of the $k_1, \dots, k_m$. $C(p_1, \dots, p_n)$ can then be defined as the conjunction, using &, of all of these formulae, one formula corresponding to each assignment of values to $p_1, \dots, p_n$, with $q_1, \dots, q_i$ taking the values 1 or 0 only. Given one such assignment, one and only one expression of the form $J_1 q_1 \,\&\, \dots \,\&\, J_i q_i \,\&\, K_1 r_1 \,\&\, \dots \,\&\, K_m r_m$ will be true while all other such expressions will be false; and also each expression $L_{j_i,\dots,j_i,\, k_1,\dots,k_m}$ appearing in the conjunction will be true or false. Hence, by 2-valued logic, the formula $J_1 q_1 \,\&\, \dots \,\&\, J_i q_i \,\&\, K_1 r_1 \,\&\, \dots \,\&\, K_m r_m \supset L_{j_1,\dots,j_i, k_1,\dots,k_m}$, corresponding to the particular assignment under consideration, will take the same value as $L_{j_1,\dots,j_i, k_1,\dots,k_m}$, while all the other conjuncts will be true. Hence, the conjunction of the formulae, defined to be $C(p_1, \dots, p_n)$, will take the value of $(q_1 \dot{\vee} {\sim} q_1) \,\&\, \dots \,\&\, (q_i \dot{\vee} {\sim} q_i)$ or $(q_1 \,\&\, {\sim} q_1) \,\&\, \dots \,\&\, (q_i \,\&\, {\sim} q_i)$, according as $C(p_1, \dots, p_n)$ takes the value 1 or 0, respectively. Hence the definition is satisfactory for $C(p_1, \dots, p_n)$, where $q_1, \dots, q_i$ takes the values 1 or 0 only.

If at least one of $q_1, \dots, q_i$ take the value n, then in the proposed definition for $C(p_1, \dots, p_n)$ each expression $L_{j_1,\dots,j_i, k_1,\dots,k_m}$ will take the value n and hence the proposed definition will be satisfactory in this case as well. Hence any connective $C(p_1, \dots, p_n)$ satisfying the property $\mathfrak{P}$ can be defined in terms of $\sim$, &, and T.

Theorem *The connectives* $\sim$, &, *and* T *are independent.*

Proof: (i) & cannot be defined in terms of $\sim$ and T, since they are both monadic.

(ii) Let $\Delta(p)$ be defined in terms of $\sim$ and & only. If p has the value n then $\Delta(p)$ has the value n. Hence T is not definable from $\sim$ and & only.

(iii) Let $\Delta(p)$ be defined in terms of & and T only. If p has the value 1 then $\Delta(p)$ has the value 1. Hence $\sim$ is not definable from & and T only.

As before, one can add the quantifier **A** to the primitives to give $\{\sim, \&, T, \mathbf{A}\}$. This is defined as in the previous system. The quantifier **E** can also be defined as in the previous system.

Now I will give an axiomatisation for the system **S1** with the primitives $\{\sim, \&, T, \mathbf{A}\}$, firstly axiomatising the sentential system and then the predicate system.

S1: *Sentential calculus* So as to be able to apply one of the rules, I need to axiomatise the 2-valued system **P** first.

System **P**

Primitives

1. $p, q, r, \ldots$ (2-valued sentential variables).
2. $\sim, \supset$ (negation and implication connectives).

Formation Rules

1. A 2-valued sentential variable is a wff.
2. If A and B are wffs, then $\sim A$ and $A \supset B$ are wffs.

Definitions

$A \;\&\; B =_{Df} \sim(A \supset \sim B)$.
$A \vee B =_{Df} \sim A \supset B$.
$A \equiv B =_{Df} (A \supset B) \;\&\; (B \supset A)$.
$TA =_{Df} A$.
$FA =_{Df} T \sim A$.
$SA =_{Df} TA \vee FA$.

Axioms

1. $p \supset . \; q \supset p$.
2. $p \supset (q \supset r) \supset . \; p \supset q \supset . \; p \supset r$.
3. $\sim p \supset \sim q \supset . \; q \supset p$.

Rules

1. Uniform Substitution for sentential variables.
2. $\vdash_{\mathbf{P}} A, \vdash_{\mathbf{P}} A \supset B \Rightarrow \vdash_{\mathbf{P}} B$.

System **S1**

The system **P** is used in the construction of the system **S1**. Note that the two systems have the common symbols p, q, r, etc. for sentential variables, and $\sim$, $\supset$ for connectives. It will be clear from the context which system the symbol is being used in.

Primitives

1. $p, q, r, \ldots$ (3-valued sentential variables).
2. $\sim, \&, T$ (connectives).

Formation Rules

1. A sentential variable is a wff.
2. If A and B are wffs then $\sim A$, $A \;\&\; B$ and TA are wffs.

Definitions

$A \dot{\vee} B \;=_{Df} \sim(\sim A \;\&\; \sim B)$.
$A \dot{\supset} B =_{Df} \sim A \dot{\vee} B$.
$A \dot{\equiv} B =_{Df} (A \dot{\supset} B) \;\&\; (B \dot{\supset} A)$.
$A \supset B =_{Df} TA \dot{\supset} TB$ } For the purpose of using '⊃'
$A \vee B \;=_{Df} A \dot{\vee} B$ } and 'v' of system **P** in Rule 3.
$FA \;=_{Df} T \sim A$.
$SA \;=_{Df} TA \vee FA$.

Axioms

1. STp.
2. $\sim Sp \supset \sim S \sim p$.
3. $\sim Sp \vee \sim Sq \supset \sim S(p \;\&\; q)$.
4. $\sim Sp \supset FTp$.

Rules

1. Uniform substitution for sentential variables.
2. $\vdash_{S1} A$, $\vdash_{S1} A \supset B \Rightarrow \vdash_{S1} B$.
3.[4] $\vdash_{P} A(p_1, \ldots, p_n) \Rightarrow \vdash_{S1} SB_1 \supset \ldots \supset .\; SB_n \supset A(B_1, \ldots, B_n)$, where $p_1, \ldots, p_n$ are *all* the variables in A.

Theorems (without proof):[5]

SFp; SSp; $Fp \supset T \sim p$; $Tp \supset Sp$; $Tp \supset F \sim p$; $Fp \supset Sp$; $Tp \;\&\; Tq \supset T(p \;\&\; q)$; $Tp \;\&\; Fq \supset F(p \;\&\; q)$; $Fp \;\&\; Sq \supset F(p \;\&\; q)$; $Tp \supset TTp$; $Fp \supset FTp$; $Tp \vee Fp \vee \sim Sp$; $(Tp \supset Tq) \;\&\; (Fp \supset Tq) \;\&\; (\sim Sp \supset Tq) \supset Tq$; $\vdash_{S1} TA \Rightarrow \vdash_{S1} A$.

The completeness proof with respect to the 3-valued matrices is one adapted from that of Church, [1], pp. 97-99, for the 2-valued sentential logic.[6]

Lemma *Let B be a wff of* **S1** *and let $p_1, \ldots, p_n$ be the distinct variables occurring in B. Let A_i be Tp_i, Fp_i or $\sim Sp_i$ according as the value a_i of p_i is* 1, 0, *or n. Let B' be TB, FB, or $\sim SB$ according as the value of B, for values $a_1, a_2, \ldots, a_n$ of $p_1, p_2, \ldots, p_n$ is* 1, 0, *or n. Then $\vdash_{S1} A_1 \;\&\; \ldots \;\&\; A_n \supset B'$.*

Proof: By induction on the number of variables. It is trivial in the case of a single variable.

(i) B is $\sim B_1$ By ind. hyp., $A_1 \;\&\; - \;\&\; A_n \supset B_1'$. If B_1 has the value 1, 0, or n, then B has the value 0, 1, or n, respectively. So, if B_1' is TB_1, FB_1 or $\sim SB_1$, then B' is $F \sim B_1$, $T \sim B_1$ or $\sim S \sim B_1$, respectively. By T.5, T.3, and A.2,

4. This is the idea of R. Routley.

5. The following theorems will be referenced in the sequel as T.1, T.2, . . . , T.13 and D.R.1, respectively.

6. This adaptation was suggested by R. Routley.

$TB_1 \supset F \sim B_1$, $FB_1 \supset T \sim B_1$ and $\sim SB_1 \supset \sim S \sim B_1$. Hence $B_1' \supset B'$ and, by using P and R.3, A_1 & ... & $A_n \supset B'$.

(ii) B is B_1 & B_2 By ind. hyp., A_1 & – & $A_n \supset B_1'$ and A_1 & – & $A_n \supset B_2'$, where $p_1, \ldots, p_n$ are all the distinct variables in B_1 & B_2.

(a) If B_1 and B_2 have the value 1, then B has the value 1. So, if B_1' is TB_1 and B_2' is TB_2, then B' is $T(B_1$ & $B_2)$. By T.7, Tp & $Tq \supset T(p$ & $q)$. Hence B_1' & $B_2' \supset B'$ and by using **P** and R.3, A_1 & ... & $A_n \supset B'$.
(b) If B_1 has the value 1 and B_2 has the value 0 then B has the value 0. So, if B_1' is TB_1 and B_2' is FB_2, then B' is $F(B_1$ & $B_2)$. By T.8, Tp & $Fq \supset F(p$ & $q)$. Hence B_1' & $B_2' \supset B'$ and, using **P** and R.3, A_1 & ... & $A_n \supset B'$.
(c) If B_1 has the value 0 and B_2 has the value 1 or 0 then B has the value 0. So, if B_1' is FB_1 and B_2' is TB_2 or FB_2, then B' is $F(B_1$ & $B_2)$. By T.9, Fp & $Sq \supset F(p$ & $q)$. By T4, **P**, and R.3, Fp & $Tq \supset Fp$ & Sq. Hence, by **P** and R.3, Fp & $Tq \supset F(p$ & $q)$. By T.6, **P**, and R.3, Fp & $Fq \supset F(p$ & $q)$. Hence B_1' & $B_2' \supset B'$ and A_1 & ... & $A_n \supset B'$.
(d) If B_1 or B_2 has the value n then B has the value n. So, if B_1' is $\sim SB_1$ or B_2' is $\sim SB_2$ then B' is $\sim S(B_1$ & $B_2)$. By A.3, $\sim Sp \vee \sim Sq \supset \sim S(p$ & $q)$. By **P** and R.3, $\sim Sp \supset \sim S(p$ & $q)$ and $\sim Sq \supset \sim S(p$ & $q)$. Hence $B_1' \supset B'$ and A_1 & ... & $A_n \supset B'$.

(iii) B is TB_1 By ind. hyp., A_1 & ... & $A_n \supset B_1'$. If B_1 has the value 1, 0, or n, then B has the value 1, 0, or 0, respectively. So, if B_1' is TB_1, FB_1, or $\sim SB_1$, then B' is TTB_1, FTB_1, or $\sim STB_1$, respectively. By T.10, T.11, and A.4, $TB_1 \supset TTB_1$, $FB_1 \supset FTB_1$, and $\sim SB_1 \supset FTB_1$. Hence $B_1' \supset B'$ and A_1 & ... & $A_n \supset B'$.

Metatheorem *If B is valid according to the 3-valued matrices, then B is a thesis of* **S1**.

Proof: Let $P_1, \ldots, P_n$ be the distinct variables of B. Let $A_1, \ldots, A_n$ be as in the Lemma. Since B is valid, B' is TB, independently of the values of $P_1, \ldots, P_n$. Hence, $\vdash_{S1} A_1$ & ... & A_{n-1} & $TP_n \supset TB$. By **P** and R.3, $\vdash_{S1} A_1$ & ... & $A_{n-1} \supset (TP_n \supset TB)$. Similarly, $\vdash_{S1} A_1$ & ... & $A_{n-1} \supset (FP_n \supset TB)$ and $\vdash_{S1} A_1$ & ... & $A_{n-1} \supset (\sim SP_n \supset TB)$. By **P** and R.3, $\vdash_{S1} A_1$ & ... & $A_{n-1} \supset .\ (TP_n \supset TB)$ & $(FP_n \supset TB)$ & $(\sim SP_n \supset TB)$. Using T.13, $\vdash_{S1} A_1$ & ... & $A_{n-1} \supset TB$. By repeating this procedure for each A_i, $1 \leqslant i \leqslant n - 1$, $\vdash_{S1} TB$. Hence, by D.R.1, $\vdash_{S1} B$.

S1: *The Predicate Calculus*

System **P**

Extra Primitives The following are added to the sentential system **P**.

3. $x, y, z, \ldots$ (individual variables).
4. $f, g, h, \ldots$ (predicate variables).
5. **A** (universal quantifier).

Extra Formation Rules

3. If f is an n-ary predicate variable and $x_1, \ldots, x_n$ are individual variables then $f(x_1, \ldots, x_n)$ is a wff.
4. If B is a wff and x an individual variable then $(\mathbf{A}x)\ B$ is a wff.

Extra Definition

$(\mathbf{S}x)\ A =_{Df} \sim(\mathbf{A}x) \sim A.$

Axioms (The axioms of the sentential system **P** are written in schematic form.)

4. $(\mathbf{A}x)A(x) \supset A(y)$, *where y is substituted for the free occurrences of x in A such that y, on substitution, does not become bound thereby.*

Extra Rule

3. $\vdash_{\mathbf{P}} A \supset B \Rightarrow \vdash_{\mathbf{P}} A \supset (\mathbf{A}x)\ B$, where x is not free in A.

System **S1**

The following are added to the sentential system **S1**:

Extra Primitives

3. $x, y, z, \ldots$ (individual variables).
4. $f, g, h, \ldots$ (predicate variables).
5. **A** (universal quantifier).

Extra Formation Rules

3. If f is an n-ary predicate variable and $x_1, \ldots, x_n$ are individual variables, then $f(x_1, \ldots, x_n)$ is a wff.

Extra Definitions

$(\mathbf{E}x)\ A =_{Df} \sim(\mathbf{A}x) \sim A.$
$(\mathbf{S}x)\ A =_{Df} (\mathbf{E}x)A.$ (For the purpose of using the 'S' of system **P** in R.3.)

Axioms (The axioms of the sentential system **S1** are written in schematic form.)

5. $(\mathbf{A}x)A(x) \supset A(y)$, *where y is substituted for the free occurrences of x in A such that y, on substitution does not become bound thereby.*
6. $(\mathbf{S}x) \sim SA(x) \supset \sim S(\mathbf{A}x)A(x)$.

Extra Rules

4. $\vdash_{\mathbf{S1}} A \supset B \Rightarrow \vdash_{\mathbf{S1}} A \supset (\mathbf{A}x)B$, where x is not free in A.
5. $\vdash_{\mathbf{P}} \mathcal{A}(A_1(x_{1,1}, \ldots, x_{1,i_1}), \ldots, A_n(x_{n,1}, \ldots, x_{n,i_n})) \Rightarrow \vdash_{\mathbf{S1}} (\mathbf{A}x_{1,1}, \ldots, x_{1,i_1}) S\mathcal{B}_1(x_{1,1}, \ldots, x_{1,i_1}) \supset \ldots\ldots\ldots \supset (\mathbf{A}x_{n,1}, \ldots, x_{n,i_n}) S\mathcal{B}_n(x_{n,1}, \ldots, x_{n,i_n}) \supset \mathcal{A}(\mathcal{B}_1(x_{1,1}, \ldots, x_{1,i_1}), \ldots, \mathcal{B}_n(x_{n,1}, \ldots, x_{n,i_n}))$, where $A_1, \ldots, A_n$ are the only wff-schemata in $\mathcal{A}$ and $x_{j,1}, \ldots, x_{j,i_j}$ are the only variables (i.e., free or bound by $\mathcal{A}$) in A_j and the only *free* variables (i.e., free or bound by $\mathcal{A}$) in $\mathcal{B}_j$. (The $\mathcal{A}$ and $\mathcal{B}_i$'s are schemata for the wff-schemata A, B, etc.) (This rule is a generalisation of Rule 3.)

Theorems (without proof):

$\vdash_{\mathbf{S1}} A \Rightarrow \vdash_{\mathbf{S1}} (\mathbf{A}x)\ A$; $\vdash_{\mathbf{S1}} A \supset B \Rightarrow \vdash_{\mathbf{S1}} (\mathbf{A}x)\ A \supset (\mathbf{A}x)\ B$; $(\mathbf{A}x)\ TA(x) \supset T(\mathbf{A}x)\ A(x)$; $S(\mathbf{A}x)A(x) \supset (\mathbf{A}x)SA(x)$; $(\mathbf{S}x)FA(x)\ \&\ (\mathbf{A}x)SA(x) \supset F(\mathbf{A}x)A(x)$; $F(\mathbf{A}x)A(x) \supset (\mathbf{S}x)FA(x)$; $T(\mathbf{S}x)A(x) \supset (\mathbf{S}x)\ TA(x)$.

Metatheorem 1 *The Deduction Theorem holds in* **S1** *for* $\supset$, i.e., *if* $A_1, \ldots, A_n \vdash_{S1} B$ *then* $A_1, \ldots, A_{n-1} \vdash_{S1} A_n \supset B$, *provided* Rule 4 *is not used to generalise on any variable of* A_n.

Metatheorem 2 *If* B *is valid according to the* 3-*valued matrices and the properties stated for* A, *then* B *is a thesis of the axiomatic system* **S1**.

Sketch of the proof: The proof is modelled on the proof for the completeness of the 2-valued predicate calculus given in Church, [1], pp. 238-245. If Γ is any class of wffs and B is any wff, then $\Gamma \vdash B$ if there is a finite number of wffs $A_1, \ldots, A_n$ of Γ such that $A_1, \ldots, A_n \vdash B$. A class Γ of wffs is called *inconsistent* if there exists a wff B such that $\Gamma \vdash B$ and $\Gamma \vdash \sim B$. If no such B exists, then Γ is *consistent.* If Γ is any class of wffs and C is any wff, then C is *consistent with* Γ if the class $\{C\} \cup \Gamma$ is consistent; otherwise C is *inconsistent with* Γ. A class Γ of wffs is called a *maximal consistent* class if Γ is consistent and if C is consistent with Γ then $C \in \Gamma$.

Lemma *Every consistent class* Γ *of wffs can be extended to a maximal consistent class* $\overline{\Gamma}$, i.e., *there exists a maximal consistent class* $\overline{\Gamma}$ *containing* Γ.

Proof: Similar to that in [1].

As in Church, we consider an infinite sequence of applied predicate logics, $\mathbf{S}_0, \mathbf{S}_1, \mathbf{S}_2, \ldots$, which have as primitive symbols all the primitive symbols of the system **S** and in addition certain individual constants. Viz., the primitive symbols of $\mathbf{S}_0$ are those of **S1** and the individual constants $W_{0,0}, W_{1,0}, W_{2,0}, \ldots$, the primitive symbols of $\mathbf{S}_{n+1}$ are those of $\mathbf{S}_n$ and the additional individual constants $W_{0,n+1}, W_{1,n+1}, W_{2,n+1}, \ldots$. Also let $\mathbf{S}_\omega$ be the applied predicate logic which has as its primitive symbols the primitive symbols of all of the systems $\mathbf{S}_0, \mathbf{S}_1, \mathbf{S}_2, \ldots$. All the wffs of $\mathbf{S}_\omega$ can be enumerated and so can the wffs of each $\mathbf{S}_n$ by deleting from the enumeration of the wffs of $\mathbf{S}_\omega$ the wffs not in $\mathbf{S}_n$. Let Γ_0 be a given consistent class of wffs of $\mathbf{S}_0$ which have no free individual variables. We define the classes Γ_n^m as follows: Γ_1^0 is $\overline{\Gamma}_0$. If the $(m+1)$'st wff $\mathbf{S}_n$, $n > 0$, has the form $(\mathsf{S}x)A(x)$ and is a member of Γ_n^0, then Γ_n^{m+1} is the class whose members are $A(W_{m,n})$ and the members of Γ_n^m; otherwise Γ_n^{m+1} is Γ_n^m. Also Γ_{n+1}^0 is $\overline{\Delta}_n$ where Δ_n is the union of the classes $\Gamma_n^0, \Gamma_n^1, \ldots$. The members of Γ_n^m are wffs of $\mathbf{S}_n$ and Γ_{n+1}^0 is a maximal consistent class of wffs of $\mathbf{S}_n$. One can prove that, if Γ_n^m is consistent then $\Gamma_n^{m\ 1}$ is consistent. This makes use of some of the theorems stated and of Rule 5, particularly.

Let Γ_ω be the union of the classes $\Gamma_1^0, \Gamma_2^0, \Gamma_3^0, \ldots$. Then Γ_ω is a maximal consistent class of wffs of $\mathbf{S}_\omega$. The following are properties of Γ_ω:

(a) If $A \in \Gamma_\omega$, then $\sim A \notin \Gamma_\omega$.
(b) If $SA \in \Gamma_\omega$ and $A \notin \Gamma_\omega$, then $\sim A \in \Gamma_\omega$.
(c) If $A \notin \Gamma_\omega$, then $\sim TA \in \Gamma_\omega$.
(d) At least one of TA, FA, $\sim SA$ is a member of Γ_ω.
(e) At most one of TA, FA, $\sim SA$ is a member of Γ_ω.

Now we can make an assignment of values to each member of $\mathbf{S}_0$. If $TA \in \Gamma_\omega$, then A has T, if $FA \in \Gamma_\omega$, then A has F, and if $\sim SA \in \Gamma_\omega$ then A has $\sim S$.

To show that the assignment is a consistent one, one must show that the primitive connectives satisfy the matrices and that the quantifier satisfies its property.

(i) $\sim$. One needs: $TA \supset F \sim A$; $FA \supset T \sim A$; $\sim SA \supset \sim S \sim A$.
(ii) &. One needs: $TA \,\&\, TB \supset T\,(A \,\&\, B)$; $TA \,\&\, FB \supset F\,(A \,\&\, B)$; $FA \,\&\, SB \supset F\,(A \,\&\, B)$; $\sim SA \vee \sim SB \supset \sim S\,(A \,\&\, B)$.
(iii) T. One needs: $TA \supset TTA$; $FA \supset FTA$; $\sim SA \supset FTA$.
(iv) $\mathbf{A}$. One needs: $(\mathbf{A}x)\,TA(x) \supset T\,(\mathbf{A}x)A(x)$; $S(\mathbf{A}x)A(x) \supset (\mathbf{A}x)\,SA(x)$; $(\mathbf{S}x)\,FA(x) \,\&\, (\mathbf{A}x)\,SA(x) \supset F(\mathbf{A}x)A(x)$.

Since Γ_0 was chosen as an arbitrary consistent class of wffs of $\mathbf{S}_0$ without free individual variables, every consistent class of wffs of $\mathbf{S}_0$ without free individual variables is simultaneously satisfiable in a denumerable domain. This extends to every consistent class of wffs of **S1**. Let B be a valid wff of **S1**. Then TB is valid and the class consisting of $\sim TB$ only is not simultaneously satisfiable and hence it is not consistent. Hence, for some wff A, $\sim TB \vdash A$ and $\sim TB \vdash \sim A$. By the Deduction Theorem, $\vdash TB$. Hence $\vdash B$. This completeness proof assumed the consistency of the system **S1**, which can easily be proved using the domain of one individual. In this proof, there is no difference between $(\mathbf{A}x)A(x)$ and $A(x_0)$, where x_0 is that individual. Thus all the quantifiers can be removed and the system reduced to a sentential one.

4 *The system* **S2** *with the positive s-n sublogic* The system **S1** allows one to deal with sentences containing the classical connectives and allows one to state their truth, falsity, or non-significance. However, there are cases of sentences of ordinary discourse that require connectives and quantifiers outside the scope of those of **S1**. To obtain the system **S2** I wish to add a disjunction and a corresponding existential quantifier. The disjunction, $\vee$, is obtained from the criterion, "If one disjunct in a disjunction is non-significant then this disjunct is ignored when assessing the value of the disjunction." That is, if 'p' is non-significant, then '$p \vee q$' has the same value as 'q'. This gives the matrix:

$\vee$	1	0	n
1	1	1	1
0	1	0	0
n	1	0	n

The disjunction is formally justified by regarding it as an existential quantifier over a finite range. This quantifier, $\mathbf{S}$, has the property that the value of $(\mathbf{S}x)\phi(x)$ is assessed by examining only the significant $\phi(x)$'s and that $(\mathbf{S}x)\phi(x)$ is only non-significant if $\phi(x)$ is non-significant for all x. When assessing the value of $(\mathbf{S}x)\phi(x)$ from the significant $\phi(x)$'s the classical 2-valued logic applies, in accordance with Goddard's criterion **II**(b).

If one formalises 'Something is happy' as $(\mathsf{S}x)\,H(x)$, using the above quantifier S, then $(\mathsf{S}x)\,H(x)$ will be true since some person in this world is happy and the quantifier has the effect of restricting the variable x to items of the "right" category (i.e., animals). 'Something is happy' seems to me to be true because there is a happy person who is certainly a "thing" in whatever broad sense this has. However, if an existential quantifier E is used in place of S, where E satisfies Goddard's criterion II(a), so that $(\mathsf{E}x)\phi(x)$ is non-significant if $\phi(x)$ is non-significant for some x, then $(\mathsf{E}x)\,H(x)$ is non-significant because it is non-significant for stones to be happy and "thing" would include all material objects. In fact, $(\mathsf{E}x)fx$ is always non-significant, except for rare predicates which have the universe as their significance range (e.g., 'is a thing'). Thus the quantifier S is necessary to represent 'Something is happy' and to ensure the significance of existential sentences. If the quantifier S is used over a finite range then $(\mathsf{S}x)\phi(x)$ would be equivalent to a finite disjunction where the connective would be '$\vee$'. Similarly to 'S' and '$\vee$', one can introduce a universal quantifier $\forall$ and a conjunction $+$, as follows:

$$p + q =_{Df} \sim(\sim p \vee \sim q);\ (\forall x)\phi(x) =_{Df} \sim(\mathsf{S}x) \sim\phi(x).$$

'+' is represented by the matrix:

+	1	0	n
1	1	0	1
0	0	0	0
n	1	0	n

Similarly to $(\mathsf{S}x)\phi(x)$, the value of $(\forall x)\phi(x)$ is assessed by examining only the significant $\phi(x)$'s and $(\forall x)\phi(x)$ is only non-significant if $\phi(x)$ is non-significant for all x. Also, when assessing the value of $(\forall x)\phi(x)$ from the significant $\phi(x)$'s the classical 2-valued logic applies, in accordance with Goddard's criterion II(b).

Consider the example, 'Not all that glitters is gold.' The intended meaning is that not all material objects that glitter are gold. By using the quantifier $\forall$ the sentence can be formalised without mention of material objects because of the automatic restriction to the significance range of 'x glitters' and 'x is gold'. If the quantifier A is used in place of $\forall$ where A satisfies Goddard's criterion II(a), so that $(\mathsf{A}x)\phi(x)$ is non-significant if $\phi(x)$ is non-significant for some x, then the variable x must be restricted to material objects otherwise the sentence formalised as '$\sim(\mathsf{A}x)(\mathrm{Gl}x \supset \mathrm{Gd}x)$' will be non-significant which it clearly is not. So the quantifier $\forall$ allows a more direct and natural formalisation of 'Not all that glitters is gold.' Again, if such quantification ranges over a finite domain then it can be replaced by a finite conjunction using the connective, $+$.

There is another use of the disjunction, $\vee$. Consider the compound predicate, 'is a holiday or likes cheese'. If 'x is a holiday' is true (say, x is New Year's Day) then 'x is a holiday or likes cheese' is true. If 'x is a holiday' and 'x likes cheese' are both non-significant (say, x is a piece of wood) then 'x is a holiday or likes cheese' is non-significant. If 'x is a holiday' is false and 'x likes cheese' is not true (say, x is my birthday) then 'x is a holiday or likes cheese' is false. Letting 'x is a holiday' be p, 'x likes cheese' be q and 'x is a holiday or likes cheese' be $p \mathbf{D} q$, the matrix for **D** as determined above is:

D	1	0	n
1	1	1	1
0	1	0	0
n	1	0	n

which is exactly the matrix for $\vee$. Thus $fx \vee gx$ has the same value as (f or g)x and the disjunction $\vee$ can be used in representing a predicate disjunction. This does not mean that (f or g)x can be interpreted as $fx \dot{\vee} gx$, where the 'or' is a classical sentential connective. The sentence 'x is a holiday or x likes cheese' would always be non-significant because whenever 'x is a holiday' is significant, 'x likes cheese' is non-significant, and vice versa.

Thus the disjunction $\vee$ is formally useful and has some application in ordinary discourse. Hence I will consider the system **S2** obtained by adding $\vee$ to the primitive connectives of **S1** and **S** to the primitive quantifier of **S1**. The system **S2** with primitives $\{\sim, \&, T, \vee, \mathbf{A}, \mathbf{S}\}$ will be shown to have all of its connectives exhausting all the "positive" connectives which can be used to form an s - n sublogic and which contain a 2-valued connective of the sentential calculus. An s - n *sublogic* is obtained by grouping together the significant values, 1 and 0, and calling it the value s, while the non-significant value n remains intact. In order to be able to perform this on a connective one must be able consistently to assign the value s or n in the 2-valued matrix of the connective. One can do this for the connectives $\sim$, &, T, and $\vee$ as follows:

$\sim$	
s	s
n	n

&	s	n
s	s	n
n	n	n

T	
s	s
n	s

$\vee$	s	n
s	s	s
n	s	n

A connective $C(p_1, \ldots, p_n)$ satisfies *the positive property* if for any i, if $C(p_1, \ldots, p_n)$ takes the value n and p_i takes the value 1 or 0 then $C(p_1, \ldots, p_n)$ takes the value n with p_i taking the value n and with the p_j's ($j \neq i$) left intact.

To give some idea, all the monadic and dyadic positive connectives which can be used to form an s - n sublogic and which contain a 2-valued connective of the sentential calculus can be represented by the following s - n matrices:

Monadic

(1)

s	s
n	s

(2)

s	s
n	n

Dyadic

(1)

	s	n
s	s	s
n	s	s

(2)

	s	n
s	s	s
n	s	n

(3)

	s	n
s	s	s
n	n	n

(4)

	s	n
s	s	n
n	s	n

(5)

	s	n
s	s	n
n	n	n

Note that, for a connective $C(p_1, \cdots, p_n)$ to contain a 2-valued connective of **S.C.**, on substituting a significant value for all the $p_1, \cdots, p_n$, $C(p_1, \cdots, p_n)$ must take a significant value.

Theorem *The set of all connectives which can be defined from the primitive set* $\{\sim, \&, T, \vee\}$ *is the set of all positive connectives which can be used to form an* s - n *sublogic and which contain a connective of the* 2-*valued* **S.C.**

Proof: Call the conjunction of these three properties, the property $\mathfrak{P}$. $\sim$, &, T, and $\vee$ contain a connective of the 2-valued **S.C.** and, as shown above, can be used to form an s - n sublogic. The positive property is vacuously true for monadic connectives and it is the case for &. If $p \vee q$ takes the value n while p takes the value 1 or 0, then $p \vee q$ still takes the value n when p takes the value n. If $C(p_1, \ldots, p_n)$ and $D(q_1, \cdots, q_m)$ satisfy the property $\mathfrak{P}$ then $\sim C(p_1, \ldots, p_n)$, $C(p_1, \cdots, p_n) \& D(q_1, \cdots, q_m)$, $TC(p_1, \ldots, p_n)$ and $C(p_1, \ldots, p_n) \vee D(q_1, \ldots, q_m)$ satisfy the property $\mathfrak{P}$.

First consider the monadic and dyadic cases for the converse result. The classical connectives $\sim$ and & can be used to define all the monadic connectives of type (2) and all the dyadic connectives of type (5). The connectives T, F, S, $Tp \dot{\vee} \sim Tp$ (taking the value 1 only) and $Tp \& \sim Tp$ (taking the value 0 only) can be defined as in **S1**. Let $\mathbf{t}p$ denote $Tp \dot{\vee} \sim Tp$ and let $\mathbf{f}p$ denote $Tp \& \sim Tp$. The monadic connectives of type (1) are obtained in the form $(\sim Tp \vee (\mathbf{t}p$ or $\mathbf{f}p)) \& (\sim Fp \vee (\mathbf{t}p$ or $\mathbf{f}p)) \& (Sp \vee (\mathbf{t}p$ or $\mathbf{f}p))$, where substitutions of $\mathbf{t}p$ or $\mathbf{f}p$ in the places indicated will yield the eight required connectives. The dyadic connectives of type (1) are obtained in the form $(\sim(Tp \& Tq) \vee (\mathbf{t}p$ or $\mathbf{f}p)) \& (\sim(Tp \& Fq) \vee (\mathbf{t}p$ or $\mathbf{f}p)) \& (\sim(Tp \& \sim Sq) \vee (\mathbf{t}p$ or $\mathbf{f}p) \& (\sim(Fp \& Tq) \vee (\mathbf{t}p$ or $\mathbf{f}p)) \& (\sim(Fp \& Fq) \vee (\mathbf{t}p \vee \mathbf{f}p)) \& (\sim(Fp \& \sim Sq) \vee (\mathbf{t}p$ or $\mathbf{f}p)) \& (\sim(\sim Sp \& Tq) \vee (\mathbf{t}p$ or $\mathbf{f}p)) \& (\sim(\sim Sp \& Fq) \vee (\mathbf{t}p$ or $\mathbf{f}p)) \& (\sim(\sim Sp \& \sim Sq) \vee (\mathbf{t}p$ or $\mathbf{f}p))$, where substitution of $\mathbf{t}p$ or $\mathbf{f}p$ in the places indicated will yield the 2^9 required connectives.

Next consider the connective defined by $(p \vee \sim p) \vee (q \vee \sim q)$. It has the matrix:

	1	0	n
1	1	1	1
0	1	1	1
n	1	1	n

Take each of the dyadic connectives of type (1) in turn and form the conjunction, using &, of it and $(p \vee \sim p) \vee (q \vee \sim q)$. All the places in the matrix of the dyadic connective of type (1) will remain intact except for the n - n place, which will be converted to an 'n'. By using this method all the dyadic connectives of type (2) can be defined.

Next consider the connective defined as $(p \vee \sim p)$ & SSq. It has the matrix:

	1	0	n
1	1	1	1
0	1	1	1
n	n	n	n

As above, take each of the dyadic connectives of the type (1) in turn and form the conjunction, using &, of it and $(p \vee \sim p)$ & SSq. All the places in the matrix of the dyadic connective of type (1) will remain intact except for the n - 1, n - 0, and n - n places, which will be converted to 'n's'. By using this method all the dyadic connectives of type (3) can be defined. Similarly, by using the connective SSp & $(q \vee \sim q)$ all the dyadic connectives of type (4) can be defined.

Next consider the generalisation to n'adic connectives. The type (1) monadic and dyadic connectives can be generalised to n'adic connectives by representing each place $(a_1, \ldots, a_n)$ in the matrix by a conjunction $K_1 p_1$ & . . . & $K_n p_n$, where K_i is T, F, or $\sim S$ according as a_i is 1, 0, or n, respectively, and by forming the conjunction of all expressions of the form $(\sim (K_1 p_1$ & . . . & $K_n p_n) \vee (\mathbf{t}p_1$ or $\mathbf{f}p_1))$, where substitutions of $\mathbf{t}p_1$ and $\mathbf{f}p_1$ in the places indicated will yield all the required connectives. By forming arbitrary disjunctions and conjunctions of the formulas $p_i \vee \sim p_i$, for $i = 1, \ldots, n$, so that if $p_j \vee \sim p_j$ does not occur in it then the formula is conjoined with SSp_j, all the formulae generalising the formulae, $(p \vee \sim p) \vee (q \vee \sim q)$, $(p \vee \sim p)$ & SSq, SSp & $(q \vee \sim q)$, and $(p \vee \sim p)$ & $(q \vee \sim q)$ of the dyadic case, can be obtained. (We could have used $(p \vee \sim p)$ & $(q \vee \sim q)$ in an alternative method of obtaining the dyadic connectives of type (5).) In the case of $n = 3$, $(p_1 \vee \sim p_1) \vee (p_2 \vee \sim p_2) \vee (p_3 \vee \sim p_3)$ has a single value n when p_1, p_2, and p_3 all take the value n, $((p_1 \vee \sim p_1) \vee (p_2 \vee \sim p_2))$ & SSp_3 has a "line" of values n when p_1 and p_2 take the value n, and $(p_1 \vee \sim p_1)$ & SSp_2 & SSp_3 has a "plane" of values n when p_1 takes the value n. By forming conjunctions of these formulae, one can form 2 or 3 "planes" of values n, 2, or 3 "lines" of values n, and a "plane" and "line" intersecting at (n, n, n). So, in general, the disjunctions of the atomic elements $p_i \vee \sim p_i$ determine the "simplex" and the conjunctions of these disjunctions superimpose the "simplexes" to form the configuration of the n's.

All of these formulae take the value 1 whenever it is significant and so, using the same method as was used in the dyadic case, any significant value can replace a value 1 in any of these formulae. This completes the proof.

It is clear for the quantifiers **A** and **S** that they contain a 2-valued quantifier of predicate logic, that they can be used to form an s - n sublogic, and that they have the positive property (in the sense that if $(\mathbf{A}x)\phi(x)$ [or $(\mathbf{S}x)\phi(x)$] takes the value n with $\phi(x_0)$ taking the value 1 or 0 then

$(\mathbf{A}x)\phi(x)$ [or $(\mathbf{S}x)\phi(x)$] still takes the value n when $\phi(x_0)$ takes the value n with the rest of the $\phi(x)$'s left intact.).

Theorem *The connectives* $\sim$, &, T, *and* $\vee$ *are independent.*

(i) Let $\Delta(p)$ be defined in terms of &, T, and $\vee$ only. If p has the value 1 then $\Delta(p)$ has the value 1. Hence $\sim$ is not definable from &, T, and $\vee$ only.
(ii) Let $\Delta(p)$ be defined in terms of $\sim$, &, and $\vee$ only. If p has the value n then $\Delta(p)$ has the value n. Hence T is not definable in terms of $\sim$, &, and $\vee$ only.
(iii) Let $\Delta(p, q)$ be defined in terms of $\sim$, &, and T only. If p takes the value n and q takes a significant value then, in order for $\Delta(p, q)$ to take a significant value all occurrences of p must be covered by a T. If one then lets p take a significant value and q take the value n then, in order for $\Delta(p, q)$ to take a significant value all occurrences of q must be covered by a T. But then $\Delta(p, q)$ takes a significant value when both p and q take the value n. Hence $\vee$ is not definable in terms of $\sim$, &, and T only.
(iv) Let $\Delta(p, q)$ be defined in terms of $\sim$, T, and $\vee$ only. If p takes a significant value and q takes the value n then $\Delta(p, q)$ takes a significant value. Hence & is not definable in terms of $\sim$, T, and $\vee$ only.

The axiomatisation of system **S2** is very similar to that of **S1**. For the sentential part one only needs to delete the definition, $A \vee B =_{Df} A \dot{\vee} B$ (since the $\vee$ of **S2** is just as good as $\dot{\vee}$ for the purpose of using Rule 3) and to add the axioms:

(5) $(\sim Sp \;\&\; Tq) \vee (Tp \;\&\; \sim Sq) \supset T(p \vee q)$.
(6) $(\sim Sp \;\&\; Fq) \vee (Fp \;\&\; \sim Sq) \supset F(p \vee q)$.
(7) $(\sim Sp \;\&\; \sim Sq) \supset \sim S(p \vee q)$,

and the Deduction Theorem and Completeness Theorem follows as for **S1**.

For the predicate system, one needs to delete the definition $(\mathbf{S}x)A =_{Df} (\mathbf{E}x)A$ and replace it by $(\forall x)A =_{Df} \sim(\mathbf{S}x) \sim A$, and to add the axioms:

(7) $(\mathbf{S}x)\, TA(x) \supset T(\mathbf{S}x)\, A(x)$.
(8) $(\mathbf{A}x) \sim TA(x) \;\&\; (\mathbf{S}x)\, FA(x) \supset F(\mathbf{S}x)\, A(x)$.
(9) $(\mathbf{A}x) \sim SA(x) \supset \sim S(\mathbf{S}x)\, A(x)$,

and the Deduction Theorem for $\supset$ and Completeness Theorem follow as for **S1**.

5 *The system* **S3** *with the* s - n *sublogic* The system **S2** (and **S1**) has the property of having an s - n sublogic, which in effect ensures a certain symmetry of truth and falsity with respect to non-significance. Instead of there being a gradation of values, starting with truth, through falsity to non-significance, non-significance is a different type of value to truth and falsity. This is indicated by the fact that true and false sentences can express propositions whereas non-significant sentences do not. The s - n sublogic is really a logic which determines whether a sentence can express a proposition or not, i.e., with the s-value designated. So it is desirable for a system to have an s - n sublogic.

The positive property of system **S2** is a rather restrictive property which was introduced to characterise the disjunction $\vee$. In forming the system **S3**, this property will be omitted so that the connectives of **S3** will consist of all of the connectives which can be used to form an s - n sublogic and which contain a 2-valued connective of the sentential calculus. The connective which is added to the primitive set $\{\sim, \&, T, \vee, \mathbf{A}, \mathbf{S}\}$ of **S2** is $\rightarrow$, which is represented as:

$\rightarrow$	1	0	n
1	1	0	n
0	1	1	n
n	1	1	1

To a certain extent, this "implication" is a 3-valued extension of the 2-valued material implication in that 'a false proposition implies any (significant) proposition' is extended to 'a non-significant sentence implies any sentence' and 'it is false for a true proposition to imply a false proposition' is extended to 'it is non-significant for a significant sentence to imply a non-significant sentence'. To a certain extent, this implication can be used in ordinary discourse reasoning involving non-significant sentences because the following are valid:

$p \& q \rightarrow p$; $p \& q \rightarrow q$; $p \rightarrow p$; $p \rightarrow p \vee q$; $q \rightarrow p \vee q$; $p \rightarrow q \rightarrow . q \rightarrow r \rightarrow . p \rightarrow r$; $p \rightarrow q \rightarrow . p \& r \rightarrow q \& r$; $p \rightarrow q \rightarrow . p \vee r \rightarrow q \vee r$; $p \& (q \vee r) \leftrightarrow (p \& q) \vee (p \& r)$; $p \vee (q \& r) \leftrightarrow (p \vee q) \& (p \vee r)$; $(p \rightarrow q \& \sim q) \rightarrow \sim Tp$; $(p \rightarrow q) \& \sim Sq \rightarrow \sim Sp$; $(p \rightarrow q \& \sim q) \& (\sim p \rightarrow r \& \sim r) \rightarrow \sim Sp$. ($A \leftrightarrow B$ is defined as $(A \rightarrow B) \& (B \rightarrow A)$.).

Using $\&$, $\vee$, and $\rightarrow$, all the theses of Hilbert's Positive Propositional Calculus[7] (the negation-less fragment of Intuitionistic Propositional Calculus) can be obtained because the matrices for $\&$, $\vee$, and $\rightarrow$ are the same as those that are the second in the infinite sequence of matrices for **I.P.C.** Also the equivalence $\leftrightarrow$, defined above, can be used for a Substitutivity of Equivalents Rule because $A \leftrightarrow B$ is true iff A and B have the same values. It is because of these nice properties that I want to maintain that $\rightarrow$ is the best implication for ordinary discourse, under the restrictions of using a 3-valued matrix.

Note that, using $\sim$, $\&$, and $\rightarrow$ only, one can define T and $\vee$ as follows:

$$TA =_{Df} \sim(A \rightarrow \sim(A \rightarrow A)).$$
$$A \vee B =_{Df} ((A \rightarrow B) \rightarrow B) \& ((B \rightarrow A) \rightarrow A).$$

Hence, let **S3** have the primitive set $\{\sim, \&, \rightarrow, \mathbf{A}, \mathbf{S}\}$.

7. *Cf.* Church [1], pp. 140-141.

Theorem *The set of all connectives which can be defined from the primitive set* $\{\sim, \&, \rightarrow\}$ *is the set of all connectives which can be used to form an* s - n *sublogic and which contain a connective of the* 2*-valued* **S.C.**

Proof: $\sim$, &, and $\rightarrow$ can be used to form an s - n sublogic and contain a connective of the 2-valued **S.C.** Assume $C(p_1, \ldots, p_n)$ and $D(q_1, \ldots, q_m)$ satisfy the two properties. Then $\sim C(p_1, \cdots, p_n)$, $C(p_1, \cdots, p_n)$ & $D(q_1, \cdots, q_m)$ and $C(p_1, \cdots, p_n) \rightarrow D(q_1, \ldots, q_m)$ also do. In order to prove the converse, consider the following n'adic connectives: $p_1 \& \ldots \& p_{n-1} \rightarrow p_n$, $p_1 \& \ldots \& p_{n-2} \rightarrow p_{n-1} \vee p_n$, $\ldots$, $p_1 \rightarrow p_2 \vee \ldots \vee p_n$, $p_1 \vee \ldots \vee p_n$. The first one has the property that it takes the value n iff $p_1, \ldots, p_{n-1}$ all take a significant value and p_n takes the value n. In general, the k'th one takes the value n iff $p_1, \cdots, p_{n-k}$ all take significant values and $p_{n-k+1}, \cdots, p_n$ all take the value n. By permuting the p_i's in each of the above and forming all possible conjunctions, using &, one obtains a set Δ of connectives which is such that each "connective" of s - n logic, with the property that if s is substituted for each variable then the value s results, is represented, with the exception of a connective taking significant values only. It remains to construct connectives which exhaust the two possibilities 1 and 0 in the cases where, in the s - n logic, s appears. Consider the example, $C(p_1, \ldots, p_n) \in \Delta$, where $C(p_1, \ldots, p_n)$ takes the value n iff $p_1, \ldots, p_k$ all take significant values and $p_{k+1}, \cdots, p_n$ all take the value n. This will be a particular connective taking certain values, 1 or 0, where it takes a significant value. Form the following: $(Sp_1 \& \ldots \& Sp_k \& \sim Sp_{k+1} \& \ldots \& \sim Sp_n \rightarrow C(p_1, \cdots, p_n))$ & $(Tp_1 \& \ldots \& Tp_n \rightarrow (\mathbf{t}p_1 \text{ or } \mathbf{f}p_1))$ & $(Tp_1 \& \ldots \& Tp_{n-1} \& Fp_n \rightarrow (\mathbf{t}p_1 \text{ or } \mathbf{f}p_1))$ & $(Tp_1 \& \ldots \& Tp_{n-1} \& \sim Sp_n \rightarrow (\mathbf{t}p_1 \text{ or } \mathbf{f}p_1))$ & $\ldots$ & $(\sim Sp_1 \& \ldots \& \sim Sp_n \rightarrow (\mathbf{t}p_1 \text{ or } \mathbf{f}p_1))$, so that any conjunction implying $Sp_1 \& \ldots \& Sp_k \& \sim Sp_{k+1} \& \ldots \& \sim Sp_n$ is omitted. By considering all possible cases of $\mathbf{t}p$ and $\mathbf{f}p$ in the indicated positions, all connectives having the same s - n logic connective as $C(p_1, \ldots, p_n)$ can be constructed. All connectives taking significant values only can be constructed as follows: $(Tp_1 \& \ldots \& Tp_n \rightarrow (\mathbf{t}p_1 \text{ or } \mathbf{f}p_1))$ & $(Tp_1 \& \ldots \& Tp_{n-1} \& Fp_n \rightarrow (\mathbf{t}p_1 \text{ or } \mathbf{f}p_1))$ & $\ldots$ & $(\sim Sp_1 \& \ldots \& \sim Sp_n \rightarrow (\mathbf{t}p_1 \text{ or } \mathbf{f}p_1))$, which gives all possible significant values for each substitution into the variables $p_1, \cdots, p_n$. Hence all connectives which can be used to form an s - n sublogic and which contain a 2-valued connective of the sentential calculus can be constructed.

Theorem *The connectives* $\sim$, &, *and* $\rightarrow$ *are independent.*

Proof: (i) Let $\Delta(p)$ be defined in terms of & and $\rightarrow$ only. If p has the value 1 then $\Delta(p)$ has the value 1. Hence $\sim$ is not definable.
(ii) Let $\Delta(p, q)$ be defined in terms of $\sim$ and & only. If p and q both take the value n then $\Delta(p, q)$ takes the value n and $\rightarrow$ is not definable.
(iii) Let $\Delta(p, q)$ be defined in terms of $\sim$ and $\rightarrow$ only. If p takes a significant value and q takes the value n then, for $\Delta(p, q)$ to take the value n, $\Delta(p, q)$ must have the form $A \rightarrow B$, preceded by a finite number of $\sim$'s, where A takes a significant value and B the value n. B must also have this form, provided B has an $\rightarrow$. Repeating this, a single variable q, with or without $\sim$'s, is obtained on the extreme right. However, if q takes a significant value, $\Delta(p, q)$ will be significant. Hence & is not definable.

The axiomatisation of **S3** is similar to that of **S1** and **S2**. One adds the definitions, $TA =_{Df} \sim(A \rightarrow \sim(A \rightarrow A))$ and $A \vee B =_{Df} ((A \rightarrow B) \rightarrow B) \,\&\, ((B \rightarrow A) \rightarrow A)$, and the axioms for $\rightarrow$, deleting the ones for T and $\vee$ in **S2**. The Deduction Theorems for $\supset$ and Completeness Theorems can then be shown for the sentential and predicate systems.

6 *The system* **S4** *containing the* 2*-valued connectives of* **S.C.** Goddard and Routley, in [5], deal with a system whose connectives are all those which contain the 2-valued connectives of **S.C.**, although initially it was given as two systems instead of one, they not realizing that they were identical. The connective that needs to be added to **S3** is $\supset$, given by the matrix:

$\supset$	1	0	n
1	1	0	n
0	1	1	1
n	1	1	1

'$p \supset q$' can be read as 'If it is true that p then q', where '$p \supset q$' takes the value of q if p is true and takes the value 1 otherwise. '$p \supset q$' is partly metatheoretic in that truth is involved. Also $\supset$ is a stronger implication than the $\rightarrow$ of system **S3** in that $p \rightarrow q \rightarrow .\ p \supset q$ is valid in **S4**. As will be shown in the next section, $\supset$ can be used to restrict the range of quantification for the quantifier **A**.

The system **S4** is a significance-preserving system in that if $C(p_1, \cdots, p_n)$ is a connective of **S4** and the p_i all take significant values then $C(p_1, \cdots, p_n)$ takes a significant value and if $A(x)$ takes significant values for all x then $(\mathbf{A}x)A(x)$ and $(\mathbf{S}x)A(x)$ take a significant value. This applies equally to the previous systems but **S4** contains all such connectives.

Note that, using $\sim$, &, and $\supset$ only, one can define $\rightarrow$ as follows:

$$TA =_{Df} \sim(A \supset \sim(A \supset A)).$$
$$FA =_{Df} T \sim A.$$
$$SA =_{Df} TA \dot{\vee} FA.$$
$$A \rightarrow B =_{Df} [\sim FA \dot{\vee} SB \supset (A \supset B)] \,\&\, [FA \,\&\, \sim SB \supset (A \,\&\, B)].$$

Hence, let **S4** have the primitive set $\{\sim, \&, \supset, \mathbf{A}, \mathbf{S}\}$.

Theorem *The set of all connectives which can be defined from the primitive set* $\{\sim, \&, \supset\}$ *is the set of all connectives which contain a connective of the* 2*-valued* **S.C.**

Proof: $\sim$, &, and $\supset$ all contain a connective of the 2-valued **S.C.** and this extends to all connectives definable in terms of them. To prove the converse, consider $\Gamma(p_1, \ldots, p_n)$, an arbitrary classical connective, and form the following: $(Sp_1 \,\&\ldots\&\, Sp_n \supset \Gamma(p_1, \ldots, p_n)) \,\&\, (Tp_1 \,\&\ldots\&\, \sim Sp_n \supset (\mathbf{t}p_1$ or $\mathbf{f}p_1)$ or $\Gamma(p_1, \ldots, p_n)) \,\&\ldots\&\, (\sim Sp_1 \,\&\ldots\&\, \sim Sp_n \supset (\mathbf{t}p_1$ or $\mathbf{f}p_1)$ or $\Gamma(p_1, \ldots, p_n))$.

The three possibilities $\mathbf{t}p_1$, $\mathbf{f}p_1$, and $\Gamma(p_1, \ldots, p_n)$ allows one to insert 1, 0, and n, respectively, in each of the positions given by the conjunctions. This gives all the connectives containing the 2-valued part of $\Gamma(p_1, \ldots, p_n)$.

Since $\Gamma(p_1, \ldots, p_n)$ is classical, all 2-valued sentential calculus connectives can be represented and hence the theorem follows.

Theorem *The connectives* $\sim$, &, *and* $\supset$ *are independent.*

Proof: (i) $\sim$ is not definable from &, and $\supset$, as for **S3**.
(ii) $\supset$ is not definable from $\sim$ and &, as for **S3**.
(iii) Let $\Delta(p, q)$ be defined in terms of $\sim$ and $\supset$ only. It is not the case that if p takes a significant value and q takes the value n then $\Delta(p, q)$ takes n and if p takes n and q a significant value then $\Delta(p, q)$ takes n. Hence & is not definable.

The axiomatisation of **S4** is obtained from **S3** by adding the definitions, $TA =_{Df} \sim(A \supset \sim(A \supset A))$, $FA =_{Df} T \sim A$, $SA =_{Df} TA \dot{\vee} FA$, $A \rightarrow B =_{Df} [\sim FA \dot{\vee} SB \supset (A \supset B)] \,\&\, [FA \,\&\, \sim SB \supset (A \,\&\, B)]$, deleting the definitions for TA and for $\supset$ in **S3**, adding the axioms for $\supset$, and deleting the axioms for $\rightarrow$. The Deduction and Completeness Theorems follow as before.

7 *The system* **S5**, *which is functionally complete* All the connectives and quantifiers, up to now, have had some intuitive appeal, in that there is some interpretation that can be given them. As soon as one deals with connectives which do not contain a 2-valued connective of **S.C.** one introduces counter-intuitive connectives, which fail Goddard's criterion **II**(b). Routley, in [8], defines the connective $\not{Q}$ as follows:

$\not{Q}$	
1	1
0	n
n	0

and uses the primitive set $\{\sim, \supset, \mathbf{S}, \not{Q}\}$ to provide a functionally complete set of connectives. He shows that $\not{Q}$ is formally useful only. I will introduce a connective T_n which will also serve the purpose and which can be used to restrict quantification.

Although, as already pointed out, restricted variables are not always necessary because of the type of quantification where variables are automatically restricted to the significance ranges of their predicates, there are many cases where they are necessary. Within the framework of a general system with variables ranging over everything or over a wide range, one may want to restrict the theory to a particular context. For example, in a theory of sets or classes, one may want to restrict consideration to ordinals, cardinals, integers, etc. To do this formally, one has to restrict the quantifiers to the required class of things so that all the original logical laws of the general system still hold in the restricted system.

In the 3-valued predicate logic we must find similar connectives to the $\supset$ and & of the 2-valued logic to restrict the variables. The 3-valued connectives required are $\supset$ and $\overline{\&}$, defined as follows:

$\supset$	1	0	n
1	1	0	n
0	1	1	1
n	1	1	1

$\bar{\&}$	1	0	n
1	1	0	n
0	n	n	n
n	n	n	n

'$\supset$' satisfies the property that if $B(x)$ is true, $B(x) \supset \phi(x)$ is equivalent to $\phi(x)$, and if $B(x)$ is not true, $B(x) \supset \phi(x)$ is true. For a particular x_0, if $B(x_0) \supset \phi(x_0)$ is true then the value of $(\mathsf{A}x)(B(x) \supset \phi(x))$, which is determined from all the values of the $B(x) \supset \phi(x)$'s, is the same whether $B(x_0) \supset \phi(x_0)$ is considered in the valuation or not. So, whenever $B(x)$ is not true, $B(x) \supset \phi(x)$ is ignored when assessing the value of $(\mathsf{A}x)(B(x) \supset \phi(x))$. Since there must be at least one x such that $B(x)$ is true, not all of the $B(x) \supset \phi(x)$'s are ignored. Hence, to evaluate $(\mathsf{A}x)(B(x) \supset \phi(x))$, one only has to consider x's such that $B(x)$ is true and the values of $\phi(x)$ for these x's.

'$\bar{\&}$' satisfies the property that if $B(x)$ is true, $B(x)\ \&\ \phi(x)$ is equivalent to $\phi(x)$, and if $B(x)$ is not true, $B(x)\ \&\ \phi(x)$ is non-significant. For a particular x_0, if $B(x_0)\ \bar{\&}\ \phi(x_0)$ is non-significant, then the value of $(\mathsf{S}x)(B(x)\ \bar{\&}\ \phi(x))$, which is determined from all the values of the $B(x)\ \bar{\&}\ \phi(x)$'s, is the same whether $B(x_0)\ \bar{\&}\ \phi(x_0)$ is considered in the valuation or not. So, whenever $B(x)$ is not true, $B(x)\ \bar{\&}\ \phi(x)$ is ignored when assessing the value of $(\mathsf{S}x)(B(x)\ \bar{\&}\ \phi(x))$. Since there is at least one x such that $B(x)$ is true, not all of the $B(x)\ \bar{\&}\ \phi(x)$'s are ignored. Hence, to evaluate $(\mathsf{S}x)(B(x)\ \bar{\&}\ \phi(x))$, one only has to consider the x's such that $B(x)$ is true and the values of $\phi(x)$ for these x's.

Since these properties uniquely determine the connectives $\supset$ and $\bar{\&}$, these are the only connectives that can satisfactorily restrict variables when the quantifiers A and S are used. It can be seen that if all variables are restricted using a given predicate $B(x)$ such that $(\mathsf{S}x)\,B(x)$ is valid, then all the valid wffs of the 3-valued predicate logic will be preserved.

Introduce the connective T_n, defined as follows:

T_n	
1	1
0	n
n	n

$\bar{\&}$ and T_n are interdefinable using $\supset$ and &. $T_nA =_{Df} A\ \bar{\&}\ (A \supset A)$; $A\ \bar{\&}\ B =_{Df} T_nA\ \&\ B$. Hence $(\mathsf{S}x)\phi(x)$ can be restricted using $B(x)$:$(\mathsf{S}x)(T_nB(x)\ \&\ \phi(x))$.

The following definitions can be given, using the primitive set, $\{\sim, \supset, T_n\}$.

$\mathbf{t}A =_{Df} A \supset A$.
$\mathbf{f}A =_{Df} \sim \mathbf{t}A$.
$\mathbf{n}A =_{Df} T_n \sim \mathbf{t}A$.
$TA =_{Df} \sim (A \supset \mathbf{f}A)$.
$FA =_{Df} T \sim A$.
$A \ddot{\vee} B =_{Df} (A \supset B) \supset B$.

$A \ddot{\&} B =_{Df} \sim(\sim A \ddot{\vee} \sim B)$.
$SA =_{Df} TA \ddot{\vee} FA$.
$A \& B =_{Df} (SA \ddot{\&} SB \supset. A \ddot{\&} B) \ddot{\&} (\sim SA \ddot{\vee} \sim SB \supset \mathbf{n}A)$.

Theorem *The set of all connectives which can be defined from the primitive set* $\{\sim, \supset, T_n\}$ *is functionally complete.*

Proof: Form the following:

$(Tp_1 \& \ldots \& Tp_n \supset (\mathbf{t}p_1$ or $\mathbf{f}p_1$ or $\mathbf{n}p_1)) \& (Tp_1 \& \ldots \& Tp_{n-1} \& Fp_n \supset (\mathbf{t}p_1$ or $\mathbf{f}p_1$ or $\mathbf{n}p_1)) \& \ldots \& (\sim Sp_1 \& \ldots \& \sim Sp_n \supset (\mathbf{t}p_1$ or $\mathbf{f}p_1$ or $\mathbf{n}p_1))$.

There is one conjunct for every assignment of values to $p_1, \ldots, p_n$ and by inserting $\mathbf{t}p_1, \mathbf{f}p_1$, or $\mathbf{n}p_1$ one can obtain every possible value for each assignment. Hence all 3-valued connectives can be constructed.

Theorem *The connectives* $\sim$, $\supset$, *and* T_n *are independent.*

Proof: (i) If $\Delta(p)$ is defined using $\supset$ and T_n only, then $\Delta(p)$ takes the value 1 when p takes the value 1. Hence $\sim$ is not definable.
(ii) $\supset$ cannot be defined from monadic connectives alone.
(iii) If $\Delta(p)$ is defined using $\sim$ and $\supset$ only, then $\Delta(p)$ takes a significant value when p takes the value 0. Hence T_n is not definable.

The axiomatisation, Deduction Theorem and Completeness Theorem follow the same pattern as before.

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