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ON THE RELATION BETWEEN FREE DESCRIPTION THEORIES AND STANDARD QUANTIFICATION THEORY

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Meyer and Lambert [2] constructed a mapping which takes formulas of free quantification theory into formulas of standard quantification theory and preserves validity. One adds a one-place predicate D to the vocabulary and translates thus:

For atomic P, $\sigma(P) = P$

$$\sigma(A \rightarrow B) = \sigma(A) \rightarrow \sigma(B)$$
$$\sigma(-A) = -\sigma(A)$$
$$\sigma((x)A) = (x)[Dx \rightarrow \sigma(A)].$$

There is also an interesting mapping τ from models of free quantification theory (**FQ**) to models of standard quantification theory (**SQ**). If $\mathfrak{M}$ is a model for **FQ** such that $\mathfrak{M} = \langle D, D^*, R\rangle$, then $\tau(\mathfrak{M}) = \langle D \cup D^*, R, D\rangle$. In other words, the domain of the **SQ** model is the union of the two **FQ** domains, each predicate letter receives the same interpretation as in **FQ** and the predicate letter D is assigned the domain of the **FQ** model. It is easy to show that for any sequence α, α satisfies A in $\mathfrak{M}$ iff α satisfies $\sigma(A)$ in $\tau(\mathfrak{M})$.[1]

One can construct a similar pair of mappings for Scott's free description theory [3], which is obtained by adding to free quantification theory the two schema

I) $(y)[y = \rceil xA] \leftrightarrow (x)[x = y \leftrightarrow A]$ *where* y *is not free in* A
II) $-(Ey)[y = \rceil xA] \rightarrow \rceil xA = \rceil x(x \neq x)$.

Models of the Scott system are simply models of **FQ** with the further requirement that one specify an element of D^* which is the denotation of all bad descriptions. In order to construct a mapping τ for this system, we

1. Thus the rather lengthy discussion of nominal interpretations in [2] could have been dispensed with since including them gives the same class of valid formulas.

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need to add a constant $\mathbf{a}$ and to extend our previous mapping by further stipulating that

$$\sigma(\urcorner xAx) = \urcorner x[(Dx \,\&\, (y)(Dy \rightarrow [Ay \leftrightarrow x = y])) \vee [-(Dx \,\&\, (y)(Dy \rightarrow [Ay \leftrightarrow x = y])) \,\&\, x = \mathbf{a}]].$$

The correlated mapping τ from models to models is the same as in the first case with the additional stipulation that $\mathbf{a}$ is assigned an element of D^*.

In [1] I presented a system of intensional free description theory. The system is intensional in that the schema $(x)[A \leftrightarrow B] \rightarrow \urcorner xA = \urcorner xB$ is not valid. The question I wish to consider now is whether that system also can be mapped in a trivial way into **SQ**. Of course one cannot show the non-existence of trivial mappings unless one has some characterization of triviality; consequently what I shall show is that there are no mappings σ and τ such that σ is a simple mapping from formulas of **IFD** to formulas of **SQ** and τ is an ultrauniform mapping from models of **IFD** to models of **SQ** such that $\mathfrak{M} \models A$ iff $\sigma(\mathfrak{M}) \models \tau(A)$. A mapping of formulas of **IFD** to formulas of **SQ** is simple iff τ has the properties (a)-(d) and $\sigma(\urcorner xA)$ is a formula whose only non-logical symbols are D, R and those of A and further $\sigma(\urcorner xA)$ is the result of substituting A for B in $\sigma(\urcorner xB \models \urcorner xB)$ if A and B have the same free variables. The extra relation R is permitted in order to attempt to characterize the definite description operator. A mapping τ from models of **IFD** to **SQ** is ultrauniform iff when $\mathfrak{M}_i/i \in I$ is a class of models of **IFD** and F an ultrafilter on I

$$\tau(\pi\mathfrak{M}_i/F) = \pi\tau(\mathfrak{M}_i)/F,$$

or, in other words, if the mapping of an ultraproduct is the ultraproduct of the mappings.

An interpretation of **IFD** is a quadruple $\langle \phi, D, D^*, \theta \rangle$, where D and D^* are disjoint non-empty sets; π is a function defined on all subsets of $D \cup D^*$ whose values are elements of $D \cup D^*$, and $\theta(x) \in D$ iff $x \cap D = \{\theta(x)\}$, ϕ is a function which is defined on all terms, wffs, predicate letters, and function symbols of **IFD** and is such that

(a) For any wff A, $\phi(A) = T$ or $\phi(A) = F$.
(b) For each variable v, $\phi(v) \in D \cup D^*$.
(c_0) For each P_i^0, $\phi(P_i^0) = T$ or F.
(c_n) For each P_i^n, $n > 0$, $\phi(P_i^n) \subseteq (D \cup D^*)^n$.
(d) For each atomic wff $P^n(s_{1i}, \ldots, s_n)$, $\phi(P^n(s_1, \ldots, s_n)) = T$ iff $\langle \phi(s_1), \ldots, \phi(s_n) \rangle \in \phi(P^n)$.
(e) $\phi(\sim A) = T$ iff $\phi(A) = F$.
(f) $\phi(A \rightarrow B) = F$ iff $\phi(A) = T \neq \phi(B)$.
(g) $\phi((v)A) = T$ iff for every interpretation $\langle \psi, D, D^*, \pi \rangle$ such that ϕ and ψ agree on all predicate and function letters and all variables except possibly v, $\psi(A) = T$.
(h_0) For each f_i^0, $\phi(f_i^0) \in D \cup D^*$.
(h_n)[11] For each f_i^n, $\phi(f_i^n)$ is a function with domain $(D \cup D^*)^n$ and range included in $D \cup D^*$.

(i) $\phi(f^n(s_1, \ldots, s_n)) = \phi(f^n)(\phi(s_1), \ldots, \phi(s_n))$.
(j) $\phi(s = t) = T$ iff $\phi(s) = \phi(t)$.
(k) $\phi(\iota xA) = \theta(\{d$: for all $\langle\psi, D, D^*, \theta\rangle$, if ψ agrees with ϕ on all predicate and function letters, and on all variables except x, and $\psi(x) = d$, then $\psi(A) = T\})$.

A wff is said to be valid if for every $\langle\phi, D, D^*, \theta\rangle$, $\phi(A) = T$.

Theorem *There is no pair* σ, τ *such that* σ *is simple,* τ *is uniform and if* $\mathfrak{M} \models_{\mathbf{IFD}} A$ *then* $\tau(\mathfrak{M}) \models_{\mathbf{SQ}} \sigma(A)$.

Proof: We must first define the notion of an ultraproduct of models of **IFD**. The usual definition of $\pi D_i/F$ can be applied also to D^* to obtain a definition of $\pi D^*_i/F$, and the interpretation of predicates will be as usual. We need only define then $\pi\theta_i/F$ where θ is the function that interprets the description operator. If X is a set of elements of $\pi D_i/F \cup \pi D^*_i/F$, and $\mathbf{a}$ is an element of $\pi D_i/F \cup \pi D^*_i/F$, then

$$\pi\theta_i/F(X) = \pi(\theta_i(X_i))/F,$$

from which it follows that $\pi\theta_i/F(X) = \mathbf{a}$ iff $\{i:\theta_i(X) = a_i\} \in F$.

Consider the following set of models $\mathfrak{M}_i$, $i \in \omega$.

$$D_i = \omega,\ D_i^* = \{a, b\}$$
$$\theta_i(X) = d \text{ if } X \cap D = \{d\}$$
$$\theta_i(X) = a \text{ if } X \cap D \text{ is finite and not a unit set}$$
$$\theta_i(X) = B \text{ otherwise.}$$

Let G be an atomic predicate and let the interpretation of G in $\mathfrak{M}_i$ be $\{n: n \leq i + 2\}$. Further let the constants $\mathbf{a}$ and $\mathbf{b}$ be assigned a and b respectively in each $\mathfrak{M}_i$.

Lemma *Łoś's theorem does not extend to* **IFD**.

Proof: Choose a non-principal ultrafilter F on ω and consider $\pi\mathfrak{M}_i/F$. The sentences $(E^n x)Gx$ each hold in at most one model and, since F is non-principal, therefore all such sentences are false in $\pi\mathfrak{M}_i/F$. Therefore G is infinite in $\pi\mathfrak{M}_i/F$. Let X be the set of elements of D_i/F which are assigned to G. By the definition $\pi\theta_i/F(X) = a$ iff $\{i: \theta_i(X_i) = a_i\} \in F$, but $\{i: \theta_i(X_i) = a_i\}$ is empty since X_i is infinite. Thus $\iota xGx = \mathbf{a}$ is false in $\pi\mathfrak{M}_i/F$. But $\iota xGx = \mathbf{a}$ does hold in all $\mathfrak{M}_i$ and thus $\{i: \mathfrak{M}_i \models xGx = \mathbf{a}\} \in F$. Thus Łoś's theorem does not extend to **IFD** and in this particular case no ultrauniform σ and simple τ exist which have the desired properties. It is perhaps worth mentioning that **IFD** is compact (by a simple modification of the completeness argument given in [1]) even though Łoś's theorem does not hold.

REFERENCES

[1] Grandy, Richard E., "A definition of truth for theories with intensional definite description operations," *Journal of Philosophical Logic*, vol. 1 (1972), pp. 137-155.

[2] Meyer, Robert K., and Karel Lambert, "Universally free logic and standard quantification theory," *The Journal of Symbolic Logic,* vol. 33 (1968), pp. 8-26,

[3] Scott, Dana, "Existence and description in formal logic," in *Bertrand Russell: Philosopher of the Century,* ed., Ralph Schoenman, George Allen and Unwin Co., London (1967).

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