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AN INDEPENDENT STATEMENT ABOUT METRIC SPACES

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In a metric space can the points near some point x pack close to each other with ever-increasing density (in the sense of cardinality or power) as x is approached, or must it always be the case that this density reaches a maximum at a certain distance from x and does not increase for smaller distances? We give a precise definition of this density concept and show that the former case can happen (for a space of cardinality $\aleph_\omega$) but that the question as to whether it can happen in a space of power less than or equal to that of the continuum cannot be answered. Our results are based on some of the recent independence results in set theory.

1 *Preliminaries* In the following (X, ρ) will be a metric space, A a subset of X, and x a point of X. We define the A-packing power near x by

$\mathrm{P}_A(x) = \sup \{a \leq \mathrm{card}\, X \mid \exists \varepsilon > 0, \mathrm{card}[(\mathrm{S}(x, \varepsilon_2) - \mathrm{S}(x, \varepsilon_1)) \cap A] \geq a$ for all ε_1, ε_2 satisfying $0 < \varepsilon_1 < \varepsilon_2 < \varepsilon\}$.

That is, if C denotes the set of cardinals $a \leq \mathrm{card}\, X$ such that between any two small enough concentric spheres about x there lie at least a points of A, then $\mathrm{P}_A(x) = \sup C$. $\mathrm{P}_X(x)$ will be written $\mathrm{P}(x)$ and called the packing power near x. A packed point of X is a point x for which $\mathrm{P}(x) > 0$. A packed space is one whose points are all packed. It is easily seen that a packed space is perfect, but that a perfect space need not be packed (take an appropriate subspace of $\mathcal{R}^1$). We remark that $\mathrm{P}_A(x)$ measures how close (in the sense of cardinality) to *each other* the points near x are packed, not how closely they pack about x itself. We will discuss this other question later.

The question here is whether or not it is always the case (i.e., for all X, A, x) that $\mathrm{P}_A(x) \in C$, i.e., whether $\sup C \in C$. We will work in Zermelo-Fraenkel set theory (**ZF**) including the axiom of choice. We will also make use of the results on the status of the continuum hypothesis (**CH**) and the generalized continuum hypothesis (**GCH**) in **ZF**, established by K. Gödel [1] and by P. J. Cohen [2]. In particular we note that $2^{\aleph_0} = \aleph_{\omega+1}$ is consistent with **ZF** [3]. The assertion $\overline{\sup C \in C}$ is taken to mean: For all (X, ρ) and for

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all $A \subseteq X$ and for all $x \in X$, $\overline{\sup C \in C}$. Its negation is $\neg(\overline{\sup C \in C})$. Similarly $\overline{\sup_0 C \in C}$ is taken to mean: For all (X, ρ) and for all $A \subseteq X$ and for all $x \in X$, $\operatorname{card} X \leqslant 2^{\aleph_0} \Longrightarrow \sup C \in C$. The following are immediate consequences of the definitions: Given X, A, x,

(1) $\mathrm{P}_A(x) = 0$ *or* $\mathrm{P}_A(x) \geqslant \aleph_0$,

(2) $\sup C \in C$ *is equivalent to* $C = \{a \leqslant \operatorname{card} X \mid 0 \leqslant a \leqslant \mathrm{P}_A(x)\}$,

(3) $\sup C \notin C$ *is equivalent to* $C = \{a \leqslant \operatorname{card} X \mid 0 \leqslant a < \mathrm{P}_A(x)\}$.

Theorem 1 *If* $\mathrm{P}_A(x) \in \{0\} \cup \{\aleph_0\} \cup \{\aleph_{\mu+1} \mid \mu$ *is an ordinal number*$\}$ *then* $\sup C \in C$.

Proof: Suppose $\sup C \notin C$. If $\mathrm{P}_A(x) = \aleph_0$ then $1 \in C$. Between any two concentric spheres about x, of small enough radii, there lies at least one point of A, hence at least a countable number of such points. It would follow that $\sup C = \aleph_0 \in C$, contradiction. If instead $\mathrm{P}_A(x) = \aleph_{\mu+1}$ then by (3) $\sup C = \aleph_\mu < \aleph_{\mu+1} = \sup C$, again a contradiction.

2 *The independence of* $\overline{\sup_0 C \in C}$ We first establish an equivalence result in **ZF**.

Theorem 2 $\overline{\sup_0 C \in C}$ *is equivalent to* $2^{\aleph_0} < \aleph_\omega$.

Proof: First suppose $2^{\aleph_0} < \aleph_\omega$. Let (X, ρ) be any metric space with $\operatorname{card} X \leqslant 2^{\aleph_0}$. Then $\sup C \leqslant 2^{\aleph_0} < \aleph_\omega$. Hence $\sup C \in \{0\} \cup \{\aleph_0\} \cup \{\aleph_{n+1} \mid n = 0, 1, 2, \ldots\}$, so that $\sup C \in C$ by Theorem 1.

Conversely, given $\overline{\sup_0 C \in C}$ we suppose $2^{\aleph_0} \geqslant \aleph_\omega$. We construct a subspace of the real line $\mathcal{R}^1$ as follows. By our supposition we may select $\aleph_1$ points in turn from each of the intervals $[\frac{1}{2}, 1]$; $[\frac{1}{2}, \frac{3}{4}]$, $[\frac{3}{4}, 1]$; $[\frac{1}{2}, \frac{5}{8}]$, $[\frac{5}{8}, \frac{3}{4}]$, $[\frac{3}{4}, \frac{7}{8}]$, $[\frac{7}{8}, 1]$; and so on, each time dividing previous intervals in half. In this way we accumulate a set $S_1 \subseteq [\frac{1}{2}, 1]$ with cardinality $\aleph_1$, such that each subinterval $[\varepsilon_1, \varepsilon_2)$ of $[\frac{1}{2}, 1]$ contains exactly $\aleph_1$ points of S_1. Similarly (this time selecting $\aleph_2$ points each time) we obtain a set $S_2 \subseteq [\frac{1}{3}, \frac{1}{2}]$ with cardinality $\aleph_2$, such that each subinterval $[\varepsilon_1, \varepsilon_2)$ of $[\frac{1}{3}, \frac{1}{2}]$ contains $\aleph_2$ points of S_2. Continuing the process we obtain $S_n \subseteq \left[\frac{1}{n+1}, \frac{1}{n}\right]$ with similar properties, for $n = 1, 2, \ldots$. Take $X = \{0\} \cup \left(\bigcup_{n=1}^{\infty} S_n\right)$ with the usual metric inherited from $\mathcal{R}^1$. Then $\operatorname{card} X = \aleph_1 + \aleph_2 + \ldots = \aleph_\omega$. Take $A = X$ and $x = 0$. Then it is easily seen that $C = \{a \mid 0 \leqslant a < \aleph_\omega\}$ so that $\sup C = \aleph_\omega \notin C$, contradicting $\overline{\sup_0 C \in C}$. Hence $2^{\aleph_0} < \aleph_\omega$ (proving the theorem).

Since $2^{\aleph_0} = \aleph_{\omega+1}$ and $2^{\aleph_0} = \aleph_1$ are both consistent with **ZF**, $2^{\aleph_0} < \aleph_\omega$ is independent. This gives us the following result.

Corollary $\overline{\sup_0 C \in C}$ *is independent.*

It is also clear by Theorem 1 that CH implies $\overline{\sup_0 C \in C}$.

3 *A counterexample to* $\overline{\sup C \in C}$ By Theorem 1 any counterexample would have to have $\operatorname{card} X \geqslant \mathrm{P}_A(x) \geqslant \aleph_\omega$. We will indeed construct one with $\operatorname{card} X = \mathrm{P}_A(x) = \aleph_\omega$.

Theorem 3 $\neg(\overline{\sup C \in C})$.

Proof: Let S be any set of cardinality $\aleph_\omega$. Let $B(S)$ be the set of bounded functions from S into $\mathcal{R}^1$ with the usual metric $\rho(f, g) = \sup_{s \in S} |f(s) - g(s)|$. For $t \in S$ let $x_t \in B(S)$ be the characteristic function of $\{t\}$. If $0 < a < b$ there are at least $\aleph_\omega$ functions of $B(S)$ between the concentric spheres $S(0, a)$ and $S(0, b)$ about the zero function of $B(S)$. (This is so because the map $t \to \frac{a+b}{2} x_t$ is one-to-one with range in $S(0, b) - S(0, a)$). As in the proof of Theorem 2 select $\aleph_1$ points (i.e., functions) in turn from each of the sets $S(0, 1) - S(0, \frac{1}{2})$; $S(0, \frac{3}{4}) - S(0, \frac{1}{2})$, $S(0, 1) - S(0, \frac{3}{4})$; $S(0, \frac{5}{8}) - S(0, \frac{1}{2})$, $S(0, \frac{3}{4}) - S(0, \frac{5}{8})$, $S(0, \frac{7}{8}) - S(0, \frac{3}{4})$, $S(0, 1) - S(0, \frac{7}{8})$; and so on, denoting the resulting set of cardinality $\aleph_1$ by S_1. Continuing this process we again obtain sets $S_n \subseteq S\left(0, \frac{1}{n}\right) - S\left(0, \frac{1}{n+1}\right)$ with $\operatorname{card} S_n = \aleph_n$ and such that if $\frac{1}{n+1} \leqslant \varepsilon_1 < \varepsilon_2 \leqslant \frac{1}{n}$, $S(0, \varepsilon_2) - S(0, \varepsilon_1)$ has $\aleph_n$ points of S_n. We take $X = \{0\} \cup \left(\bigcup_{n=1}^{\infty} S_n\right)$ as our metric subspace of $B(S)$. Evidently $\operatorname{card} X = \aleph_\omega$. Choosing $x = 0$ and $A = X$ it follows that $C = \{a \mid 0 \leqslant a < \aleph_\omega\}$. Hence $\sup C = \aleph_\omega \notin C$ and $\neg(\sup C \in C)$.

4 *Comments and possible generalizations* One course of further investigation would be the characterization of those cardinalities that $P_A(x)$ can assume, especially those for which $\sup C \notin C$ can occur. For example if a cardinal $\aleph_\mu$ is the sum of an increasing sequence $\aleph_{\nu_1}, \aleph_{\nu_2}, \ldots$ of infinite cardinals then it is possible to carry out the construction in Theorem 3 (taking $\operatorname{card} S = \aleph_\mu$) to obtain a case where $\aleph_\mu = \sup C \notin C$. Also it may be possible to generalize these concepts to first-countable topological spaces (where each point x has a *decreasing* sequence of neighborhoods constituting a base at x) or even to general topological spaces (using a generalized sequence or net of neighborhoods of x).

According to the definition of C, each $a \in C$ has an $\varepsilon > 0$ corresponding to it. If one is interested in the finer details of the packing it would be useful to investigate, for any particular space, how ε depends on a. For example if ε corresponds to a then any positive $\eta < \varepsilon$ also corresponds to a. What is the maximum ε corresponding to a (especially in the case that $a = \sup C$)? For special types of metric spaces such as Banach and Hilbert spaces most of these questions reduce to trivialities.

To investigate the packing *at* x rather than near x we define the A-packing power at x as $R_A(x) = \sup D$ where $D = \{a \leqslant \operatorname{card} X \mid \exists \varepsilon > 0, \operatorname{card}[S(x, \eta) \cap A] \geqslant a$ for all η satisfying $0 < \eta < \varepsilon\}$. Assertions (1), (2), (3), and Theorem 1 hold for $R_A(x)$, except that in (1) and Theorem 1 we also allow the possibility of $R_A(x) = 1$. There are differences between $P_A(x)$ and $R_A(x)$. For one thing if $a \in D$ then every $\varepsilon > 0$ will correspond to a. For another we *always* have $\sup D \in D$, because if $S(x, \eta)$ is any given sphere and $a \in D$ then $\operatorname{card}[S(x, \eta) \cap A] \geqslant a$. Hence $\operatorname{card}[S(x, \eta) \cap A] \geqslant \sup D$; and this is true for all $\eta > 0$.

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