Notre Dame Journal of Formal Logic
Volume 23, Number 1, January 1982

Generalized S2-Like Systems of Propositional Modal Logic

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In the usual semantics for *S2*, the models are permitted to contain "abnormal" worlds, i.e., possible worlds meeting two conditions: (1) they are terminal (no worlds are accessible from them), and (2) at the abnormal worlds the usual truth conditions for $\Box p$ and $\Diamond p$ are replaced by the conditions that $\Box p$ be false and $\Diamond p$ be true.[1] If a normal world were terminal, then $\Box p$ would be vacuously true at that world and $\Diamond p$ trivially false. Hence we might consider that abnormal worlds are ones at which the usual meanings of the two modal operators are simply interchanged, and that in *S2* it happens that such abnormal worlds are always terminal. This way of looking at *S2* semantics suggests that it might be of some interest to weaken the conditions on a model for modal logic by permitting worlds like *S2*'s abnormal worlds without demanding that they be terminal. Then models for *S2* and its various extensions might be seen as special cases of models of this very general sort.

For each model of this sort, there would be a complementary model, in which there would be abnormal worlds at all those "places" at which the first model had normal worlds, and vice versa, so that in the two models taken as wholes the meanings of the two modal operators would be exactly reversed. The collection of all modal propositions which hold in all such models must therefore be closed under interchange of $\Box$ and $\Diamond$. Indeed it seems likely that this collection of propositions will consist of just those theorems of *S2* which would remain theorems of *S2* under an interchange of modal operators.

In the present paper we shall establish completeness results for a variety of systems of propositional modal logic employing such models. Both the style of presentation and the methods of proof we shall employ owe much to the work of E. J. Lemmon and Dana Scott [5]. In that work, however, Lemmon and Scott do not consider models involving abnormal worlds.[2]

Received October 6, 1980; revised February 18, 1981

The weakest system considered in [5] (namely K) can easily appear to be the weakest system for which the methods of proof of completeness there used would be suitable. The work presented below can be understood as showing that, and (at least approximately) how far, these methods can be generalized so as to extend to systems weaker than K. It is noteworthy that the weakest system considered here (the system J) though it is weaker than $S2$, is still too strong to admit interpretation as an epistemic or doxastic logic. Syntactically, the system J does seem to admit interpretation as a deontic logic, in some contrast to K, whose deontic interpretation is at best a bit strained.

1 The system J: proof theory Consider a language for modal logic whose primitive symbols are the propositional constants $P_1, P_2, \ldots, P_n, \ldots$ $(n < \omega)$, and whose connectives are $\bot$, $\rightarrow$, and $\Box$. We define $\neg A$ to be $A \rightarrow \bot$, and define $\vee$, $\wedge$, $\longleftrightarrow$, and $\Diamond$ as usual. We let[3]

$\Box^n \triangleq \Box\Box \ldots \Box$ (n occurrences of '$\Box$')
$\Diamond^n \triangleq \Diamond\Diamond \ldots \Diamond$ (n occurrences of '$\Diamond$').

Now consider a system J of modal logic, in the language just described, whose axiom schemata and rules are:

(AJ1) $A \rightarrow [B \rightarrow A]$
(AJ2) $[A \rightarrow [B \rightarrow C]] \rightarrow [[A \rightarrow B] \rightarrow [A \rightarrow C]]$
(AJ3) $\neg\neg A \rightarrow A$
(AJ4) $[[\Box A \wedge \Box B] \rightarrow \Box[A \wedge B]] \vee [[\Diamond C \wedge \Diamond D] \rightarrow \Diamond [C \wedge D]]$
(RJ1) From A and $A \rightarrow B$, to infer B
(RJ2) From $A \rightarrow B$, to infer $\Box A \rightarrow \Box B$.

We identify J with the set of wffs A such that $\vdash A$, where this latter expression is defined in the obvious way.

By an *extension* of J (or a *J-extension*) we mean any set S of wffs of our language such that S is closed under (RJ1) and $J \subseteq S$. By an *ordinary J-extension* we mean any J-extension closed under (RJ2). If S is any J-extension, A any wff, and T any set of wffs, we write:

$\vdash_S A$ iff $A \in S$.
$T \vdash_S A$ iff for some $B_1, B_2, \ldots, B_k \in T$ $(k < \omega)$
$\vdash_S [B_1 \wedge [B_2 \ldots \wedge [B_{k-1} \wedge B_k] \ldots]] \rightarrow A$.

It is evident that for any J-extension S, if $T \cup \{A\} \vdash_S B$ then $T \vdash_S A \rightarrow B$.

If T is any set of wffs and S is any J-extension, we may say T is *S-consistent* if any only if we do not have $T \vdash_S \bot$. We say S is *consistent* if any only if $\bot \notin S$. If S is any J-extension, we say S' is an *S-extension* if any only if $S \subseteq S'$ and S' is a J-extension. We say S' is an *S-extension of T* (for any set T of wffs) if and only if S' is an S-extension and $T \subseteq S'$. Finally, if S is any J-extension, we say S is *maximal* if any only if for each wff A either $A \in S$ or $(\neg A) \in S$, and we say that S' is a maximal consistent extension of S if and only if it is maximal, consistent, and an extension of S.

Lemma 1 (The Lindenbaum Extension Theorem) *Every S-consistent set T of wffs has a maximal consistent S-extension.* (Theorem 0.1 in [5].)

Lemma 2 *For any wff A, any set T of wffs, and any J-extension S, $T \vdash_S A$ iff A is an element of every maximal consistent S-extension of T.*

Proof: Directly from Lemma 1. The proof is analogous to that for Theorem 0.2 in [5].

Lemma 3 *Let S be any J-extension, A any wff. Then $\vdash_S A$ iff A is an element of every maximal consistent S-extension.*

Proof: Directly from Lemma 2.

Lemma 4 *In any ordinary J-extension S, if $\vdash_S A \to B$ then $\vdash_S \Diamond A \to \Diamond B$.*

Proof: If $\vdash_S A \to B$ then $\vdash_S \neg B \to \neg A$, so $\vdash_S \Box\neg B \to \Box\neg A$, and thus $\vdash_S \neg\Box\neg A \to \neg\Box\neg B$, i.e., $\Diamond A \to \Diamond B$.

Lemma 5 *In any ordinary J-extension S, if A, B are any wffs, and $C(A)$ is any wff containing (zero or more) occurrences of A, and $C(B)$ is the result of replacing (zero or more of) A's occurrences in $C(A)$ by occurrences of B, then if $\vdash_S A \to B$, then $\vdash_S C(A) \to C(B)$.*

Proof: By routine induction on the complexity of $C(A)$.

2 ***The system J: soundness*** By a *basic model* we shall mean any ordered quintuple $M = \langle U, V, P, R, \phi\rangle$ such that if $W \triangleq U \cup V$ (and if 0 is the null set and N the set of nonnegative integers) then

(BM1) $0 \subset P \subseteq W$
(BM2) $R \subseteq W \times W$
(BM3) $U \cap V = 0$
(BM4) $\phi: N \to \mathcal{P}(W)$.

We call the elements of W worlds, those of U usual worlds, those of V variant worlds, and those of P preferred worlds. R is called the accessibility relation, and ϕ the truth function, for M.

We employ the following truth conditions for a basic model $M = \langle U, V, P, R, \phi\rangle$:

(TD1) $\forall w \in W$ [it is false that $w, M \vDash \bot$]
(TD2) $\forall w \in W\ [w, M \vDash P_i$ iff $w \in \phi(i)]$
(TD3) $\forall w \in W\ [w, M \vDash A \to B$ iff [if $w, M \vDash A$ then $w, M \vDash B$]]
(TD4) $\forall u \in U\ [u, M \vDash \Box A$ iff $\forall w \in W$ [if uRw then $w, M \vDash A$]]
(TD5) $\forall v \in V\ [v, M \vDash \Box A$ iff $\exists w \in W$ [vRw and $w, M \vDash A$]].

We then set $M \vDash A \triangleq \forall w \in P\ [w, M \vDash A]$, and set $\vDash A \triangleq \forall M\ [M \vDash A]$.

The usual Kripke models for S2 are then basic models with the additional conditions that $P = U$, $U \times U \subseteq R$, and $(V \times W) \cap R = 0$.

Lemma 6 $\vDash [[\Box A \wedge \Box B] \to \Box[A \wedge B]] \vee [[\Diamond C \wedge \Diamond D] \to \Diamond[C \wedge D]]$.

Proof: (TD4) guarantees that the left-hand disjunct will be satisfied at all usual worlds, while (TD5) assures that the right-hand disjunct will hold at all variant worlds.

Lemma 7 *If $\vDash A$ then for any basic model M, $\forall w \in W[w, M \vDash A]$.*

Proof: Suppose $\vDash A$, and let M be any basic model, with w any world in M. Compare M to M' in which $U' = U$, $V' = V$, $R' = R$, and $\phi' = \phi$, but in which $P' = W$. Then by induction on the complexity of A, $w, M \vDash A$ iff $w, M' \vDash A$, since the truth conditions make no mention of P. But $w, M' \vDash A$, since M' is a basic model, and so $w, M \vDash A$.

Lemma 8 *If* $\vDash A \to B$ *then* $\vDash \Box A \to \Box B$.

Proof: Assume $\vDash A \to B$. Let M be any basic model and let $w \in W$. Assume $w, M \vDash \Box A$. *Case 1:* $w \in U$. Let w' be any world from W such that wRw'. Then $w', M \vDash A$. By Lemma 7, $w', M \vDash A \to B$, so by (TD3), $w', M \vDash B$. *Case 2:* $w \in V$. Then for some $w' \in W$ we have wRw' and $w', M \vDash A$. By Lemma 7 we also have $w', M \vDash A \to B$, so there is a world w' with wRw' and $w', M \vDash B$.

Theorem 1 *If* $\vdash A$ *then* $\vDash A$.

Proof: By routine induction on the length of the deduction of A, using the usual propositional calculus arguments, augmented by Lemmas 6 and 8.

3 The system J: completeness If S is any consistent ordinary J-extension, then by the *characteristic model* for S we shall mean the basic model $M[S] = \langle U[S], V[S], P[S], R[S], \phi[S]\rangle$ defined by the following conditions:

(CM1) $P[S] \triangleq$ the set of all maximal consistent S-extensions

(CM2) $U[S] \triangleq \{u \in P[S]$: for all wffs A, B, if $\Box A \in u$ and $\Box B \in u$ then $\Box[A \wedge B] \in u\}$

(CM3) $V[S] \triangleq \{v \in P[S]: v \notin U[S]\}$

(CM4) $w_1 R[S] w_2 \triangleq \{A: \Box A \in w_1\} \subseteq w_2$ and $w_1 \in U[S]$, or $\{A: \Diamond A \in w_1\} \subseteq w_2$ and $w_1 \in V[S]$

(CM5) $\phi[S](i) \triangleq \{w \in P[S]: P_i \in w\}$.

Lemma 9 *Let* $u, v \in P[S]$, *for any ordinary J-extension S. Then* $\{A: \Diamond A \in v\} \subseteq u$ *iff* $\{\Box A: A \in u\} \subseteq v$.

Proof: Suppose $\{A: \Diamond A \in v\} \subseteq u$. Choose any $A \in u$, and suppose $\Box A \notin v$. Since v is maximal, $\neg\Box A \in v$. But $\vdash_S \neg\Box A \to \neg\Box\neg\neg A$ (because $\vdash_S \neg\neg A \to A$ and hence by (RJ2) $\vdash_S \Box\neg\neg A \to \Box A$), and hence $\neg\Box\neg\neg A \in v$, i.e., $\Diamond\neg A \in v$. It then follows from our first assumption that $\neg A \in u$, contradicting the consistency of u. The converse argument is similar.

Lemma 10 *Let* $u \in U[S]$, *for any ordinary J-extension S. Then* $\Box A \in u$ *iff* $\forall w \in W[S]$ [*if* $uR[S]w$ *then* $A \in w$] *(where* $W[S] \triangleq P[S]$*).*

Proof: Suppose $\Box A \in u$ and $uR[S]w$. From the latter, $\{A: \Box A \in u\} \subseteq w$, so $A \in w$. For the converse, suppose that $\forall w \in W[S]$ [if $uR[S]w$ then $A \in w$]. This means that for all maximal consistent S-extensions w of $\{B: \Box B \in u\}$, we have $A \in w$. Hence by Lemma 2 $\{B: \Box B \in u\} \vdash_S A$. Thus for some $B_1, B_2, \ldots, B_k \in \{B: \Box B \in u\}$ we have $\{B_1, B_2, \ldots, B_k\} \vdash_S A$. Thus $\vdash_S [B_1 \wedge B_2 \ldots \wedge B_k] \to A$, and since S is an ordinary J-extension, $\vdash_S \Box[B_1 \wedge B_2 \ldots \wedge B_k] \to \Box A$. But also, by the choice of $B_1, \ldots, B_k$, we have $\Box B_1, \ldots, \Box B_k \in u$, and since $u \in U[S]$, the definition of $U[S]$ gives us $\Box[B_1 \wedge \ldots \wedge B_k] \in u$. Hence $\Box A \in u$.

Lemma 11 *Let S be any ordinary J-extension, with $v \in V[S]$. Then $\Box A \in v$ iff $\exists w \in W[S]$ $[vR[S]w$ and $A \in w]$.*

Proof: Suppose $\Box A \in v$. We must show that $\exists w[\{B: \Diamond B \in v\} \subseteq w$ and $A \in w]$. It suffices, then, to show that $\{B: \Diamond B \in v\} \cup \{A\}$ is S-consistent. Suppose it were not. Then for some $B_1, \ldots, B_k \in \{B: \Diamond B \in v\}$, $\{B_1, \ldots, B_k\} \cup \{A\} \vdash_S \bot$, i.e., $\vdash_S [B_1 \wedge \ldots \wedge B_k] \rightarrow \neg A$, and since S is an ordinary J-extension, Lemma 4 gives us $\vdash_S \Diamond[B_1 \wedge \ldots \wedge B_k] \rightarrow \Diamond\neg A$. Since $v \in V[S]$, there must be some C, D such that $\Box C \in v$ and $\Box D \in v$, but $\Box[C \wedge D] \notin v$. Then $[\Box C \wedge \Box D] \rightarrow \Box[C \wedge D] \notin v$. But by (AJ4) we do have $[[\Box C \wedge \Box D] \rightarrow \Box[C \wedge D]] \vee [[\Diamond B_1 \wedge \Diamond B_2] \rightarrow \Diamond[B_1 \wedge B_2]] \in v$. Hence $[\Diamond B_1 \wedge \Diamond B_2] \rightarrow \Diamond[B_1 \wedge B_2] \in v$, and so $\Diamond[B_1 \wedge B_2] \in v$. By repeated use of this argument, we get $\Diamond[B_1 \wedge B_2 \wedge B_3] \in v$, and eventually $\Diamond[B_1 \wedge \ldots \wedge B_k] \in v$. Hence $\Diamond\neg A \in v$, i.e., $\neg\Box\neg\neg A \in v$. Clearly, however, $\neg\Box\neg\neg A \rightarrow \neg\Box A \in v$, so $\neg\Box A \in v$, contradicting our initial assumption. For the converse, suppose $\exists w[vR[S]w$ and $A \in w]$. Then $\{A: \Diamond A \in v\} \subseteq w$. So by Lemma 9 $\{\Box A: A \in w\} \subseteq v$. Hence $\Box A \in v$.

Theorem 2 *Let $w \in W[S]$, for any ordinary J-extension S. Then $w, M[S] \models A$ iff $A \in w$.*

Proof: By induction on the number of occurrences of the arrow and the box in A. The basis is trivial in the light of (TD1) and (CM5). For the induction step we distinguish three cases:

Case 1. A is $B \rightarrow C$. Then $w, M[S] \models A$

iff $w, M[S] \models B \rightarrow C$	
iff [if $w, M[S] \models B$ then $w, M[S] \models C$]	
iff [if $B \in w$ then $C \in w$]	(by the induction hypothesis)
iff $B \rightarrow C \in w$	(since w is maximal consistent)
iff $A \in w$.	

Case 2. A is $\Box B$, $w \in U[S]$. Then $w, M[S] \models A$

iff $w, M[S] \models \Box B$	
iff $\forall w' \in W[S]$ [if $wR[S]w'$ then $w', M[S] \models B$]	(by (TD4))
iff $\forall w' \in W[S]$ [if $wR[S]w'$ then $B \in w'$]	(by the induction hypothesis)
iff $\Box B \in w$	(by Lemma 10)
iff $A \in w$.	

Case 3. A is $\Box B$, $w \notin U[S]$. Then $w, M[S] \models A$

iff $w, M[S] \models \Box B$	
iff $\exists w' \in W[S]$ $[wR[S]w'$ and $w', M[S] \models B]$	(by (TD5))
iff $\exists w' \in W[S]$ $[wR[S]w'$ and $B \in w']$	(by the induction hypothesis)
iff $\Box B \in w$	(by Lemma 11)
iff $A \in w$.	

Theorem 3 *If $\models A$ then $\vdash A$.* (Completeness for J.)

Proof: Assume $\models A$. Since $M[J]$ is a basic model, $M[J] \models A$, i.e., A is an element of all maximal consistent extensions of J, by Theorem 2. Hence by Lemma 3 $\vdash A$.

4 Extending J with axioms We now set about extending these results to all ordinary J-extensions. For any wff A, and any world w, let the characteristic condition $C(A, w)$ on w be defined as follows:

(CC1) $C(\perp, w) \triangleq w \neq w$
(CC2) $C(P_i, w) \triangleq w \in \phi(i)$
(CC3) $C(B \rightarrow C, w) \triangleq$ if $C(B, w)$ then $C(C, w)$
(CC4) for $w \in U$: $C(\Box B, w) \triangleq \forall w' \in W$ [if wRw' then $C(B, w')$]
(CC5) for $w \in V$: $C(\Box B, w) \triangleq \exists w' \in W[wRw'$ and $C(B, w')]$.

In stating clauses (CC4) and (CC5) for particular cases, we must be sure that the quantified variable is chosen so as to have no free occurrences in $C(B, w)$.

If we let $wR^0w' \triangleq w = w'$ and let $wR^{n+1}w' \triangleq \exists w''[wRw''$ and $w''R^nw']$, then we may easily establish that, for $w \in U$, $C(\Box^n B, w)$ is equivalent to

(a) $\forall w'$[if wR^nw' then $C(B, w')$]

and $C(\Diamond^n B, w)$ is equivalent to

(b) $\exists w'[wR^nw'$ and $C(B, w')]$.

For $w \in V$, $C(\Box^n B, w)$ is equivalent to (b), while $C(\Diamond^n B, w)$ is equivalent to (a).

Lemma 12 *Let $M = \langle U, V, P, R, \phi\rangle$ be any basic model, $w \in W$, and A any wff. Then $w, M \models A$ iff $C(A, w)$.*

Proof: By routine induction on the construction of A, noting the agreement between the truth conditions and the definition of the characteristic condition.

Lemma 13 *Under the conditions of Lemma 12, $M \models A$ iff $(\forall w \in P)[C(A, w)]$.*

Proof: Trivial, from Lemma 12.

Now if $M = \langle U, V, P, R, \phi\rangle$ is any basic model, then we call M an *ordinary* model iff $U \subseteq P$ and $V \subseteq P$, i.e., iff $P = W$. Given any wff A, let $J(A)$ be the ordinary J-extension which results from adding A to J (as an axiom, *not* as an axiom schema) and closing under (RJ2). The meaning of the expression '$\vdash_{\overline{J(A)}} B$' is then already fixed by our previous conventions. Let '$M(A)$' and '$M_1(A)$', . . ., '$M_n(A)$', . . . designate ordinary models which satisfy the characteristic condition $C(A, w)$ at all $w \in P$, i.e., at all $w \in W$. (In context, we shall be able to designate the components of $M(A)$ as U, V, P, R, and ϕ, rather than as $U(A)$, $V(A)$, etc.) Then the meaning of the expression '$M(A) \models B$' is fixed by previous conventions. We let $\models_{\overline{J(A)}} B \triangleq \forall M(A)[M(A) \models B]$.

Lemma 14 *If $M(A) \models B \rightarrow C$ then $M(A) \models \Box B \rightarrow \Box C$.*

Proof: Assume $M(A) \models B \rightarrow C$. Let $w \in P$, and suppose $w, M(A) \models \Box B$. It will suffice to show that under these two assumptions $w, M(A) \models \Box C$.

Case 1. $w \in U$. Let w' be any element of W such that wRw'. It will suffice to show that $w', M(A) \models C$. But since wRw', by our second assumption $w', M(A) \models B$. Since $M(A)$ is ordinary, $w' \in P$, so by our first assumption $w', M(A) \models B \rightarrow C$. Hence $w', M(A) \models C$, by (TD3).

Case 2. $w \in V$. Then $\exists w' \in W[wRw'$ and $w', M(A) \models B]$, by our second assumption. Since $M(A)$ is ordinary, $w' \in P$, so that our first assumption yields $w', M(A) \models B \rightarrow C$. Hence $\exists w' \in W[wRw'$ and $w', M(A) \models C]$, i.e., $w, M(A) \models \Box C$.

Theorem 4 *If* $\vdash_{\overline{J(A)}} B$ *then* $\models_{\overline{\overline{J(A)}}} B$.

Proof: By induction on the number of steps in the deduction of B in the system $J(A)$, using Lemma 13 to cover uses of A, Lemma 14 to cover uses of (RJ2), and otherwise arguing as in the proof for Theorem 1.

Lemma 15 *Let S be any consistent ordinary J-extension. If* $\vdash_S A$, *then* $M[S]$ *is an ordinary model such that* $\forall w \in P[S]\,[C(A, w)]$.

Proof: Let S be any consistent ordinary J-extension, and suppose $\vdash_S A$. Then $A \in w$ for all maximal consistent S-extensions w, by Lemma 4. So $M[S] \models A$ by Theorem 2. Hence by Lemma 13 $\forall w \in P[S]\,[C(A, w)]$. And $M[S]$ has $P[S] = W[S]$ by definition, so $M[S]$ is an ordinary model.

Theorem 5 *If* $\models_{\overline{\overline{J(A)}}} B$ *then* $\vdash_{\overline{J(A)}} B$.

Proof: Suppose $\models_{\overline{\overline{J(A)}}} B$. Since by definition $\vdash_{\overline{J(A)}} A$, Lemma 15 tells us that $M[J(A)]$ is a model, like $M(A)$, of the type by reference to which the expression '$\models_{\overline{\overline{J(A)}}} B$' is defined, so that from our assumption we may conclude that $M[J(A)] \models B$. But then B is an element of all maximal consistent $J(A)$-extensions, by the definition of characteristic models, and hence by Lemma 3 we get $\vdash_{\overline{J(A)}} B$.

5 ***Extending J with axiom schemata*** We now extend the results in Theorems 4 and 5 to deal with J-extensions created by the addition of axiom schemata. If A is a wff involving as its atomic components only formulas from among $P_{k_1}, \ldots, P_{k_n}$, and A^* is the wff which results when we simultaneously replace P_{k_i} by B_{k_i} $(0 \leqslant i \leqslant n)$ for some wffs $B_{k_1}, \ldots, B_{k_n}$, then we shall call A^* a *substitution instance of A employing the substitutions* B_j/P_j $(0 \leqslant j < \omega)$, where for j other than $k_1, \ldots, k_n$ we take $B_j = P_j$.

Lemma 16 *Let* A^* *be a substitution instance of A, employing the substitutions* B_j/P_j $(0 \leqslant j < \omega)$. *Let M be any basic model, and let* $M^* = \langle U, V, P, R, \phi^*\rangle$, *where* $\phi^*(i) \triangleq \{w \in W\colon w, M \models B_i\}$. *Then for any* $w \in W$, $w, M^* \models A$ *iff* $w, M \models A^*$.

Proof: By induction on the structure of A.

Consider now any nonrepetitive enumeration $A_0, A_1, \ldots, A_n, \ldots$ of all wffs, and suppose we are given any basic model M. We define a collection $Z(M) \subseteq \mathcal{P}(W)$ by setting $Y_i \triangleq \{w \in W\colon w, M \models A_i\}$ $(i < \omega)$, and setting $Z(M) \triangleq \{Y_i\colon i < \omega\}$. Since one of the A_i will be $\perp$, the corresponding Y_i will be empty, and likewise since some of the A_i will be tautologies, the corresponding Y_i will be W. It can easily be verified that $Z(M)$ is closed under union, intersection, and complementation relative to W. But in general $Z(M)$ will be a proper subset of $\mathcal{P}(W)$.

Now we say that $M^* = \langle U, V, P, R, \phi^*\rangle$ is *derivative from* $M = \langle U, V, P, R, \phi\rangle$ iff for each $i < \omega$, $\phi^*(i) \in Z(M)$. Note that for any ordinary model M, M is derivative from itself. Note too that if A^* results from A by the substitutions B_j/P_j ($j < \omega$) then if we set $\phi^*(i) = \{w \in W\colon w, M \models B_i\}$ we get a model M^* which is derivative from M. In the other direction, we note that if M^* is derivative from M, and A is any wff, then for each P_j in A the set $\phi^*(j)$ will correspond to at least one sentence B_j such that $\phi^*(j) = \{w \in W\colon w, M \models B_j\}$ and thus will lead us to at least one substitution instance A^* of A even though, for a given A, the wff A^* may not be uniquely determined.

We say that M *strongly satisfies* $C(A, w)$ iff every M^* derivative from M satisfies $C(A, w)$. Note that, since M is derivative from itself if it is an ordinary model, if M strongly satisfies $C(A, w)$ and is an ordinary model, then it also satisfies $C(A, w)$. By $J[A]$ we mean that ordinary J-extension which results when A is added as an axiom schema to J, and the resulting collection of wffs is closed under (RJ1) and (RJ2). Finally, for any wff A, we let $I(A)$ be the set of all substitution instances of A.

Lemma 17 *If M strongly satisfies $C(A, w)$, then for all $A^* \in I(A)$, $M \models A^*$.*

Proof: Let M strongly satisfy $C(A, w)$, and let $A^* \in I(A)$ be the result of employing substitutions B_j/P_j in A. Set $\phi^*(i) \triangleq \{w \in W\colon w, M \models B_i\}$ and set $M^* \triangleq \langle U, V, P, R, \phi^*\rangle$. Then M^* is derivative from M, and M strongly satisfies $C(A, w)$, so M^* satisfies $C(A, w)$. By Lemma 13, $M^* \models A$. By Lemma 16, $M \models A^*$.

We write $\models\!\!\!=_{J[A]} B$ iff for all ordinary models M which strongly satisfy $C(A, w)$, $M \models B$.

Theorem 6 *If $\vdash_{J[A]} B$ then $\models\!\!\!=_{J[A]} B$.*

Proof: By induction on the length of the demonstration of B, using Lemmas 14 and 17.

Lemma 18 *If S is any consistent ordinary J-extension, and A any wff, then if $\vdash_S A^*$ for all $A^* \in I(A)$, then $M[S]$ strongly satisfies $C(A, w)$.*

Proof: Suppose S is a consistent ordinary J-extension, with $\vdash_S A^*$ for all $A^* \in I(A)$. First note that, by definition, $M[S]$ is an ordinary model. Let M^* be any ordinary model derivative from $M[S]$. It will suffice to show that M^* satisfies $C(A, w)$. As we remarked earlier, M^* will determine at least one substitution instance A^* of A, because to have M^* derivative from $M[S]$ we must have $\phi(i) \in Z(M[S])$ for each i, i.e., for each i there must be at least one B_i such that $\phi(i)$ is $\{w \in W\colon w, M[S] \models B_i\}$. (If there is more than one such B_i for a given i, pick the first to appear in the standard enumeration of all wffs.) Then the substitutions B_i/P_i which result will turn A into A^*. By Lemma 16, for any $w \in W[S]$, $w, M^* \models A$ iff $w, M[S] \models A^*$; but $\vdash_S A^*$, so $w, M[S] \models A^*$ (by Theorem 6), and hence $w, M^* \models A$. From this, by Lemma 13, it follows that $C(A, w)$ in M^*, i.e., M^* satisfies $C(A, w)$.

Theorem 7 *If $\models\!\!\!=_{J[A]} B$ then $\vdash_{J[A]} B$.*

Proof: Suppose $\vDash_{\overline{J[A]}} B$. By definition, for all $A^* \epsilon I(A)$, $\vdash_{\overline{J[A]}} A^*$. Hence, by Lemma 18, $M[J[A]]$ strongly satisfies $C(A, w)$. Hence by definition $M[J[A]] \vDash B$. Hence B is an element of all maximal consistent extensions of $J[A]$, so that by Lemma 3 $\vdash_{\overline{J[A]}} B$.

6 Relations to S0.5, S1, S2, and K It is obvious that *S2* is a *J*-extension. *S0.5* is not, however, since $\Diamond\Box[P_1 \wedge \neg P_1] \rightarrow \Diamond\Box P_2$ is a theorem of *J*, but not of *S0.5*. The proof of this in *J* consists in applying (RJ2) and then Lemma 4 to the obvious tautology. The disproof in *S0.5* consists of an appropriate assignment of truth conditions in the simplest possible two-world model for *S0.5*. *J* is not an extension of *S0.5*, either, for $\Box P_1 \rightarrow P_1$ is a theorem of *S0.5*, but not of *J*. As a consequence, *J* is not an extension of *S1* either, but I am not at present able to say whether *S1* is an extension of *J*. My conjecture is that it is not.

Since the Lemmon system *K* in [5] is an ordinary *S2*-extension, it is an ordinary *J*-extension as well. It is trivial to show that every ordinary *K*-extension is normal in the sense of [5], so for the case of ordinary *K*-extensions completeness results have already been given in [5].

NOTES

1. The usual formal semantics for *S2* was first given by Kripke in [3], and is presented by Hughes and Cresswell in [2]. The axiomatic presentations of *S2* most often used are variants on that presented by Lemmon in [4].
2. Chellas [1] also follows the style and methods of Lemmon and Scott [5], and adds considerable material not in [5]. It is therefore a useful alternative to [5], which has been out of print recently. Chellas does not consider *S2* or abnormal worlds, however.
3. We use the symbol '$\triangleq$' throughout in place of '$=_{df}$', to mean *is by definition.*

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